ACTA ARITHMETICA
LXXIV.4 (1996)

Ergodic properties of generalized Lüroth series

by

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1. Introduction

1.1. *Lüroth series.* In 1883 J. Lüroth [Lu] introduced and studied the following series expansion, which can be viewed as a generalization of the decimal expansion. Let $x \in (0,1]$. Then

$$(1)\quad x = \frac{1}{a_1} + \frac{1}{a_1(a_1-1)a_2} + \ldots + \frac{1}{a_1(a_1-1)\ldots a_{n-1}(a_{n-1}-1)a_n} + \ldots,$$

where $a_n \geq 2$, $n \geq 1$. Lüroth showed, among other things, that each irrational number has a unique infinite expansion (1), and that each rational number has either a finite or a periodic expansion.

Underlying the series expansion (1) is the operator $T_L : [0,1] \to [0,1]$, defined by

$$(2)\qquad T_L x := \left\lfloor \frac{1}{x} \right\rfloor \left(\left\lfloor \frac{1}{x} \right\rfloor + 1 \right) x - \left\lfloor \frac{1}{x} \right\rfloor, \quad x \neq 0; \quad T0 := 0$$

(see also Figure 1), where $\lfloor \xi \rfloor$ denotes the greatest integer not exceeding ξ. For $x \in [0,1]$ we define $a(x) := \lfloor 1/x \rfloor + 1$, $x \neq 0$; $a(0) := \infty$ and $a_n(x) = a(T^{n-1}x)$ for $n \geq 1$. From (2) it follows that $T_L x = a_1(a_1-1)x - (a_1-1)$, and therefore

$$\begin{aligned} x &= \frac{1}{a_1} + \frac{1}{a_1(a_1-1)} T_L x \\ &= \frac{1}{a_1} + \frac{1}{a_1(a_1-1)a_2} + \ldots + \frac{T_L^n x}{a_1(a_1-1)\ldots a_n(a_n-1)}. \end{aligned}$$

Since $a_n \geq 2$ and $0 \leq T_L^n x \leq 1$ one easily sees that the infinite series from (1) converges to x.

1991 *Mathematics Subject Classification*: 28D05, 11K55.

Recently S. Kalpazidou, A. and J. Knopfmacher introduced and studied in [K³1,2] the so-called *alternating Lüroth series*. For each $x \in (0,1]$ one has

$$x = \frac{1}{a_1 - 1} - \frac{1}{a_1(a_1-1)(a_2-1)} + \dots + \frac{(-1)^{n+1}}{a_1(a_1-1)\dots a_{n-1}(a_{n-1}-1)(a_n-1)} + \dots,$$

where $a_n \geq 2$, $n \geq 1$. Dynamically the alternating series expansion is generated by the operator $S_A : [0,1] \to [0,1]$ defined by

$$S_A x := 1 + \left\lfloor \frac{1}{x} \right\rfloor - \left\lfloor \frac{1}{x} \right\rfloor \left(\left\lfloor \frac{1}{x} \right\rfloor + 1 \right) x, \qquad x \neq 0, \tag{3}$$

and $S_A 0 := 0$, i.e. $S_A x = 1 - T_L x$ (see also Figure 1).

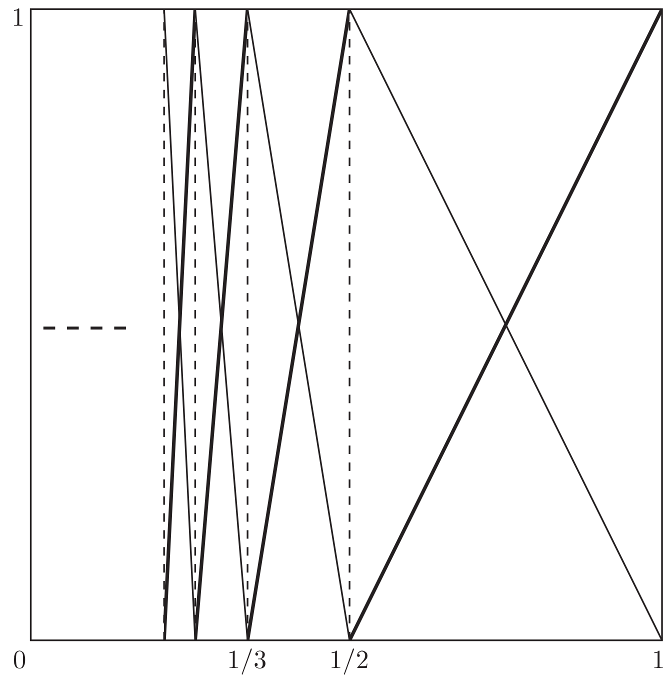

Fig. 1

Lüroth series have been extensively studied; for further reference we mention here papers by H. Jager and C. de Vroedt [JdV], T. Šalát [Sa], and books by J. Galambos [G], O. Perron [Pe] and W. Vervaat [V].

In [JdV] it was shown that the stochastic variables $a_1(x), a_2(x), \dots$ are independent with $\lambda(a_n = k) = 1/(k(k-1))$ for $k \geq 2$, and that T_L is measure preserving with respect to Lebesgue measure ([1]) and ergodic ([2]). Using similar techniques analogous results were obtained in [K^{3}1,2] for the operator S_A from (3). In fact, much stronger results can be obtained easily, not only in the case of the (alternating) Lüroth series, but also in a more general setting.

1.2. *Generalized Lüroth series*. Let $I_n = (l_n, r_n]$, $n \in \mathcal{D} \subset \mathbb{N} = \{0, 1, 2, \ldots\}$, be a finite or infinite collection of disjoint intervals of length $L_n := r_n - l_n$, such that

$$\sum_{n\in\mathcal{D}} L_n = 1 \tag{4}$$

and

$$0 < L_i \le L_j < 1 \qquad \text{for all } i, j \in \mathcal{D},\ i > j. \tag{5}$$

The set $\mathcal{D}$ is called the *digit set*. Usually such a digit set is either a finite or infinite set of consecutive positive integers, see also the examples at the end of this section.

Furthermore, let $I_\infty := [0,1] \setminus \bigcup_{n\in\mathcal{D}} I_n, L_\infty := 0$ and define the maps $T, S : [0,1] \to [0,1]$ by

$$Tx := \begin{cases} \dfrac{x - l_n}{r_n - l_n}, & x \in I_n,\ n \in \mathcal{D}, \\ 0, & x \in I_\infty, \end{cases}$$

$$Sx := \begin{cases} \dfrac{r_n - x}{r_n - l_n}, & x \in I_n,\ n \in \mathcal{D}, \\ 0, & x \in I_\infty. \end{cases}$$

Define for $x \in \Omega := [0,1] \setminus I_\infty = \bigcup_{n\in\mathcal{D}} I_n$,

$$s(x) := \frac{1}{r_n - l_n} \quad \text{and} \quad h(x) := \frac{l_n}{r_n - l_n}, \quad \text{in case } x \in I_n,\ n \in \mathcal{D},$$

$$s_n = s_n(x) := \begin{cases} s(T^{n-1}x), & T^{n-1}x \notin I_\infty, \\ \infty, & T^{n-1}x \in I_\infty, \end{cases}$$

and

$$h_n = h_n(x) := \begin{cases} h(T^{n-1}x), & T^{n-1}x \notin I_\infty, \\ 1, & T^{n-1}x \in I_\infty. \end{cases}$$

For $x \in (0,1)$ such that $T^{n-1}x \notin I_\infty$, one has

$$Tx = s(x)x - h(x) = s_1 x - h_1.$$

([1]) That is, $\lambda(T_L^{-1}(A)) = \lambda(A)$ for every Lebesgue measurable set A.

([2]) That is, $\lambda(A \mathbin{\triangle} T_L^{-1}A) = 0 \Rightarrow \lambda(A) \in \{0,1\}$.

Inductively we find

$$x = \frac{h_1}{s_1} + \frac{1}{s_1}Tx = \frac{h_1}{s_1} + \frac{1}{s_1}\left(\frac{h_2}{s_2} + \frac{1}{s_2}T^2x\right) = \ldots \tag{6}$$
$$= \frac{h_1}{s_1} + \frac{h_2}{s_1s_2} + \ldots + \frac{h_n}{s_1\ldots s_n} + \frac{1}{s_1\ldots s_n}T^nx.$$

Since $Sx = 1 - Tx = -s_1x + 1 + h_1$, for $x \in \Omega$, one finds

$$x = \frac{h_1+1}{s_1} - \frac{Sx}{s_1}. \tag{7}$$

Continuing in this way we obtain an alternating series expansion (see also [K^{3}1,2]). Figure 1 illustrates the case $\mathcal{D} = \mathbb{N}\setminus\{0,1\}$, $I_n := (1/n, 1/(n-1)]$.

Now let $\varepsilon = (\varepsilon(n))_{n\in\mathcal{D}}$ be an arbitrary, fixed sequence of zeroes and ones. We define the map $T_\varepsilon : [0,1] \to [0,1]$ by

$$T_\varepsilon x := \varepsilon(x)Sx + (1-\varepsilon(x))Tx, \qquad x \in [0,1], \tag{8}$$

where

$$\varepsilon(x) := \begin{cases} \varepsilon(n), & x \in I_n,\ n \in \mathcal{D},\\ 0, & x \in I_\infty. \end{cases}$$

Let $\varepsilon_n := \varepsilon(T_\varepsilon^{n-1}x)$,

$$s_n = s_n(x) := \begin{cases} s(T_\varepsilon^{n-1}x), & T_\varepsilon^{n-1}x \notin I_\infty,\\ \infty, & T_\varepsilon^{n-1}x \in I_\infty, \end{cases}$$

and h_n defined similarly. By (6) and (7) one finds that

$$\begin{aligned} x &= \frac{h_1+\varepsilon_1}{s_1} + \frac{(-1)^{\varepsilon_1}}{s_1}T_\varepsilon x\\ &= \frac{h_1+\varepsilon_1}{s_1} + \frac{(-1)^{\varepsilon_1}}{s_1}\left(\frac{h_2+\varepsilon_2}{s_2} + \frac{(-1)^{\varepsilon_2}}{s_2}T_\varepsilon^2 x\right) = \ldots\\ &= \frac{h_1+\varepsilon_1}{s_1} + (-1)^{\varepsilon_1}\frac{h_2+\varepsilon_2}{s_1s_2} + (-1)^{\varepsilon_1+\varepsilon_2}\frac{h_3+\varepsilon_3}{s_1s_2s_3} + \ldots\\ &\quad + (-1)^{\varepsilon_1+\ldots+\varepsilon_{n-1}}\frac{h_n+\varepsilon_n}{s_1\ldots s_n} + \frac{(-1)^{\varepsilon_1+\ldots+\varepsilon_n}}{s_1\ldots s_n}T_\varepsilon^n x. \end{aligned}$$

For each $k \ge 1$ and $1 \le i \le k$ one has $s_i \ge 1/L > 1$, where $L = \max_{n\in\mathcal{D}} L_n$, and $|T_\varepsilon^k x| \le 1$. Thus,

$$\left|x - \frac{p_k}{q_k}\right| = \frac{T_\varepsilon^k x}{s_1\ldots s_k} \le L^k \to 0 \qquad \text{as } k \to \infty, \tag{9}$$

where

$$\frac{p_k}{q_k} = \frac{h_1+\varepsilon_1}{s_1} + (-1)^{\varepsilon_1}\frac{h_2+\varepsilon_2}{s_1s_2} + (-1)^{\varepsilon_1+\varepsilon_2}\frac{h_3+\varepsilon_3}{s_1s_2s_3} + \ldots \tag{10}$$
$$+ (-1)^{\varepsilon_1+\ldots+\varepsilon_{k-1}}\frac{h_k+\varepsilon_k}{s_1\ldots s_k}$$

and $q_k = s_1 \ldots s_k$. In general p_k and q_k need not be relatively prime (see also Section 3.1). Let $\varepsilon_0 := 0$, then for each $x \in [0,1]$ one has

$$x = \sum_{n=1}^{\infty} (-1)^{\varepsilon_0 + \ldots + \varepsilon_{n-1}} \frac{h_n + \varepsilon_n}{s_1 \ldots s_n}. \tag{$*$}$$

For each $x \in [0,1]$ we define its sequence of *digits* $a_n = a_n(x)$, $n \geq 1$, as follows:

$$a_n = k \Leftrightarrow T_\varepsilon^{n-1} x \in I_k,$$

for $k \in \mathcal{D} \cup \{\infty\}$. The expansion $(*)$ is called the (I,ε)-*generalized Lüroth series* (*GLS*) of x. Notice that for each $x \in [0,1] \setminus I_\infty$ one finds a unique expansion $(*)$, and therefore a unique sequence of digits a_n, $n \geq 1$. Conversely, each sequence of digits a_n, $n \geq 1$, with $a_n \in \mathcal{D} \cup \{\infty\}$ and $a_1 \neq \infty$ defines a unique series expansion $(*)$. We denote $(*)$ by

$$x = \begin{bmatrix} \varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_n, \ldots \\ a_1, a_2, a_3, \ldots, a_n, \ldots \end{bmatrix}. \tag{11}$$

Since $\varepsilon_n = \varepsilon(a_n)$, $n \geq 1$, we might as well replace (11) by

$$x = [\, a_1, a_2, a_3, \ldots, a_n, \ldots] \tag{12}$$

No new information is obtained using (11) instead of (12). However, we will see in Section 3 that it is sometimes adventageous to use (11) instead of (12).

Finite truncations of the series in $(*)$ yield the sequence p_n/q_n of (I,ε)-GLS convergents of x. We denote p_n/q_n by

$$\frac{p_n}{q_n} = \begin{bmatrix} \varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_n \\ a_1, a_2, a_3, \ldots, a_n \end{bmatrix}.$$

Examples. 1. Let $I_n := (1/n, 1/(n-1)]$, $n \geq 2$. In case $\varepsilon_n = 0$ for $n \geq 2$, one gets the classical Lüroth series, while $\varepsilon_n = 1$ for $n \geq 2$ yields the alternating Lüroth series.

2. For $n \in \mathbb{N}$, $n \geq 2$, put $I_i = (i/n, (i+1)/n]$, $i = 0, 1, \ldots, n-1$. In case $\varepsilon(i) = 0$ for all i, T_ε yields the n-adic expansion. In case $n = 2$ and $\varepsilon(0) = 0$, $\varepsilon(1) = 1$, T_ε is the tent map.

See also Sections 3.2 and 3.3 for more intricate examples.

2. Ergodic properties of generalized Lüroth series. Let Ω be as in Section 1.2, $\mathcal{B}$ be the collection of Borel subsets of Ω, and λ be the Lebesgue measure on $(\Omega, \mathcal{B})$. Let (I,ε) be as in the previous section, viz. $I = (I_n)_{n \in \mathcal{D}}$ satisfies (4) and (5), while $\varepsilon = (\varepsilon(n))_{n \in \mathcal{D}}$ is a sequence of zeroes and ones. We have the following lemma.

Lemma 1. *The stochastic variables* $a_1(x), a_2(x), \ldots,$ *corresponding to the* (I,ε)-*GLS operator* T_ε *from* (8) *are i.i.d. with respect to the Lebesgue* *measure* λ, *and*

$$\lambda(a_n = k) = L_k \quad \textit{for } k \in \mathcal{D} \cup \{\infty\}.$$

Furthermore, $(I_n)_{n\in\mathcal{D}}$ *is a generating partition.*

Proof. Define for $(k_1, \ldots, k_n) \in \mathcal{D}^n$, $n \geq 1$, the so-called *fundamental intervals of order* n by

$$\Delta^{\varepsilon}_{k_1\ldots k_n} := \{x \in \Omega : a_1(x) = k_1, \ldots, a_n(x) = k_n\}. \tag{13}$$

Let $p_n/q_n \in \mathbb{Q}$ be defined as in (10) (and recall that the s_i's, h_i's and ε_i's are uniquely determined by $k_1, \ldots, k_n$), then obviously one has

$$x \in \Delta^{\varepsilon}_{k_1\ldots k_n} \Leftrightarrow \exists y \in [0,1] \text{ such that } x = \frac{p_n}{q_n} + \frac{(-1)^{\varepsilon_1+\ldots+\varepsilon_n}}{s_1 \ldots s_n} y.$$

Thus $\Delta^{\varepsilon}_{k_1\ldots k_n}$ is an interval with p_n/q_n as one endpoint, and having length $1/(s_1 \ldots s_n)$. Therefore,

$$\lambda(\Delta^{\varepsilon}_{k_1\ldots k_n}) = \lambda(a_1 = k_1, \ldots, a_n = k_n) = \frac{1}{s_1 \ldots s_n}.$$

Since

$$s_i = \frac{1}{r_{k_i} - l_{k_i}} = \frac{1}{L_{k_i}}, \qquad i = 1, \ldots, n,$$

one finds

$$\lambda(\Delta^{\varepsilon}_{k_1\ldots k_n}) = \lambda(a_1 = k_1, \ldots, a_n = k_n) = \prod_{i=1}^{n} L_{k_i}.$$

The independence of the $a_n(x)$'s and the equality $\lambda(a_n = k) = L_k$ for $k \in \mathcal{D} \cup \{\infty\}$ now easily follow from

$$\sum_{k_i \in \mathcal{D}} L_{k_i} = 1 \qquad \text{for all } n \geq 1 \text{ and all } 1 \leq i \leq n.$$

That $(I_n)_{n\in\mathcal{D}}$ is a generating partition is immediate from (9). ∎

THEOREM 1. *The* (I,ε)*-GLS operator* T_ε *from* (8) *is measure preserving with respect to Lebesgue measure and Bernoulli.*

Proof. For any $k_1, \ldots, k_n \in \mathcal{D}$, $n \geq 1$,

$$T_\varepsilon^{-1} \Delta^{\varepsilon}_{k_1\ldots k_n} = \bigcup_{k\in\mathcal{D}} \Delta^{\varepsilon}_{kk_1\ldots k_n}$$

is a disjoint union of fundamental intervals of order $n+1$, so that

$$\begin{aligned}\lambda(T_\varepsilon^{-1} \Delta^{\varepsilon}_{k_1\ldots k_n}) &= \sum_{k\in\mathcal{D}} \lambda(\Delta^{\varepsilon}_{kk_1\ldots k_n}) = L_{k_1} \ldots L_{k_n} \sum_{k\in\mathcal{D}} L_k \\ &= L_{k_1} \ldots L_{k_n} = \lambda(\Delta^{\varepsilon}_{k_1\ldots k_n}).\end{aligned}$$

Since the collection $\{\Delta^{\varepsilon}_{k_1\ldots k_n} : n \geq 1,\ k_i \in \mathcal{D}\}$ generates $\mathcal{B}$, it follows from [W, Theorem 1.1, p. 20] that λ is T_ε-invariant. From Lemma 1, viz.

$$\lambda(\Delta^{\varepsilon}_{k_1\dots k_n}) = \prod_{i=1}^{n} \lambda(\Delta^{\varepsilon}_{k_i}),$$

we conclude that $([0,1],\mathcal{B},\lambda,T_\varepsilon)$ is a Bernoulli system. ■

R e m a r k s. 1. The Bernoullicity of the Lüroth operator T_L was already noticed by P. Liardet in [Li].

2. From the fact that T_ε is Bernoulli, and therefore ergodic, one can draw a great number of easy consequences, using Birkhoff's Ergodic Theorem. See also [JdV] and [K^{3}2]. We mention here some results:

For almost every x the sequence $(T_\varepsilon^n x)_{n=0}^{\infty}$ is uniformly distributed over $[0,1]$. Furthermore,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} T_\varepsilon^k x = \frac{1}{2}, \qquad \lim_{n\to\infty} \Big(\prod_{k=0}^{n-1} T_\varepsilon^k x\Big)^{1/n} = \frac{1}{e} \ \ a.e.$$

and

$$\lim_{n\to\infty} (a_1 \dots a_n)^{1/n} = e^c \qquad a.e.,$$

where $c = \sum_{k\in\mathcal{D}} L_k \log k$ (3).

Define the map $\mathcal{T}_\varepsilon : [0,1]\times[0,1] \to [0,1]\times[0,1]$ by

$$\mathcal{T}_\varepsilon(x,y) := \Big(T_\varepsilon(x),\ \frac{h(x)+\varepsilon(x)}{s(x)} + \frac{(-1)^{\varepsilon(x)}}{s(x)}y\Big). \tag{14}$$

Notice that for

$$x = \begin{bmatrix} \varepsilon_1,\ \varepsilon_2,\ \varepsilon_3,\ \dots,\ \varepsilon_n,\ \dots \\ a_1,\ a_2,\ a_3,\ \dots,\ a_n,\ \dots \end{bmatrix}$$

one has

$$\mathcal{T}_\varepsilon(x,0) = \left(\begin{bmatrix} \varepsilon_2,\ \varepsilon_3,\ \dots,\ \varepsilon_n,\ \dots \\ a_2,\ a_3,\ \dots,\ a_n,\ \dots \end{bmatrix}, \begin{bmatrix} \varepsilon_1 \\ a_1 \end{bmatrix} \right),$$

where

$$\begin{bmatrix} \varepsilon_1 \\ a_1 \end{bmatrix} = (-1)^{\varepsilon_0}\frac{h_1+s_1}{s_1} = \frac{h_1+s_1}{s_1}.$$

Now

$$\mathcal{T}_\varepsilon^2(x,0) = \left(\begin{bmatrix} \varepsilon_3,\ \varepsilon_4,\ \dots,\ \varepsilon_n,\ \dots \\ a_3,\ a_4,\ \dots,\ a_n,\ \dots \end{bmatrix}, \begin{bmatrix} \varepsilon_2,\ \varepsilon_1 \\ a_2,\ a_1 \end{bmatrix} \right),$$

where

$$\begin{bmatrix} \varepsilon_2,\ \varepsilon_1 \\ a_2,\ a_1 \end{bmatrix} = (-1)^{\varepsilon_0}\frac{h_2+\varepsilon_2}{s_2} + (-1)^{\varepsilon_0+\varepsilon_2}\frac{h_1+\varepsilon_1}{s_1 s_2} = \frac{h_2+\varepsilon_2}{s_2} + (-1)^{\varepsilon_2}\frac{h_1+\varepsilon_1}{s_1 s_2}.$$

(3) In case $0\in\mathcal{D}$ we put $e^c := 0$.

Putting $\mathcal{T}_\varepsilon^n(x,0) =: (T_n, V_n)$, $n \ge 0$, where

$$T_n = T_\varepsilon^n x = \begin{bmatrix} \varepsilon_{n+1},\ \varepsilon_{n+2},\ \ldots \\ a_{n+1},\ a_{n+2},\ \ldots \end{bmatrix}, \qquad n \ge 0,$$

and

$$V_n = \begin{bmatrix} \varepsilon_n,\ \varepsilon_{n-1},\ \ldots,\ \varepsilon_1 \\ a_n,\ a_{n-1},\ \ldots,\ a_1 \end{bmatrix}, \qquad n \ge 1, \qquad V_0 := 0,$$

we see inductively that

$$V_{n+1} = \frac{h_{n+1} + \varepsilon_{n+1}}{s_{n+1}} + \frac{(-1)^{\varepsilon_{n+1}}}{s_{n+1}} V_n.$$

As in the case of the continued fraction we will call $T_n = T_\varepsilon^n x$ the *future of x at time n*, while $V_n = V_n(x)$ is *the past of x at time n* (see also [K]). We have the following theorem.

THEOREM 2. *The system* $([0,1]\times[0,1], \mathcal{B}\times\mathcal{B}, \lambda\times\lambda, \mathcal{T}_\varepsilon)$ *is the natural extension of* $([0,1], \mathcal{B}, \lambda, T_\varepsilon)$. *Furthermore*, $([0,1]\times[0,1], \mathcal{B}\times\mathcal{B}, \lambda\times\lambda, \mathcal{T}_\varepsilon)$ *is Bernoulli.*

Proof. For any two vectors $(k_1,\ldots,k_n) \in \mathcal{D}^n$, $(l_1,\ldots,l_m) \in \mathcal{D}^m$ one has

$$\Delta^\varepsilon_{k_1\ldots k_n} \times \Delta^\varepsilon_{l_1\ldots l_m} = \mathcal{T}_\varepsilon^m(\Delta^\varepsilon_{l_m\ldots l_1 k_1\ldots k_n} \times [0,1]).$$

Since $\{\Delta^\varepsilon_{k_1\ldots k_n} \times \Delta^\varepsilon_{l_1\ldots l_m} : k_i, l_j \in \mathcal{D},\ 1 \le i \le n,\ 1 \le j \le m,\ n,m \ge 1\}$ generates $\mathcal{B}\times\mathcal{B}$, it follows that

$$\bigvee_{m\ge 0} \mathcal{T}_\varepsilon^m(\mathcal{B}\times[0,1]) = \mathcal{B}\times\mathcal{B}.$$

Now, for any $\Delta^\varepsilon_{k_1\ldots k_n} \times \Delta^\varepsilon_{l_1\ldots l_m}$ one has

$$\mathcal{T}_\varepsilon^{-1}(\Delta^\varepsilon_{k_1\ldots k_n} \times \Delta^\varepsilon_{l_1\ldots l_m}) = \Delta^\varepsilon_{l_1 k_1\ldots k_n} \times \Delta^\varepsilon_{l_2\ldots l_m}.$$

Thus,

$$\begin{aligned}\lambda\times\lambda(\mathcal{T}_\varepsilon^{-1}(\Delta^\varepsilon_{k_1\ldots k_n} \times \Delta^\varepsilon_{l_1\ldots l_m})) &= \lambda(\Delta^\varepsilon_{l_1 k_1\ldots k_n})\lambda(\Delta^\varepsilon_{l_2\ldots l_m}) \\ &= \lambda(\Delta^\varepsilon_{k_1\ldots k_n})\lambda(\Delta^\varepsilon_{l_1\ldots l_m}) \\ &= \lambda\times\lambda(\Delta^\varepsilon_{k_1\ldots k_n} \times \Delta^\varepsilon_{l_1\ldots l_m}).\end{aligned}$$

Since cylinders of the form $\Delta^\varepsilon_{k_1\ldots k_n} \times \Delta^\varepsilon_{l_1\ldots l_m}$ generate the σ-algebra $\mathcal{B}\times\mathcal{B}$, it follows that $\mathcal{T}_\varepsilon$ is measure preserving with respect to Lebesgue measure. Thus, $\mathcal{T}_\varepsilon$ is the natural extension of T_ε (see [R] for details). Since T_ε is Bernoulli it is an exercise to show that $\mathcal{T}_\varepsilon$ is Bernoulli (see also [B]). ■

COROLLARY 1. *For almost all x the two-dimensional sequence*

$$\mathcal{T}_\varepsilon^n(x,0) = (T_n, V_n), \qquad n \ge 0, \tag{15}$$

is uniformly distributed over $[0,1]\times[0,1]$.

P r o o f. Denote by $\mathtt{A}$ that subset of $[0,1]$ for which the sequence $(T_n, V_n)_{n=0}^{\infty}$ is *not* uniformly distributed over $[0,1]\times[0,1]$. It follows from Lemma 1 and the definition of $\mathcal{T}_\varepsilon$ that for all $x, y, y^* \in [0,1]$ one has

$$|\mathcal{T}_\varepsilon^n(x,y) - \mathcal{T}_\varepsilon^n(x,y^*)| < L^n, \qquad n \ge 0,$$

and we see that $(\mathcal{T}_\varepsilon^n(x,y) - \mathcal{T}_\varepsilon^n(x,y^*))_{n=0}^{\infty}$ is a null-sequence. Hence, if $\mathcal{A} := \mathtt{A} \times [0,1]$, then for every pair $(x,y) \in \mathcal{A}$ the sequence $\mathcal{T}_\varepsilon^n(x,y)$, $n \ge 0$, is not uniformly distributed over $[0,1]\times[0,1]$. Now, if $\mathtt{A}$ had, as a subset of $[0,1]$, positive Lebesgue measure, so would $\mathcal{A}$ as a subset of $[0,1]\times[0,1]$. However, this is impossible in view of Theorem 2. ■

The partition $\xi = \{I_k \times [0,1]\}_{k\in\mathcal{D}}$ is a generator for $\mathcal{T}_\varepsilon$, which implies that the entropy $h(\mathcal{T}_\varepsilon)$ of $\mathcal{T}_\varepsilon$ equals $h(\mathcal{T}_\varepsilon, \xi)$ (see also [W], p. 96). Therefore,

$$h(\mathcal{T}_\varepsilon) = -\sum_{k\in\mathcal{D}} L_k \log L_k.$$

Now let $(I_k)_{k\in\mathcal{D}}$ and $(I_k^*)_{k\in\mathcal{D}}$ be two partitions of $[0,1]$ satisfying (4) and (5), and suppose that $L_k = L_k^*$ for $k \in \mathcal{D}$. Furthermore, let $\varepsilon = (\varepsilon_k)_{k\in\mathcal{D}}$ and $\varepsilon^* = (\varepsilon_k^*)_{k\in\mathcal{D}}$ be two arbitrary sequences of zeroes and ones. It follows at once from Ornstein's Isomorphism Theorem (see [W], p. 105) and Theorem 2 that $\mathcal{T}_\varepsilon$ and $\mathcal{T}_{\varepsilon^*}$ are metrically isomorphic. We conclude this section with the following theorem, which gives a concrete isomorphism.

THEOREM 3. *Let $(I_k)_{k\in\mathcal{D}}$ and $(I_k^*)_{k\in\mathcal{D}}$ be two partitions of $[0,1]$, satisfying* (4) *and* (5). *Suppose that $L_k = L_k^*$ for $k \in \mathcal{D}$. Furthermore, let $\varepsilon = (\varepsilon_k)_{k\in\mathcal{D}}$ and $\varepsilon^* = (\varepsilon_k^*)_{k\in\mathcal{D}}$ be two sequences of zeroes and ones. Let $\mathcal{T}_\varepsilon$ and $\mathcal{T}_{\varepsilon^*}$ be defined as in* (8). *Finally, define $\Psi : [0,1]\times[0,1] \to [0,1]\times[0,1]$ by*

$$\Psi\left(\begin{bmatrix}\varepsilon_1, \varepsilon_2, \ldots \\ a_1, a_2, \ldots\end{bmatrix}, \begin{bmatrix}\varepsilon_0, \varepsilon_{-1}, \ldots \\ a_0, a_{-1}, \ldots\end{bmatrix}\right) := \left(\begin{bmatrix}\varepsilon^*(a_1), & \varepsilon^*(a_2), & \ldots \\ a_1, & a_2, & \ldots\end{bmatrix}, \begin{bmatrix}\varepsilon^*(a_0), & \varepsilon^*(a_{-1}), & \ldots \\ a_0, & a_{-1}, & \ldots\end{bmatrix}\right).$$

Then Ψ is a measure preserving isomorphism.

P r o o f. Since almost every $x \in [0,1]$ has unique (I,ε)-, (I^*,ε^*)-GLS expansions, it follows that Ψ is injective. Now, for any cylinder $\Delta_{k_1\ldots k_n}^{\varepsilon^*} \times \Delta_{l_1\ldots l_m}^{\varepsilon^*}$,

$$\Delta_{k_1\ldots k_n}^{\varepsilon} \times \Delta_{l_1\ldots l_m}^{\varepsilon} = \Psi^{-1}(\Delta_{k_1\ldots k_n}^{\varepsilon^*} \times \Delta_{l_1\ldots l_m}^{\varepsilon^*})$$

and

$$\begin{aligned}(\lambda\times\lambda)(\Delta^{\varepsilon^*}_{k_1\ldots k_n}\times\Delta^{\varepsilon^*}_{l_1\ldots l_m}) &= L^*_{k_1}\ldots L^*_{k_n}L^*_{l_1}\ldots L^*_{l_m}\\ &= L_{k_1}\ldots L_{k_n}L_{l_1}\ldots L_{l_m}\\ &= (\lambda\times\lambda)(\Delta^{\varepsilon}_{k_1\ldots k_n}\times\Delta^{\varepsilon}_{l_1\ldots l_m})\\ &= (\lambda\times\lambda)\Psi^{-1}(\Delta^{\varepsilon^*}_{k_1\ldots k_n}\times\Delta^{\varepsilon^*}_{l_1\ldots l_m}).\end{aligned}$$

This shows that Ψ is measurable and measure preserving.

Finally, let $(x,y)\in[0,1]\times[0,1]$ with

$$x=\begin{bmatrix}\varepsilon_1,\ \varepsilon_2,\ \ldots\\ a_1,\ a_2,\ \ldots\end{bmatrix},\qquad y=\begin{bmatrix}\varepsilon_0,\ \varepsilon_{-1},\ \ldots\\ a_0,\ a_{-1},\ \ldots\end{bmatrix}.$$

Then

$$\begin{aligned}\Psi\mathcal{T}_\varepsilon(x,y) &= \left(\begin{bmatrix}\varepsilon^*(a_2), & \varepsilon^*(a_3), & \ldots\\ a_2, & a_3, & \ldots\end{bmatrix},\begin{bmatrix}\varepsilon^*(a_1), & \varepsilon^*(a_0), & \varepsilon^*(a_{-1}), & \ldots\\ a_1, & a_0, & a_{-1}, & \ldots\end{bmatrix}\right)\\ &= \mathcal{T}_{\varepsilon^*}\Psi(x,y),\end{aligned}$$

therefore Ψ is a measure preserving isomorphism. ■

3. Applications and examples

3.1. *Approximation coefficients and their distribution*. As before let $I=(I_n)_{n\in\mathcal{D}}$ be a partition of [0,1] which satisfies (4) and (5), and let $\varepsilon=(\varepsilon(n))_{n\in\mathcal{D}}$ be a sequence of zeroes and ones. Putting $q_k=s_1\ldots s_k$, it follows from (9) and Corollary 1 that for a.e. x the *approximation coefficients* θ_n, $n\ge 0$, defined by

$$\theta_{\varepsilon,n}=\theta_{\varepsilon,n}(x):=q_n\left|x-\frac{p_n}{q_n}\right|,\qquad n\ge 0,$$

have the same distribution as $T^n_\varepsilon x$, $n\ge 0$. Viz., for a.e. x the sequence $(\theta_{\varepsilon,n})_n$ is uniformly distributed on [0,1].

In fact, for many partitions $(I_n)_{n\in\mathcal{D}}$ more information on the distribution of the θ_n's can be obtained by a more careful definition of q_n. As an example we will treat here the case of the classical Lüroth series, and all other GLS expansions with the same partition $(I_n)_{n\in\mathcal{D}}$ (see also the examples at the end of Section 1.2).

In this case

$$s_n=s(T^{n-1}_\varepsilon x)=\frac{1}{\frac{1}{a_n-1}-\frac{1}{a_n}}=a_n(a_n-1),\qquad h_n=a_n-1,$$

and

$$\frac{h_n+\varepsilon_n}{s_1\ldots s_n}=\frac{a_n-1+\varepsilon_n}{a_1(a_1-1)a_2(a_2-1)\ldots a_n(a_n-1)}=\frac{1}{a_1(a_1-1)\ldots(a_n-\varepsilon_n)}.$$

Therefore it is more appropriate to put

$$q_1=a_1-\varepsilon_1,\quad q_n=a_1(a_1-1)a_2(a_2-1)\ldots a_{n-1}(a_{n-1}-1)(a_n-\varepsilon_n),\quad n\ge 2,$$

and we see

$$\theta_n(x) = \frac{a_n - \varepsilon_n}{a_n(a_n - 1)} T_\varepsilon^n x, \qquad n \geq 1. \tag{16}$$

We have the following theorem.

THEOREM 4. *Let* $(I_n)_{n\in\mathcal{D}}$ *be the Lüroth partition, that is,* $I_n := (1/n, 1/(n-1)]$ *for* $n \geq 2$, *and let* $\varepsilon(n) \in \{0,1\}$ *for* $n \geq 2$. *Then for a.e.* x *and for every* $z \in (0,1]$ *the limit*

$$\lim_{N\to\infty} \frac{1}{N} \#\{1 \leq j \leq N : \theta_j(x) < z\}$$

exists and equals $F_\varepsilon(z)$, *where*

$$F_\varepsilon(z) := \sum_{k=2}^{\lfloor 1/z\rfloor+1-\varepsilon(\lfloor 1/z\rfloor+1)} \frac{z}{k-\varepsilon(k)} + \frac{1}{\lfloor 1/z\rfloor + 1 - \varepsilon(\lfloor 1/z\rfloor+1)}, \qquad 0 < z \leq 1.$$

P r o o f. Let $z \in (0,1]$. From (15) and (16) it follows that

$$\theta_n < z \Leftrightarrow (T_n, V_n) \in \Xi(z) = \bigcup_{k=2}^{\infty} \Xi_k(z), \tag{17}$$

where

$$\Xi_k(z) := \left(\left[0, \frac{k(k-1)}{k-\varepsilon(k)} z\right] \cap [0,1]\right) \times \Delta_k, \qquad k \geq 2.$$

For $k \geq 2$ we have the following two cases (of which the first one might be void).

(A) $2 \leq k \leq \lfloor 1/z\rfloor + 1 - \varepsilon(\lfloor 1/z\rfloor + 1)$. In this case

$$\Xi_k(z) = \left[0, \frac{k(k-1)}{k-\varepsilon(k)} z\right] \times \Delta_k.$$

(B) $k > \lfloor 1/z\rfloor + 1 - \varepsilon(\lfloor 1/z\rfloor + 1)$. In this case $\Xi_k(z) = [0,1] \times \Delta_k$.

From (A) and (B) one finds, that

$$\begin{aligned}
(\lambda\times\lambda)(\Xi(z)) &= \sum_{k=2}^{\lfloor 1/z\rfloor+1-\varepsilon(\lfloor 1/z\rfloor+1)} (\lambda\times\lambda)\left(\left[0, \frac{k(k-1)}{k-\varepsilon(k)} z\right] \times \Delta_k\right) \\
&\quad + \frac{1}{\lfloor 1/z\rfloor + 1 - \varepsilon(\lfloor 1/z\rfloor+1)} \\
&= \sum_{k=2}^{\lfloor 1/z\rfloor+1-\varepsilon(\lfloor 1/z\rfloor+1)} \frac{z}{k-\varepsilon(k)} + \frac{1}{\lfloor 1/z\rfloor + 1 - \varepsilon(\lfloor 1/z\rfloor+1)}.
\end{aligned}$$

The theorem at once follows from Corollary 1. ∎

Remarks. 1. Although the map $x \to (1/x) \bmod 1$ generating the continued fraction is not piecewise linear, which leads to complications in estimations, a similar result as in Theorem 4 was obtained for continued fractions (see also [BJW]).

2. If $\varepsilon(n) = 0$, $n \geq 2$ (the classical Lüroth case) ([4]), the distribution function F_ε reduces to

$$F_{\mathrm{L}}(z) = \sum_{k=2}^{\lfloor 1/z \rfloor + 1} \frac{z}{k} + \frac{1}{\lfloor 1/z \rfloor + 1}, \qquad 0 < z \leq 1.$$

Furthermore

$$F_{\mathrm{A}}(z) = \sum_{k=2}^{\lfloor 1/z \rfloor} \frac{z}{k-1} + \frac{1}{\lfloor 1/z \rfloor}, \qquad 0 < z \leq 1;$$

see also Figure 2. Notice that $F_{\mathrm{A}} \leq F_\varepsilon \leq F_{\mathrm{L}}$.

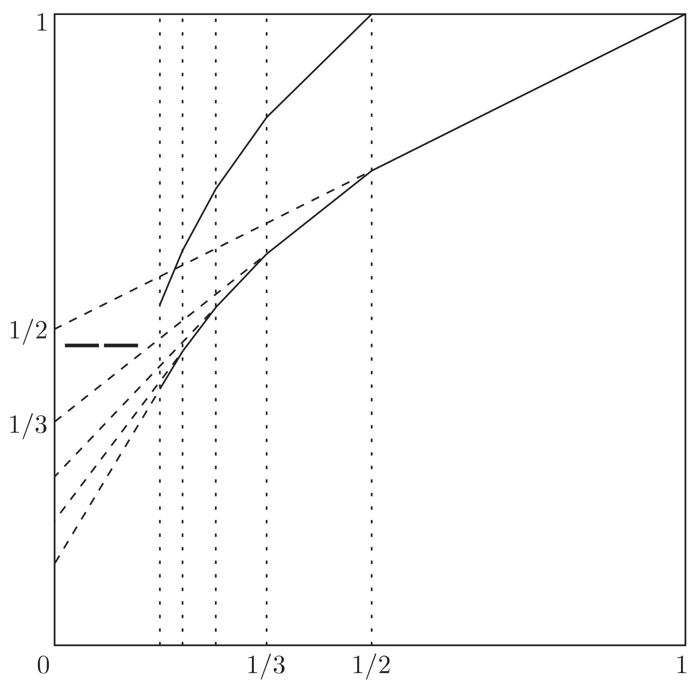

Fig. 2

We have the following corollary.

([4]) From now on the classical (resp. the alternating) Lüroth case will be indicated by a subscript L (resp. A).

COROLLARY 2. *Let $(I_n)_{n\in\mathcal{D}}$ be the Lüroth partition and let $\varepsilon(n)\in\{0,1\}$ for $n\ge 2$. Then there exists a constant M_ε such that for a.e. x,*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\theta_{\varepsilon,i}=M_\varepsilon.$$

Moreover, M_ε can be calculated explicitly, and $M_{\rm A}\le M_\varepsilon\le M_{\rm L}$, where $M_{\rm A}=1-\frac{1}{2}\zeta(2)=0.177533\ldots$ and $M_{\rm L}=\frac{1}{2}(\zeta(2)-1)=0.322467\ldots$

Proof. By definition M_ε is the first moment of F_ε and thus $M_\varepsilon=\int_0^1(1-F_\varepsilon(x))\,dx$. ∎

Remarks. 1. From Corollary 2 it follows that among all (I,ε)-GLS expansions with I the Lüroth partition the alternating Lüroth series has the best approximation properties.

2. The presentation of Corollary 2 suggests that by choosing $\varepsilon=(\varepsilon(n))_{n\ge2}$ appropriately, each value in the interval $[M_{\rm A},M_{\rm L}]=[0.177533\ldots,0.322467\ldots]$ might be attained. This is certainly incorrect, as the following example shows. Let $\varepsilon=(\varepsilon(n))_{n\ge2}$ be given by $\varepsilon(2)=1$ and $\varepsilon(n)=0$ for $n\ge3$, and let $\varepsilon^*=(\varepsilon^*(n))_{n\ge2}$ be given by $\varepsilon^*(2)=0$ and $\varepsilon^*(n)=1$ for $n\ge3$. A simple calculation yields that $M_\varepsilon=M_{\rm L}-1/8=0.197467\ldots$ and $M_{\varepsilon^*}=M_{\rm A}+1/8=0.302533\ldots$; we thus see that $M_\varepsilon<M_{\varepsilon^*}$ and from this one can easily deduce that there does not exist a sequence $\varepsilon^\flat=(\varepsilon^\flat(n))_{n\ge2}$ of zeroes and ones for which $M_{\varepsilon^\flat}\in[M_\varepsilon,M_{\varepsilon^*}]$. Some further investigations even suggest that the set

$$\Upsilon:=\{M_\varepsilon:\varepsilon(n)\in\{0,1\}\text{ for }n\ge2\}$$

is a Cantor set.

3.2. *Jump transformations.* Let T_ε be a (I,ε)-GLS operator with digit set $\mathcal{D}$, and let $a\in\mathcal{D}$. For $x\in\Omega$, put

$$n_a=n_a(x):=\min_{n\ge1}\{a_n(x):a_n(x)=a\}$$

(and $n_a=\infty$ in case $a_n(x)\ne a$ for all $n\ge1$). Define the *jump transformation* $J_a:\Omega\to\Omega$ by

$$J_ax:=\begin{cases}T_\varepsilon^{n_a}x, & n_a\in\mathbb{N},\\ 0, & n_a=\infty.\end{cases}\tag{18}$$

Jump transformations were first studied by H. Jager [J] for the particular case that $T_\varepsilon x=10x\pmod 1$. Jager showed that such jump transformations are stronly mixing. Here, in the more general setting, we have a stronger result.

THEOREM 5. *Let T_ε be an (I,ε)-GLS operator with digit set $\mathcal{D}$. For each $a\in\mathcal{D}$ the corresponding jump transformation J_a, as defined in* (18), *is an*

(I^*, ε^*)-GLS operator, with

$$I^* = \{\Delta_{a_1\ldots a_n} : n \geq 1,\ a_n = a \text{ and } a_i \neq a \text{ for } 1 \leq i \leq n-1\}$$

and for each $\Delta_{a_1\ldots a_n} \in I^*$ the corresponding value of ε is given by

$$\varepsilon^*(\Delta_{a_1\ldots a_n}) = \varepsilon(a_1) + \ldots + \varepsilon(a_n) \pmod 2.$$

3.3. *β-expansions and pseudo golden mean numbers*. For an irrational number $\beta > 1$ the β-transformation $T_\beta : [0,1] \to [0,1]$ is defined by

$$T_\beta x = \beta x \pmod 1$$

(see also [FS] for further references). Clearly, β-transformations do not belong to the class of GLS-transformations. However, in some cases there exists an intimate relation between both types of transformations, as the following example shows.

Let $\beta > 1$ be the positive root of the polynomial $X^m - X^{m-1} - \ldots - X - 1$, where [5] $m \geq 2$. Due to C. Frougny and B. Solomyak [FS] we know that such β's are Pisot numbers and that the β-expansion $d(1,\beta)$ is finite, and equals

$$1 = \frac{1}{\beta} + \frac{1}{\beta^2} + \ldots + \frac{1}{\beta^m}.$$

Notice that $T_\beta^i 1 = \beta^{-1} + \ldots + \beta^{-(m-i)}$, $0 \leq i \leq m-1$, and $T_\beta^i 1 = 0$ for $i \geq m$. Furthermore, let

$$X := \bigcup_{k=0}^{m-1} (T_\beta^{m-k} 1, T_\beta^{m-k-1} 1] \times [0, T_\beta^k 1]$$

(see also Figure 3 for $m = 4$), and define $\mathcal{T}_\beta : X \to X$ by

$$\mathcal{T}_\beta(x,y) := \left(T_\beta x, \frac{1}{\beta}(\lfloor \beta x \rfloor + y)\right).$$

Let $Y := [0,1] \times [0, 1/\beta]$ and $\mathcal{W} : Y \to Y$ the corresponding induced transformation under $\mathcal{T}_\beta$, that is

$$\mathcal{W}(x,y) = \mathcal{T}_\beta^{n(x,y)}(x,y),$$

where $n(x,y) = \min\{s > 0 : \mathcal{T}_\beta^s(x,y) \in Y\}$. Clearly one has

$$\mathcal{W}(x,y) = \mathcal{T}_\beta^{k+1}(x,y),$$

where $k = k(x) \in \{0, 1, \ldots, m-1\}$ is such that $x \in (T_\beta^{m-k} 1, T_\beta^{m-k-1} 1]$. Notice that $\mathcal{W}$ maps rectangles to rectangles; see also Figure 3.

Finally, let T_ε be the (I,ε)-GLS operator with partition I given by $(T_\beta^{m-i} 1, T_\beta^{m-i-1} 1]$, $0 \leq i \leq m-1$ (see also Figure 3), and $\varepsilon(n) = 0$ for

[5] For $m = 2$ one has $\beta = (\sqrt{5}+1)/2$, which is the golden mean. For $m \geq 3$ we call these β's *pseudo golden mean* numbers.

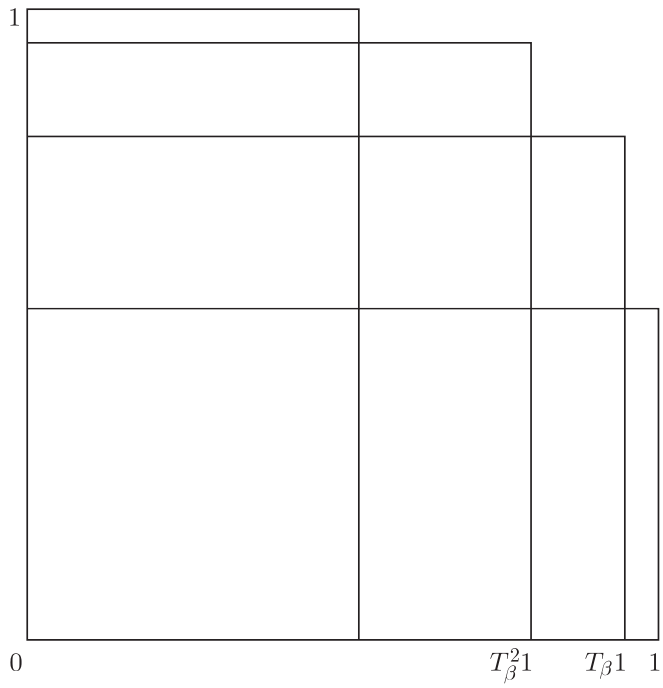

Fig. 3

each digit n. Notice that for $x \in (T_\beta^{m-i}1, T_\beta^{m-(i+1)}1]$, $0 \le i \le m-1$, one has

$$T_\varepsilon x = T_\beta^{i+1} x. \tag{19}$$

We have the following lemma.

LEMMA 2. *Let $\Psi : [0,1] \times [0,1] \to Y$ be defined by $\Psi(x,y) := (x, y/\beta)$. Then Ψ is a measurable bijection which satisfies $\Psi \circ \mathcal{T}_\varepsilon = \mathcal{W} \circ \Psi$.*

Proof. For $(x,y) \in [0,1] \times [0,1]$ let $i = i(x) \in \{0,1,\ldots,m-1\}$ be such that $x \in (T_\beta^{m-i}1, T_\beta^{m-(i+1)}1]$. From (14) it then follows that

$$\mathcal{T}_\varepsilon(x,y) = \left(T_\varepsilon x, T_\beta^{m-i}1 + \frac{y}{\beta^{i+1}}\right]$$

and therefore

$$\Psi(\mathcal{T}_\varepsilon(x,y)) = \left(T_\varepsilon x, \frac{1}{\beta} T_\beta^{m-i}1 + \frac{y}{\beta^{i+2}}\right]. \tag{20}$$

From (19), (20) and the definitions of $\mathcal{W}$ and Ψ it now follows that $\mathcal{W}(\Psi(x,y)) = \Psi(\mathcal{T}_\varepsilon(x,y))$ for $i = 0$ and in case $i \ne 0$ one has

$$\begin{aligned}\mathcal{W}(\Psi(x,y)) &= \mathcal{W}\left(x,\frac{y}{\beta}\right) = \mathcal{T}_\beta^{i+1}\left(x,\frac{y}{\beta}\right)\\ &= \left(T_\beta^{i+1}x, \frac{1}{\beta}\left(0+\left(\underbrace{\frac{1}{\beta}\left(1+\ldots\frac{1}{\beta}\left(1+\frac{1}{\beta}\left(1\right.\right.\right.}_{i\text{ times}}+\frac{y}{\beta}\right)\right)\ldots\right)\right)\\ &= \left(T_\varepsilon x, \frac{1}{\beta^2}+\ldots+\frac{1}{\beta^{i+1}}+\frac{y}{\beta^{i+2}}\right) = \Psi(\mathcal{T}_\varepsilon(x,y)). \ \blacksquare\end{aligned}$$

Let ϱ be the measure on Y defined by

$$\varrho(A) := (\lambda\times\lambda)(\Psi^{-1}(A)) \quad \text{for each Borel set } A\subset Y.$$

It follows from Lemma 2 and the fact that $\lambda\times\lambda$ is an invariant measure for $\mathcal{T}_\varepsilon$ that ϱ is invariant with respect to $\mathcal{W}$, and $\varrho = \beta(\lambda\times\lambda)$. Lemma 2 now at once yields the following proposition.

Proposition. *The dynamical systems* $([0,1]\times[0,1],\lambda\times\lambda,\mathcal{T}_\varepsilon)$ *and* $(Y,\varrho,\mathcal{W})$ *are isomorphic.*

Using standard techniques (see [CFS], p. 21) one obtains the measure μ on X which is invariant with respect to $\mathcal{T}_\beta$, viz.

$$\mu(A) = \frac{\beta}{\frac{1}{\beta}+\frac{2}{\beta^2}+\ldots+\frac{m}{\beta^m}}(\lambda\times\lambda)(A)$$

for each Borel set $A\subset X$. One also easily shows that $(X,\mu,\mathcal{T}_\beta)$ forms the natural extension of $([0,1],\nu,T_\beta)$, where ν is the invariant measure with respect to T_β [So]. Projecting μ on the first coordinate of X yields ν; one finds that ν has density $h(x)$, where

$$h(x) = \frac{1}{\frac{1}{\beta}+\frac{2}{\beta}+\ldots+\frac{m}{\beta^m}}\sum_{x<T_\beta^i 1}\frac{1}{\beta^i},$$

as given by W. Parry [Pa].

Acknowledgements. We want to thank the referee for several helpful suggestions concerning the presentation of this paper.

References

[B] J. R. Brown, *Ergodic Theory and Topological Dynamics*, Academic Press, New York, 1976.

[BJW] W. Bosma, H. Jager and F. Wiedijk, *Some metrical observations on the approximation by continued fractions*, Indag. Math. 45 (1983), 281–299.

[CFS] I. P. Cornfeld, S. V. Fomin and Ya. G. Sinai, *Ergodic Theory*, Grundlehren Math. Wiss. 245, Springer, New York, 1982.

[FS] C. Frougny and B. Solomyak, *Finite beta-expansions*, Ergodic Theory Dynamical Systems 12 (1992), 713–723.

[G] J. Galambos, *Representations of Real Numbers by Infinite Series*, Lecture Notes in Math. 502, Springer, Berlin, 1982.

[J] H. Jager, *On decimal expansions*, Zahlentheorie, Berichte aus dem Mathematische Forschungsinstitut Oberwolfach 5 (1971), 67–75.

[JdV] H. Jager and C. de Vroedt, *Lüroth series and their ergodic properties*, Indag. Math. 31 (1968), 31–42.

[$K^3 1$] S. Kalpazidou, A. Knopfmacher and J. Knopfmacher, *Lüroth-type alternating series representations for real numbers*, Acta Arith. 55 (1990), 311–322.

[$K^3 2$] —, —, —, *Metric properties of alternating Lüroth series*, Portugal. Math. 48 (1991), 319–325.

[K] C. Kraaikamp, *A new class of continued fraction expansions*, Acta Arith. 57 (1991), 1–39.

[Li] P. Liardet, MR: 93m:11077.

[Lu] J. Lüroth, *Ueber eine eindeutige Entwickelung von Zahlen in eine unendliche Reihe*, Math. Ann. 21 (1883), 411–423.

[Pa] W. Parry, *On the β-expansions of real numbers*, Acta Math. Acad. Sci. Hungar. 11 (1960), 401–416.

[Pe] O. Perron, *Irrationalzahlen*, de Gruyter, Berlin, 1960.

[R] V. A. Rohlin, *Exact endomorphisms of a Lebesgue space*, Izv. Akad. Nauk SSSR Ser. Mat. 24 (1960) (in Russian); English translation: Amer. Math. Soc. Transl. Ser. 2, 39 (1969), 1–36.

[Sa] T. Šalát, *Zur metrischen Theorie der Lürothschen Entwicklungen der reellen Zahlen*, Czech. Math. J. 18 (1968), 489–522.

[So] B. Solomyak, Personal communication with C. Kraaikamp, Seattle, July 9, 1991.

[V] W. Vervaat, *Success Epochs in Bernoulli Trails with Applications in Number Theory*, Math. Centre Tracts 42, Amsterdam, 1972.

[W] P. Walters, *An Introduction to Ergodic Theory*, Grad. Texts in Math. 79, Springer, New York, 1982.

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Received on 29.12.1994
and in revised form on 1.8.1995 (2721)