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Bounds for the solutions of Thue–Mahler equations and norm form equations

by

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To the memory of N. I. Feldman and Z. Z. Papp

1. Introduction. Several effective bounds have been established for the heights of the solutions of Thue equations, Thue–Mahler equations and, more generally, norm form equations (for references, see e.g. [1], [11], [21], [4], [22], [2]). Except in [2], their proofs involved the theory of linear forms in logarithms and its p-adic analogue as well as certain quantitative results concerning independent units of number fields. The best known explicit estimates for Thue equations and norm form equations in integers are due to Győry and Papp [14], and for Thue–Mahler equations and norm form equations in S-integers to Győry [9], [13]. These led to many applications.

The bounds in [14] depend among other things on some parameters of the number field $\mathbb{M}$ involved in our equations (2.1) and (3.1), respectively. These results of [14] were extended in [9] and [13] to wider classes of norm forms, to equations of Mahler type and to S-integral solutions. However, the estimates in [9] and [13] are weaker in terms of certain parameters because the corresponding bounds depend on the normal closure of $\mathbb{M}$, too. The main purpose of this paper is to give considerable improvements (cf. Theorems 1, 2) of the estimates of [9] and [13]. Our bounds are independent of the normal closure of $\mathbb{M}$. In particular, for the equations considered in [14], our Theorems 1 and 2 provide much better estimates than those in [14].

We give some applications of Theorems 1 and 2 as well. In Section 3, we improve (cf. Theorems 3, 4) the best known bounds for the solutions of Thue equations and Thue–Mahler equations over $\mathbb{Z}$. This section can be read independently of the other parts of the paper. In Section 4, we derive some improvements (cf. Theorem 5) of the previous explicit lower estimates for

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linear forms with algebraic coefficients at integral points. In particular, our estimates improve upon the best known explicit improvement of Liouville's approximation theorem.

Sections 5 and 6 are devoted to the proofs of our theorems. In the proofs we extend and generalize some arguments of [14]. Further, we utilize among others some recent improvements of Waldschmidt [24] and Kunrui Yu [25] concerning linear forms in logarithms, some recent estimates of Hajdu [15] concerning fundamental systems of S-units, some recent estimates of the authors [3] for S-regulators and some new ideas of Schmidt [20] and the authors [3].

2. Bounds for the solutions of norm form equations. Let $\mathbb{K}$ be an algebraic number field and let $\mathbb{M}$ be a finite extension of $\mathbb{K}$ with $[\mathbb{K}:\mathbb{Q}]=k$, $[\mathbb{M}:\mathbb{Q}]=d$ and $[\mathbb{M}:\mathbb{K}]=n\geq 3$. Let $R_{\mathbb{M}}$ be the regulator, $h_{\mathbb{M}}$ the class number and $r=r_{\mathbb{M}}$ the unit rank of $\mathbb{M}$. Let S be a finite set of places on $\mathbb{K}$ containing the set of infinite places S_∞, and let T be the set of all extensions to $\mathbb{M}$ of the places in S. Let P denote the largest of the rational primes lying below the finite places of S (with the convention that $P=1$ if $S=S_\infty$). Denote by $s\geq 0$ the number of finite places in S, by t the cardinality of T and by R_T the T-regulator of $\mathbb{M}$ (for the definition and properties of the T-regulator, see Section 5). Let O_S denote the ring of S-integers in $\mathbb{K}$.

For any algebraic number α, we denote by $h(\alpha)$ the (absolute) height of α (cf. Section 5). Throughout this paper we write $\log^* a$ for $\max\{\log a, 1\}$.

Let $\alpha_1=1,\alpha_2,\ldots,\alpha_m$ ($m\geq 2$) be elements of $\mathbb{M}$, linearly independent over $\mathbb{K}$ and having (absolute) heights at most A ($\geq e$). Let β be a nonzero element of $\mathbb{K}$ with (absolute) height at most B and with S-norm (cf. Section 5) not exceeding B^* ($\geq e$). Consider the norm form equation

$$N_{\mathbb{M}/\mathbb{K}}(x_1\alpha_1+\ldots+x_m\alpha_m)=\beta \quad \text{in } x_1,\ldots,x_m\in O_S. \tag{2.1}$$

Theorem 1. *Suppose that α_m is of degree ≥ 3 over $\mathbb{K}(\alpha_1,\ldots,\alpha_{m-1})$. Then all solutions $x_1,\ldots,x_m$ of* (2.1) *with $x_m\neq 0$ satisfy*

$$\max_{1\leq i\leq m} h(x_i) < B^{(m-1)/n} \times \exp\{c_1 P^N R_T(\log^* R_T)(\log^*(PR_T)/\log^* P)(R_{\mathbb{M}}+sh_{\mathbb{M}}+\log(AB^*))\}, \tag{2.2}$$

where $N=d(n-1)(n-2)$ and

$$c_1=c_1(d,t,N)=3^{t+25}t^{5t+12}N^{3t+4d}.$$

Further, if in particular $S=S_\infty$ (i.e. $s=0$), then the bound in (2.2) *can be replaced by*

$$B^{(m-1)/n}\exp\{c_2R_{\mathbb{M}}(\log^* R_{\mathbb{M}})(R_{\mathbb{M}}+\log(AB^*))\}, \tag{2.3}$$

where

$$c_2 = c_2(d, r, n) = 3^{r+26}(r+1)^{7r+19} d^{4r+2} n^{2(n+r+6)}.$$

It is clear that $t \le r + 1 + ns$. Further, the factor $\log^*(PR_T)/\log^* P$ in (2.2) does not exceed $2\log^* R_T$, and, if $\log^* R_T \le \log^* P$, then it is at most 2. Finally, by Lemma 3 (cf. Section 5), we have

$$R_T \le R_{\mathbb{M}} h_{\mathbb{M}} (d \log^* P)^{sn}. \tag{2.4}$$

From Theorem 1 we deduce the following

THEOREM 2. *Suppose that*, *in* (2.1), α_{i+1} *has degree* ≥ 3 *over* $\mathbb{K}(\alpha_1, \ldots, \alpha_i)$ *for* $i = 1, \ldots, m-1$. *Then all solutions of* (2.1) *satisfy* (2.2). *Further*, *if* $S = S_\infty$ (*i.e.* $s = 0$), *then the bound in* (2.2) *can be replaced by* (2.3).

Theorems 1 and 2 considerably improve Corollaries 2 and 3 of Győry [13] in terms of $R_{\mathbb{M}}$, $h_{\mathbb{M}}$, P, d, r, t and n. Further, they imply significant improvements of Corollary 2 of Győry [9], Theorems 3, 4 of Győry [12] and Theorem 1 of Kotov [16]. In contrast with the bounds in [9], [13], [12] and [16], our estimates do not depend on the parameters of the normal closure of $\mathbb{M}$ over $\mathbb{K}$. For $S = S_\infty$, Theorem 2 is an extension and, in terms of n, r and $R_{\mathbb{M}}$, a considerable improvement of Theorem 1 of Győry and Papp [14].

Our general bounds are still large for practical use. However, some new ideas in our proofs can be useful in the resolution of concrete equations.

Sprindžuk [22] and Gaál (see e.g. [7]) established some effective results for inhomogeneous norm form equations as well. By combining the arguments of [22], [7] with those of the present paper, the bounds obtained in [22] and [7] for the solutions can also be improved.

3. Bounds for the solutions of Thue equations and Thue–Mahler equations. In this section we apply Theorem 2 to Thue equations and Thue–Mahler equations over $\mathbb{Z}$.

Let $F(X, Y) \in \mathbb{Z}[X, Y]$ be an irreducible binary form of degree $n \ge 3$, and let b be a non-zero rational integer with absolute value at most B ($\ge e$). Let $\mathbb{M} = \mathbb{Q}(\alpha)$ for some zero α of $F(X, 1)$, and denote by $R_{\mathbb{M}}$, $h_{\mathbb{M}}$ and $r = r_{\mathbb{M}}$ the regulator, class number and unit rank of $\mathbb{M}$. Further, let H (≥ 3) be an upper bound for the height (i.e. the maximum absolute value of the coefficients) of F. The Thue equation

$$F(x, y) = b \quad \text{in } x, y \in \mathbb{Z} \tag{3.1}$$

is a special case of equation (2.1).

The first estimate in Theorem 3 below is a special case of Theorem 2. The second estimate easily follows from the first one (see Section 6).

THEOREM 3. *All solutions* x, y *of equation* (3.1) *satisfy*

$$\max\{|x|, |y|\} < \exp\{c_3 R_{\mathbb{M}} (\log^* R_{\mathbb{M}})(R_{\mathbb{M}} + \log(HB))\} \tag{3.2}$$

and

$$\max\{|x|,|y|\} < \exp\{c_4 H^{2n-2}(\log H)^{2n-1}\log B\}, \tag{3.3}$$

where

$$c_3 = c_3(n,r) = 3^{r+27}(r+1)^{7r+19}n^{2n+6r+14}, \quad c_4 = c_4(n) = 3^{3(n+9)}n^{18(n+1)}.$$

Let $p_1,\ldots,p_s$ ($s > 0$) be distinct rational primes not exceeding P. Consider now the Thue–Mahler equation

$$F(x,y) = bp_1^{z_1}\ldots p_s^{z_s} \quad \text{in } x,y,z_1,\ldots,z_s \in \mathbb{Z} \tag{3.4}$$

with $(x,y,p_1\ldots p_s) = 1$ and $z_1,\ldots,z_s \ge 0$.

The following theorem is a consequence of Theorem 2.

Theorem 4. *All solutions of equation* (3.4) *satisfy*

$$\begin{aligned}\max\{|x|,|y|,p_1^{z_1}\ldots p_s^{z_s}\} \\ < \exp\{c_5 P^N(\log^* P)^{ns+2}R_{\mathbb{M}}h_{\mathbb{M}}(\log^*(R_{\mathbb{M}}h_{\mathbb{M}}))^2(R_{\mathbb{M}} + sh_{\mathbb{M}} + \log(HB))\}\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}\max\{|x|,|y|,p_1^{z_1}\ldots p_s^{z_s}\} \\ < \exp\{c_6 P^N(\log^* P)^{ns+2}H^{2n-2}(\log H)^{2n-1}\log B\},\end{aligned} \tag{3.6}$$

where $N = n(n-1)(n-2)$ *and*

$$c_5 = c_5(n,s) = 3^{n(2s+1)+27}n^{2n(7s+13)+13}(s+1)^{5n(s+1)+15},$$
$$c_6 = c_6(n,s) = 2^{5n}n^{3n}c_5(n,s).$$

In case of equations considered over $\mathbb{Z}$, Theorem 3 improves the Corollary of Győry and Papp [14] in terms of n, r and $R_{\mathbb{M}}$. Further, for irreducible F, our estimate (3.5) gives a significant improvement of Corollary 1 of Győry [9] in $R_{\mathbb{M}}$, $h_{\mathbb{M}}$, P, n, r and s. Theorem 4 can be regarded as an explicit version of Theorem 1.1 in Chapter V of Sprindžuk [22].

4. Lower bounds for some linear forms with algebraic coefficients. The bounds obtained in [14], [10], [22] for the solutions of norm form equations implied lower bounds for linear forms with algebraic coefficients at integral points. As consequences of our Theorem 2 we considerably improve upon these lower estimates.

Let again $\mathbb{K}$ and $\mathbb{M}$ be algebraic number fields with $\mathbb{K} \subset \mathbb{M}$ and with the same parameters as in Section 2. Let $O_{\mathbb{K}}$ and $O_{\mathbb{M}}$ denote the rings of integers of $\mathbb{K}$ and $\mathbb{M}$, respectively. Let $R_{\mathbb{K}}$ and $r_{\mathbb{K}}$ be the regulator and unit rank of $\mathbb{K}$. Denote by $\Omega_{\mathbb{M}}$ (resp. Ω_∞) the set of all (resp. all infinite) places on $\mathbb{M}$. For $v \in \Omega_{\mathbb{M}}$, denote by $|\cdot|_v$ the corresponding valuation normalized as in Section 5 below. Let Γ_∞ and Γ_0 be finite subsets of Ω_∞ and $\Omega_{\mathbb{M}} \setminus \Omega_\infty$, respectively, and put $\Gamma = \Gamma_\infty \cup \Gamma_0$. We denote by r_1 and r_2 the numbers of real and complex places in Γ_∞. Further, suppose that Γ_0 contains $t_0 \ge 0$ finite places and that the corresponding prime ideals of $O_{\mathbb{M}}$ lie above rational primes not exceeding P (for $t_0 = 0$, let $P = 1$). Let S denote the set of places on $\mathbb{K}$, induced by the places in $\Gamma_0 \cup \Omega_\infty$. Further, let T be the set of all extensions to $\mathbb{M}$ of the places in S, and let R_T denote the T-regulator of $\mathbb{M}$.

We recall that the *size* of an algebraic number α, denoted by $\overline{|\alpha|}$, is the maximum of the absolute values of the conjugates of α.

Using the above notations, we deduce from Theorem 2 the following result.

THEOREM 5. *Suppose that $\alpha_0 = 1, \alpha_1, \ldots, \alpha_m$ are elements in $\mathbb{M}$ with* (*absolute*) *heights at most A ($\ge e$) such that $\mathbb{M} = \mathbb{K}(\alpha_1, \ldots, \alpha_m)$ and that α_{i+1} is of degree ≥ 3 over $\mathbb{K}(\alpha_0, \ldots, \alpha_i)$ for $i = 0, \ldots, m-1$. Then for any $\mathbf{x} = (x_0, \ldots, x_m) \in O_{\mathbb{K}}^{m+1} \setminus \{\mathbf{0}\}$ there exists an S-unit $\varepsilon \in O_{\mathbb{K}}$ such that $\varepsilon^{-1}\mathbf{x} \in O_{\mathbb{K}}^{m+1} \setminus \{\mathbf{0}\}$ and*

$$\prod_{v\in\Gamma} |(\varepsilon^{-1}x_0)\alpha_0 + \ldots + (\varepsilon^{-1}x_m)\alpha_m|_v > \kappa_1 X^{-d+r_1+2r_2+\tau_1}, \tag{4.1}$$

$$X = \max_{0\le i\le m} \overline{|\varepsilon^{-1}x_i|},$$

where

$$\kappa_1 = (2m)^{-d+r_1+2r_2} A^{-(d^2+1)(dm+1)} \exp\{-(R_{\mathbb{M}} + t_0 h_{\mathbb{M}})\},$$
$$\tau_1 = ((t_0+1)2^{k-1}k^{2k-1}c_1 P^N (\log^* P) R_T (\log^* R_T)^2 R_{\mathbb{K}} h_{\mathbb{K}})^{-1}.$$

Further, *if Γ contains only infinite places*, *κ_1 and τ_1 can be replaced by*

$$\kappa_2 = (2m)^{-d+r_1+2r_2} A^{-(d^2+1)(dm+1)} \exp\{-R_{\mathbb{M}}\},$$
$$\tau_2 = (2kc_2 R_{\mathbb{M}} \log^* R_{\mathbb{M}})^{-1},$$

respectively. (*Here c_1, c_2 denote the constants occurring in Theorem* 1.)

Our Theorem 1 has a similar consequence. Theorem 5 generalizes and improves Theorem 2 of [14]. Further, it is an improvement of Corollary 1 of [10].

The next corollary is concerned with the case $\mathbb{K} = \mathbb{Q}$. For any complex number ξ, we denote by $\|\xi\|$ the distance from ξ to the nearest rational integer.

COROLLARY 1. *Let $\alpha_0 = 1, \alpha_1, \ldots, \alpha_m$ be algebraic numbers with* (*absolute*) *heights at most A ($\ge e$) such that α_{i+1} is of degree ≥ 3 over $\mathbb{Q}(\alpha_0, \ldots, \alpha_i)$ for $i = 0, \ldots, m-1$. Further*, *let $\mathbb{M} = \mathbb{Q}(\alpha_1, \ldots, \alpha_m)$ with degree n and regulator $R_{\mathbb{M}}$. Then*, *putting $\sigma = 1$ or 2 according as $\mathbb{M}$ is real or not*, *we have for any $(x_1, \ldots, x_m) \in \mathbb{Z}^m \setminus \{\mathbf{0}\}$*,

$$\|x_1\alpha_1 + \ldots + x_m\alpha_m\| > \kappa_3 X^{-(n-\sigma-\tau_3)/\sigma}, \qquad X = \max_{1\le i\le m} |x_i|,$$

where

$$\kappa_3 = (2m)^{-2(n-\sigma)/\sigma} A^{-(n^2+1)(nm+2)/\sigma} \exp\{-R_{\mathbb{M}}/\sigma\},$$
$$\tau_3 = (3^{n+26} n^{15n+20} R_{\mathbb{M}} \log^* R_{\mathbb{M}})^{-1}.$$

This is an extension and improvement of the Corollary in [14]. For $m = 1$, this result of [14] provided an explicit version of a theorem of Feldman [6]. Our Corollary 1 above gives the best (up to now) effective improvement of Liouville's approximation theorem: If α is a real algebraic number of degree $n \ge 3$ with (absolute) height at most A $(\ge e)$ then, putting $\mathbb{M} = \mathbb{Q}(\alpha)$, we have

$$\left|\alpha - \frac{y}{x}\right| > \frac{2^{-2n+2} A^{-(n^2+1)(n+2)} \exp\{-R_{\mathbb{M}}\}}{x^{n-\tau_3}}$$

for every rational y/x with $x > 0$.

Corollary 2 below considerably improves Corollary 3 of [10] which was an explicit version of a previous theorem of Kotov and Sprindžuk [17].

Corollary 2. *Let* $\mathbb{K}$, $\mathbb{M}$ *and* Γ *be as in Theorem* 5, *and let* $\theta \in \mathbb{M}$ *with* $\mathbb{M} = \mathbb{K}(\theta)$ *and with* (*absolute*) *height at most* A $(\ge e)$. *Then for all* $\alpha \in \mathbb{K}$ *we have*

$$\prod_{v \in \Gamma} |\theta - \alpha|_v > \kappa_4 (h(\alpha))^{-kd+\tau_1/2} \tag{4.2}$$

while for all $\alpha \in O_{\mathbb{K}}$ *we have*

$$\prod_{v \in \Gamma} |\theta - \alpha|_v > \kappa_4 (h(\alpha))^{k(-d+r_1+2r_2)+\tau_1/2}, \tag{4.3}$$

where $\kappa_4 = \kappa_1 (2A^d)^{-d+r_1+2r_2}$ *with the choice* $m = 1$.

5. Auxiliary results. In this section, we introduce some notation. Further, we formulate some lemmas and two estimates for linear forms in logarithms which will be used in the next section, in the proofs of our theorems.

For an algebraic number field $\mathbb{K}$, we denote by $O_{\mathbb{K}}$ the ring of integers of $\mathbb{K}$ and by $\Omega_{\mathbb{K}}$ the set of places on $\mathbb{K}$. Put $k = [\mathbb{K} : \mathbb{Q}]$. We choose a valuation $|\cdot|_v$ for every $v \in \Omega_{\mathbb{K}}$ in the following way : if v is infinite and corresponds to $\sigma : \mathbb{K} \to \mathbb{C}$ then we put, for $\alpha \in \mathbb{K}$, $|\alpha|_v = |\sigma(\alpha)|^{k_v}$, where $k_v = 1$ or 2 according as $\sigma(\mathbb{K})$ is contained in $\mathbb{R}$ or not; if v is finite and corresponds to the prime ideal $\mathfrak{p}$ in $\mathbb{K}$ then we put $|\alpha|_v = N(\mathfrak{p})^{-\operatorname{ord}_{\mathfrak{p}}(\alpha)}$ for $\alpha \in \mathbb{K} \setminus \{0\}$ and $|0|_v = 0$. The set of valuations thus normalized is uniquely determined and satisfies the product formula for valuations

$$\prod_{v \in \Omega_{\mathbb{K}}} |\alpha|_v = 1 \quad \text{for any } \alpha \text{ in } \mathbb{K} \setminus \{0\}. \tag{5.1}$$

We shall assume throughout the paper that the valuations under consideration are normalized in the above sense. Further, it will be frequently used that if a place w on a finite extension $\mathbb{M}$ of $\mathbb{K}$ is an extension of a place v on $\mathbb{K}$ then

$$|\alpha|_w = |\alpha|_v^{[\mathbb{M}_w:\mathbb{K}_v]} \quad \text{for } \alpha \in \mathbb{K} \tag{5.2}$$

and, if the place v is finite,

$$|N_{\mathbb{M}/\mathbb{K}}(\alpha)|_v = \prod_{\substack{w \text{ place on } \mathbb{M} \\ w|v}} |\alpha|_w \quad \text{for } \alpha \in \mathbb{M}. \tag{5.3}$$

Here $\mathbb{K}_v$ and $\mathbb{M}_w$ denote the completions of $\mathbb{K}$ and of $\mathbb{M}$ at the places v and w, respectively.

The (absolute) *height* of $\alpha \in \mathbb{K}$ is defined by

$$h(\alpha) = \prod_{v \in \Omega_{\mathbb{K}}} \max\{1, |\alpha|_v\}^{1/k}.$$

It depends only on α, and not on the choice of $\mathbb{K}$. We shall frequently use that $h(\alpha^{-1}) = h(\alpha)$ for $\alpha \in \mathbb{K} \setminus \{0\}$, and

$$h(\alpha_1 \ldots \alpha_m) \le h(\alpha_1) \ldots h(\alpha_m), \qquad h(\alpha_1 + \ldots + \alpha_m) \le m h(\alpha_1) \ldots h(\alpha_m)$$

for $\alpha_1, \ldots, \alpha_m \in \mathbb{K}$. Further, we have

$$\sum_{v \in \Omega_{\mathbb{K}}} |\log |\alpha|_v| = 2k \log h(\alpha) \quad \text{for } \alpha \in \mathbb{K} \setminus \{0\}. \tag{5.4}$$

There exists a $\lambda(d) > 0$, depending only on d, such that $\log h(\alpha) \ge \lambda(d)$ for any non-zero algebraic number α of degree d which is not a root of unity. For $d = 1$, we can take $\lambda(d) = \log 2$. For $d \ge 2$, Stewart, Dobrowolski and others gave lower bounds for $\lambda(d)$; very recently, Voutier [23] has improved these bounds by showing that one can take

$$\lambda(d) = \frac{2}{d(\log(3d))^3} \quad \text{for } d \ge 2. \tag{5.5}$$

Let S be a finite subset of $\Omega_{\mathbb{K}}$ containing the set of infinite places S_∞. Denote by O_S the ring of S-integers, and by O_S^* the group of S-units in $\mathbb{K}$. For $\alpha \in \mathbb{K} \setminus \{0\}$, the ideal (α) generated by α can be uniquely written in the form $\mathfrak{a}_1\mathfrak{a}_2$ where the ideal $\mathfrak{a}_1$ (resp. $\mathfrak{a}_2$) is composed of prime ideals outside (resp. inside) S. The S-norm of α, denoted by $N_S(\alpha)$, is defined as $N(\mathfrak{a}_1)$. The S-norm is multiplicative, and, for $S = S_\infty$, $N_S(\alpha) = |N_{\mathbb{K}/\mathbb{Q}}(\alpha)|$. For any $\alpha \in \mathbb{K} \setminus \{0\}$, we have $N_S(\alpha) = \prod_{v \in S} |\alpha|_v$. Further, if $\alpha \in O_S \setminus \{0\}$, then $N_S(\alpha)$ is a positive integer and $N_S(\alpha) \le (h(\alpha))^k$.

Let q be the cardinality of S. Let $v_1, \ldots, v_{q-1}$ be a subset of S, and let $\{\varepsilon_1, \ldots, \varepsilon_{q-1}\}$ be a fundamental system of S-units in $\mathbb{K}$. Denote by R_S the absolute value of the determinant of the matrix $(\log |\varepsilon_i|_{v_j})_{i,j=1,\ldots,q-1}$. It is easy to verify that R_S is a positive number which is independent of the choice of $v_1, \ldots, v_{q-1}$ and of the fundamental system of S-units $\{\varepsilon_1, \ldots, \varepsilon_{q-1}\}$. R_S

is called the *S-regulator* of $\mathbb{K}$. If in particular $S = S_\infty$, then R_S is just the regulator, $R_{\mathbb{K}}$, of $\mathbb{K}$.

For the proofs of Lemmas 1 to 3 below we refer the reader to [3].

Put

$$c_7 = c_7(k, q) = ((q-1)!)^2/(2^{q-2}k^{q-1})$$

and

$$c_8 = c_8(k, q) = c_7(\lambda(k))^{2-q}, \qquad c_9 = c_9(k, q) = c_7 k^{q-2}/\lambda(k).$$

Lemma 1. *There exists in $\mathbb{K}$ a fundamental system $\{\varepsilon_1, \ldots, \varepsilon_{q-1}\}$ of S-units with the following properties*:

(i) $$\prod_{i=1}^{q-1} \log h(\varepsilon_i) \le c_7 R_S;$$

(ii) $$\log h(\varepsilon_i) \le c_8 R_S, \quad i = 1, \ldots, q-1;$$

(iii) *the absolute values of the entries of the inverse matrix of $(\log|\varepsilon_i|_{v_j})_{i,j=1,\ldots,q-1}$ do not exceed c_9.*

This is a slight improvement of a theorem of Hajdu [15].

Denote by $h_{\mathbb{K}}$ and $r = r_{\mathbb{K}}$ the class number and unit rank of $\mathbb{K}$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ be the prime ideals corresponding to the finite places in S, and denote by P the largest of the rational primes lying below $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$. Put $c_{10} = c_{10}(k, r) = r^{r+1}(k\lambda(k))^{-(r-1)}/2$.

Lemma 2. *For every $\alpha \in O_S \setminus \{0\}$ and every integer $n \ge 1$ there exists an S-unit ε such that*

$$h(\varepsilon^n \alpha) \le N_S(\alpha)^{1/k} \exp\{n(c_{10} R_{\mathbb{K}} + s h_{\mathbb{K}} \log^* P)\}.$$

Lemma 3. *If $s > 0$, then we have*

$$R_S \le R_{\mathbb{K}} h_{\mathbb{K}} \prod_{i=1}^{s} \log N(\mathfrak{p}_i) \le R_{\mathbb{K}} h_{\mathbb{K}} (k \log^* P)^s$$

and

$$R_S \ge R_{\mathbb{K}} \prod_{i=1}^{s} \log N(\mathfrak{p}_i) \ge c_{11}(\log 2)(\log^* P),$$

where $c_{11} = 0.2052$.

We remark that, in our Theorems 1 and 2, the improvements of the previous bounds in terms of $R_{\mathbb{M}}$, $h_{\mathbb{M}}$ and P are mainly due to the use of fundamental systems of S-units, S-regulators as well as Lemmas 1 to 3.

We also need an explicit version of a lemma due to Sprindžuk [22]. Let $\mathbb{M}$ be an extension of $\mathbb{K}$ with $[\mathbb{M} : \mathbb{K}] = n$. Denote by d, $R_{\mathbb{M}}$, $h_{\mathbb{M}}$ and $r_{\mathbb{M}}$ the degree, regulator, class number and unit rank of $\mathbb{M}$.

LEMMA 4. *With the above notations, we have*

$$h_{\mathbb{K}} \le n h_{\mathbb{M}} \quad \textit{and} \quad R_{\mathbb{K}} \le r_{\mathbb{M}}! n(2\lambda(d))^{-(r_{\mathbb{M}}-1)} R_{\mathbb{M}}.$$

Proof. For the first inequality, see [22], page 21; the second inequality can be easily derived from Lemma 2.3 in Chapter II of [22]. ■

The application of Propositions 1 and 2 below enables us to considerably improve the previous bounds for the solutions of equation (2.1) in terms of d, r, n and t. Moreover, we shall pay a particular attention to the dependence on these parameters.

Let $\alpha_1, \ldots, \alpha_m$ ($m \ge 2$) be non-zero algebraic numbers such that $\mathbb{K} = \mathbb{Q}(\alpha_1, \ldots, \alpha_m)$. Let $H_1, \ldots, H_m$ be real numbers such that

$$\log H_i \ge \max\left\{\log h(\alpha_i), \frac{|\log \alpha_i|}{3.3k}, \frac{1}{k}\right\}, \qquad i = 1, \ldots, m, \tag{5.6}$$

where log denotes the principal value of the logarithm. Let $b_1, \ldots, b_m$ be rational integers and put $B = \max\{|b_1|, \ldots, |b_m|, 3\}$. Further, set

$$\Lambda = \alpha_1^{b_1} \ldots \alpha_m^{b_m} - 1.$$

In Proposition 1, it will be convenient to use the following technical conditions :

$$B \ge \log H_m \exp\{4(m+1)(7 + 3\log(m+1))\}, \tag{5.7}$$

$$7 + 3\log(m+1) \ge \log k. \tag{5.8}$$

As was shown in [3], Proposition 1 is a consequence of Corollary 10.1 of Waldschmidt [24].

PROPOSITION 1 (M. Waldschmidt [24]). *If* $\Lambda \ne 0$, $b_m = 1$ *and* (5.7), (5.8) *hold*, *then*

$$|\Lambda| \ge \exp\left\{-c_{12}(m) k^{m+2} \log H_1 \ldots \log H_m \log\left(\frac{2mB}{\log H_m}\right)\right\},$$

where $c_{12}(m) = 1500 \cdot 38^{m+1}(m+1)^{3m+9}$.

In Proposition 2, let $v = v_{\mathfrak{p}}$ be a finite place on $\mathbb{K}$, corresponding to the prime ideal $\mathfrak{p}$ of $O_{\mathbb{K}}$. Let p denote the rational prime lying below $\mathfrak{p}$, and denote by $|\cdot|_v$ the non-archimedean valuation normalized as above. Instead of (5.6), assume now that $H_1, \ldots, H_m$ are real numbers such that

$$\log H_i \ge \max\{\log h(\alpha_i), |\log \alpha_i|/(10k), \log p\}, \qquad i = 1, \ldots, m.$$

The following proposition is a simple consequence of the main result of Kunrui Yu [25].

PROPOSITION 2 (Kunrui Yu [25]). *Let*

$$\Phi = c_{13}(m)(k/\sqrt{\log p})^{2(m+1)} p^k \log H_1 \ldots \log H_m \log(10mk \log H),$$

where $c_{13}(m) = 22000(9.5(m+1))^{2(m+1)}$ *and* $H = \max\{H_1, \ldots, H_m, e\}$. *If* $\Lambda \neq 0$ *then*

$$|\Lambda|_v \geq \exp\{-k(\log p)\Phi \log(kB)\}.$$

Further, *if* $b_m = 1$ *and* $H_m \geq H_i$ *for* $i = 1, \ldots, m-1$, *then* H *can be replaced by* $\max\{H_1, \ldots, H_{m-1}, e\}$ *and for any* δ *with* $0 < \delta \leq 1$, *we have*

$$|\Lambda|_v \geq \exp\{-k(\log p) \max\{\Phi \log(\delta^{-1}\Phi/\log H_m), \delta B\}\}.$$

Remark. We remark that, in Propositions 1 and 2, the condition $\mathbb{K} = \mathbb{Q}(\alpha_1, \ldots, \alpha_m)$ can be removed. It is enough to assume that $\mathbb{K}$ is an algebraic number field of degree k which contains $\alpha_1, \ldots, \alpha_m$. This observation will be needed in Section 6.

6. Proofs

Proof of Theorem 1. We keep the notation of Section 2 and use some ideas of [3]. Further, we generalize some arguments of [14]. We may and shall assume that $\alpha_1, \ldots, \alpha_m$ are algebraic integers in (2.1) with $\alpha_1 \in \mathbb{Z}\setminus\{0\}$. This can be achieved by multiplying equation (2.1) by the nth power of the product of the denominators of $\alpha_2, \ldots, \alpha_m$ and replacing the bounds A, B for the heights of the α_i and β by $A_1 = A^{d(m-1)+1}$ and $B_1 = BA^{dn(m-1)}$, respectively, and the bound B^* for the S-norm of β by $B_1^* = B^* A^{kdn(m-1)}$. Further, we assume that $\beta \in O_S \setminus \{0\}$ since otherwise (2.1) is not solvable.

Let now $\mathbf{x} = (x_1, \ldots, x_m) \in O_S^m$ be an arbitrary but fixed solution of (2.1) with $x_m \neq 0$. Denote by $\mathbb{L}$ the number field $\mathbb{K}(\alpha_1, \ldots, \alpha_{m-1})$, by $T_{\mathbb{L}}$ the set of all extensions to $\mathbb{L}$ of the places in S, and by $O_{\mathbb{L},T_{\mathbb{L}}}$ the ring of $T_{\mathbb{L}}$-integers in $\mathbb{L}$. Putting

$$x = x_1\alpha_1 + \ldots + x_{m-1}\alpha_{m-1}, \quad y = x_m \quad \text{and} \quad \tau = \alpha_m,$$

equation (2.1) can be written as

$$N_{\mathbb{L}/\mathbb{K}}(N_{\mathbb{M}/\mathbb{L}}(x + y\tau)) = \beta \quad \text{in } x \in O_{\mathbb{L},T_{\mathbb{L}}},\ y \in O_S \setminus \{0\}, \tag{6.1}$$

whence

$$N_{\mathbb{M}/\mathbb{L}}(x + y\tau) = \beta_1 \tag{6.2}$$

with some $\beta_1 \in O_{\mathbb{L},T_{\mathbb{L}}} \setminus \{0\}$. Since β_1 is a divisor of β in $O_{\mathbb{L},T_{\mathbb{L}}}$, its $T_{\mathbb{L}}$-norm satisfies $N_{T_{\mathbb{L}}}(\beta_1) \leq N_{T_{\mathbb{L}}}(\beta) \leq (B_1^*)^{l/k}$, where $l = [\mathbb{L} : \mathbb{Q}]$. It follows from Lemmas 2 and 4 that there exist a unit ε in $O_{\mathbb{L},T_{\mathbb{L}}}$ and $\beta_2 \in O_{\mathbb{L},T_{\mathbb{L}}}$ such that $\beta_2 = \beta_1\varepsilon^{n_1}$ with $n_1 = [\mathbb{M} : \mathbb{L}]$ and

$$\begin{aligned} &h(\beta_2) \\ &\quad \leq (B_1^*)^{1/k} \exp\{n^2(r!r^{r+1}(d\lambda^2(d))^{-(r-1)}2^{-r}R_{\mathbb{M}} + sh_{\mathbb{M}} \log^* P)\} =: C_1. \end{aligned} \tag{6.3}$$

From (6.2) we get

$$N_{\mathbb{M}/\mathbb{L}}((\varepsilon x) + (\varepsilon y)\tau) = \beta_2. \tag{6.4}$$

We are going to give an upper bound for $h(\varepsilon y)$ and $h(\varepsilon x)$. Denote by $\mathbb{G}$ the normal closure of $\mathbb{M}$ over $\mathbb{K}$, and by $T_{\mathbb{G}}$ the set of all extensions to $\mathbb{G}$ of the elements of S. Putting $t_1 = \mathrm{Card}(T_{\mathbb{G}})$ and $g = [\mathbb{G} : \mathbb{K}]$, we have $t_1 \le tg/n$ and $g \le n!$. Assume that

$$h(\varepsilon y) > \max\{C_1^{2t_1}, (2^{t_1} A_1^2)^{3n_1 t_1}\}. \tag{6.5}$$

Let τ_i, $\mu_i = (\varepsilon x) + (\varepsilon y)\tau_i$, $i = 1, \ldots, n_1$, denote the corresponding conjugates of τ and $\mu = (\varepsilon x) + (\varepsilon y)\tau$, respectively, over $\mathbb{L}$. There is no loss of generality in assuming that $\mu_1, \ldots, \mu_{n_0}$ are distinct, where, by assumption, $n_0 \ge 3$. Let $v^* \in T_{\mathbb{G}}$ for which $|\varepsilon y|_{v^*}$ is maximal. We may assume that $|\mu_1|_{v^*} \le \ldots \le |\mu_{n_0}|_{v^*}$. Then $|\mu_1 - \mu_i|_{v^*} \le 2|\mu_i|_{v^*}$ for $i = 1, \ldots, n_0$, and so

$$|\mu_i|_{v^*} \ge \tfrac{1}{2}|\mu_1 - \mu_i|_{v^*} = \tfrac{1}{2}|\varepsilon y|_{v^*}|\tau_1 - \tau_i|_{v^*} \ge \tfrac{1}{2}|\tau_1 - \tau_i|_{v^*} h(\varepsilon y)^{gk/t_1}. \tag{6.6}$$

Hence it follows from (6.4) and (6.5) that

$$|\mu_1|_{v^*} \le |\beta_2|_{v^*}^{n_0/n_1} \Big(\prod_{i=2}^{n_0} |\mu_i|_{v^*}\Big)^{-1} \le h(\varepsilon y)^{-gk/t_1}. \tag{6.7}$$

Fix a $v \in T_{\mathbb{G}}$ with $v \ne v^*$, and take $j \in \{1, \ldots, n_0\}$ for which $|\mu_j|_v$ is minimal. Then, by (6.4) and (6.3), $|\mu_j|_v \le |\beta_2|_v^{1/n_1} \le h(\beta_2)^{gk/n_1} \le C_1^{gk/n_1}$ and so, using (6.5) and $|\mu_1 - \mu_j|_v = |\varepsilon y|_v|\alpha_1 - \alpha_j|_v$, we obtain

$$|\mu_1|_v \le |\mu_1 - \mu_j|_v + |\mu_j|_v \le h(\varepsilon y)^{2gk}. \tag{6.8}$$

Further, $|\mu_i|_v \le |\mu_1 - \mu_i|_v + |\mu_1|_v \le 2h(\varepsilon y)^{2gk}$ for each i and $v \in T_{\mathbb{G}}$. Hence, for each $v \in T_{\mathbb{G}}$, we have

$$|\mu_1|_v = |\beta_2|_v \Big(\prod_{i=2}^{n_1} |\mu_i|_v\Big)^{-1} \ge h(\varepsilon y)^{-2gkn_1}. \tag{6.9}$$

Let O_T denote the ring of T-integers in $\mathbb{M}$. Since, by (6.4), μ_1 is a divisor of β_2 in O_T, we have $N_T(\mu_1) \le N_T(\beta_2) \le h(\beta_2)^d \le C_1^d$. We recall that t denotes the cardinality of T. Let $\varepsilon_1, \ldots, \varepsilon_{t-1}$ be T-units in $\mathbb{M}$ with the properties specified in Lemma 1. By Lemma 2, there are rational integers $z_1, \ldots, z_{t-1}$ and $\gamma_1 \in O_T$ such that

$$\mu_1 = \gamma_1 \varepsilon_1^{z_1} \ldots \varepsilon_{t-1}^{z_{t-1}} \tag{6.10}$$

and that

$$h(\gamma_1) \le C_1 \exp\{c_{10}(d, r) R_{\mathbb{M}} + nsh_{\mathbb{M}} \log^* P\} =: C_2. \tag{6.11}$$

It follows from (6.10) that

$$z_1 \log|\varepsilon_1|_w + \ldots + z_{t-1} \log|\varepsilon_{t-1}|_w = \log|\mu_1/\gamma_1|_w$$

for each $w \in T$. Put $Z = \max\{|z_1|, \ldots, |z_{t-1}|, 3\}$ and $T = \{w_1, \ldots, w_t\}$. On applying now Lemma 2 and Lemma 1(iii) and using (6.8), (6.9), (5.4), (6.5) and (6.11), we infer that [1]

$$(6.12)\qquad \begin{aligned} Z &\le c_{14}\sum_{i=1}^{t-1} |\log|\mu_1/\gamma_1|_{w_i}| \\ &\le 2dc_{14}(\log h(\mu_1)+\log h(\gamma_1)) \le c_{15}\log h(\varepsilon y), \end{aligned}$$

where $c_{14}=c_9(d,t)$ and $c_{15}=5d(tg/n)n_1c_{14}$.

Consider the identity

$$(6.13)\qquad (\tau_2-\tau_3)\mu_1+(\tau_3-\tau_1)\mu_2+(\tau_1-\tau_2)\mu_3=0.$$

In view of (6.6), (6.7) and (6.5), we get

$$\left|1+\frac{(\tau_1-\tau_2)\mu_3}{(\tau_3-\tau_1)\mu_2}\right|_{v^*} = \left|\frac{(\tau_2-\tau_3)\mu_1}{(\tau_3-\tau_1)\mu_2}\right|_{v^*} \le h(\varepsilon y)^{-gk/t_1}.$$

Denote by $\varepsilon_{i,j}$ and γ_j the conjugates of ε_i and γ_1 over $\mathbb{K}$ corresponding to μ_j, and put

$$\eta_i=\varepsilon_{i,3}/\varepsilon_{i,2}\quad \text{for } i=1,\dots,t-1 \quad\text{and}\quad \eta_t=\frac{\tau_2-\tau_1}{\tau_3-\tau_1}\cdot\frac{\gamma_3}{\gamma_2}.$$

Then $h(\eta_t)\le 4A_1^4C_2^2 =: C_3$. Further, putting

$$\Lambda=\eta_1^{z_1}\dots\eta_{t-1}^{z_{t-1}}\eta_t^{z_t}-1 \qquad \text{with } z_t=1$$

we get

$$(6.14)\qquad 0<|\Lambda|_{v^*}<h(\varepsilon y)^{-gk/t_1}.$$

Denote by u^* the restriction of v^* to the field $\mathbb{M}_{123}=\mathbb{M}(\tau_1,\tau_2,\tau_3)$, and normalize $|\cdot|_{u^*}$ as above. Then, by (6.14) and (5.2), we have

$$(6.15)\qquad 0<|\Lambda|_{u^*}<h(\varepsilon y)^{-gk/t_1}.$$

First assume that u^* is infinite. To apply Proposition 1, we define $d_1=[\mathbb{M}_{123}:\mathbb{Q}]$ and

$$(6.16)\qquad \begin{aligned} \log H_i &= (d_1\lambda(d_1))^{-1}\log h(\eta_i), \qquad i=1,\dots,t-1,\\ \log H_t &= (d_1\lambda(d_1))^{-1}\log C_3. \end{aligned}$$

Then $d_1\le N$. Further, it is easy to check that $7+3\log(t+1)\ge \log d_1$. We may assume that

$$(6.17)\qquad Z\ge \log H_t\exp\{4(t+1)(7+3\log(t+1))\}.$$

Indeed, it follows from (6.10), (6.11) and Lemma 1(ii) that

$$h(\mu_1)\le C_2\prod_{i=1}^{t-1} h(\varepsilon_i)^{|z_i|}\le C_2\exp\{(t-1)c_{16}R_TZ\},$$

[1] In certain applications, it can be more useful to work with our upper bounds of Z, provided by (6.12), (6.19), (6.22) and (6.24).

where $c_{16} = c_8(d,t) = ((t-1)!)^2/d(2d\lambda(d))^{t-2}$. Hence, if (6.17) does not hold, we get an upper bound for $h(\mu_1)$ and also for $h(\mu_2)$. Then we can derive from $\mu_1 = (\varepsilon x) + (\varepsilon y)\tau_1$, $\mu_2 = (\varepsilon x) + (\varepsilon y)\tau_2$ an explicit bound for $h(\varepsilon y)$ which is better than that occurring in (6.19) below.

We have $|\cdot|_{u^*} = |\sigma(\cdot)|^{d_{u^*}}$ for some $\sigma : \mathbb{M}_{123} \to \mathbb{C}$ and $d_{u^*} \le 2$. Applying σ to equation (6.13) and then omitting σ everywhere, we may assume that $|\cdot|_{u^*} = |\cdot|^{d_{u^*}}$. On applying now Proposition 1 to $|\Lambda|_{u^*}$ and using (6.12) and Lemma 1(i), we derive that

$$|\Lambda|_{u^*} \ge \exp\left\{-c_{17}R_T \log H_t \log\left(\frac{2tc_{15}\log h(\varepsilon y)}{\log H_t}\right)\right\}, \tag{6.18}$$

where $c_{17} = d_{u^*}c_{12}(t)d_1^{t+2}(d_1\lambda(d_1))^{-(t-1)}2^{t-1}c_{18}$, with $c_{18} = c_7(d_1,t) = ((t-1)!)^2/(2^{t-2}d_1^{t-1})$. Now (6.15) and (6.18) imply that

$$\frac{\log h(\varepsilon y)}{\log H_t} \le \frac{t_1}{gk}c_{17}R_T \log\left(\frac{2tc_{15}\log h(\varepsilon y)}{\log H_t}\right).$$

Together with (6.16), (5.5) and the inequalities $(t-1)! \le (t-1)^t e^{-t+2}$, $t+1 \le e^{1/t}t$, this gives

$$\log h(\varepsilon y) \le c_{19}R_T(\log^* R_T)\log C_3 =: C_4 \tag{6.19}$$

with $c_{19} = 3^{t+22}t^{5t+11}N^3(\log 3N)^{3t+1}$.

When $S = S_\infty$, we have $t = r+1$ and we get the upper bound

$$\log h(\varepsilon y) \le c'_{19}R_{\mathbb{M}}(\log^* R_{\mathbb{M}})\log C_3 =: C'_4$$

with $c'_{19} = 3^{r+23}(r+1)^{5r+16}N^3(\log 3N)^{3r+4}$.

Next assume that u^* is finite. To apply Proposition 2, we put now

$$\log H_i = (d_1\lambda(d_1))^{-1}\log h(\eta_i) + \log^* P, \qquad i = 1,\ldots,t-1,$$
$$\log H_t = (d_1\lambda(d_1))^{-1}\log C_3 + \log^* P.$$

Then we get (cf. [3])

$$\log H_1 \ldots \log H_{t-1} \le 2c_{20}R_T(\log^* P)^{t-2}, \tag{6.20}$$

where $c_{20} = t((t-1)!)^2 d_1^{-t}(\lambda(d_1))^{-t+1}$.

We distinguish two cases. First assume that $\log C_3 < c_{16}R_T$. Then, by Lemmas 1 and 3, we have

$$\log H := \max_{1\le i\le t}\log H_i \le c_{21}R_T \tag{6.21}$$

with $c_{21} = c_{16}(d_1\lambda(d_1))^{-1} + (c_{11}\log 2)^{-1}$. We now apply to $|\Lambda|_{u^*}$ the first part of Proposition 2. Putting

$$\Phi = c_{22}\frac{P^{d_1}}{(\log^* P)^{t+1}}\log H_1 \ldots \log H_t \log(10td_1\log H)$$

with $c_{22} = c_{13}(t)(d_1^2/\log 2)^{t+1}$, we infer that

$$|\Lambda|_{u^*} \geq \exp\{-d_1(\log^* P)\Phi \log(d_1 Z)\}.$$

Together with (6.15) and (6.12) this implies that

$$\log h(\varepsilon y) \leq d_1 \frac{t_1}{gk}(\log^* P)\Phi \log(c_{15} d_1 \log h(\varepsilon y)).$$

By combining this with (6.20), (6.21), (6.5), (5.5) and the inequalities $\log C_3 < c_{16} R_T$ and $(t-1)! \leq (t-1)^t e^{-t+2}$, we get

$$\begin{aligned} (6.22) \qquad & \log h(\varepsilon y) \\ & \leq c_{23} P^{d_1} R_T (\log^* R_T)(\log^*(P R_T)/(\log^* P)^2)(\log C_3 + \log^* P), \end{aligned}$$

where $c_{23} = 3^{t+25} t^{5t+11} N^{3t}$.

Finally, assume that $\log C_3 \geq c_{16} R_T$. Then, by Lemmas 1 and 3, we have $H_t \geq H_i$ for $i = 1, \ldots, t-1$ and

$$\log H := \max_{1 \leq i \leq t-1} \log H_i \leq c_{21} R_T.$$

Consider now the above defined Φ with this value of $\log H$. First we give an upper bound for $h(\varepsilon y)$ in terms of Φ.

If $Z < \Phi(\log^* P)/(c_{16} R_T)$ then, by (6.10), (6.11), Lemma 1(ii) and (6.20),

$$\begin{aligned} h(\mu_1) &\leq C_2 h(\varepsilon_1)^{|z_1|} \ldots h(\varepsilon_{t-1})^{|z_{t-1}|} \\ &\leq C_2 \exp\{(t-1) c_{16} R_T Z\} \leq C_2 \exp\{\Phi(\log^* P)\}. \end{aligned}$$

Together with $\varepsilon y = (\mu_1 - \mu_2)/(\tau_1 - \tau_2)$ this gives

$$\log h(\varepsilon y) \leq 3\Phi(\log^* P). \tag{6.23}$$

Assume now that $Z \geq \Phi(\log^* P)/(c_{16} R_T)$. We apply the second part of Proposition 2 to $|\Lambda|_{u^*}$. Putting $\delta = \Phi(\log^* P)/(Z c_{16} R_T)$ we obtain

$$|\Lambda|_{u^*} \geq \exp\left\{-d_1(\log^* P)\Phi \log\left(\frac{Z c_{16} R_T}{\log^* P \log H_t}\right)\right\}.$$

Hence, by (6.12) and (6.15), we get

$$\log h(\varepsilon y) \leq \frac{t_1}{gk} d_1 (\log^* P)\Phi \log\left(\frac{c_{15} c_{16} R_T \log h(\varepsilon y)}{\log^* P \log H_t}\right).$$

This implies that

$$\log h(\varepsilon y) \leq 2\frac{t_1}{gk} d_1 (\log^* P)\Phi \log\{(t_1/gk) d_1 c_{15} c_{16} R_T (\Phi/\log H_t)\}.$$

Together with (6.23) this yields

$$\begin{aligned} (6.24) \qquad & \log h(\varepsilon y) \\ & \leq c_{23} P^{d_1} R_T (\log^* R_T)(\log^*(P R_T)/(\log^* P)^2)(\log C_3 + \log^* P) =: C_5 \end{aligned}$$

with the constant c_{23} defined above.

We note that $C_5 \geq C_4$. In what follows, we denote by C_6 the expression C_4' or C_5 according as $S = S_\infty$ or $S \neq S_\infty$. Then C_6 is larger than the bound in (6.5). Thus $\log h(\varepsilon y) \leq C_6$ in each case considered above.

It follows from (6.7) and (6.8) that $h(\mu_1) \leq h(\varepsilon y)^{2gkt_1}$ and so, from $\mu_1 = (\varepsilon x) + (\varepsilon y)\tau$ we infer that

$$h(\varepsilon x) \leq 2A_1 \exp\{(2gkt_1 + 1)C_6\}.$$

For $\kappa = x/y$, we have $h(\kappa) \leq 2A_1 \exp\{(4gkt_1 + 1)C_6\}$. Then it follows from (6.1) that $y^n N_{\mathbb{M}/\mathbb{K}}(\kappa + \tau) = \beta$, whence

$$h(y) \leq 4A_1^2 B_1^{1/n} \exp\{(4gkt_1 + 1)C_6\}$$

and

$$h(x) \leq 8A_1^3 B_1^{1/n} \exp\{(8gkt_1 + 2)C_6\} =: C_7.$$

We recall that

$$y = x_m, \quad x = x_1\alpha_1 + \ldots + x_{m-1}\alpha_{m-1}.$$

Taking the conjugates of x over $\mathbb{K}$ and using Cramer's rule, we get

$$\max_{1\leq i\leq m-1} h(x_i) \leq ((m-1)!)^{2(m-1)} A_1^{2(m-1)^2(m-1)!} C_7^{m-1}.$$

Now, using (5.5), it is easily seen that in Theorem 1 the estimate (2.2) follows with

$$c_1 = 3^{t+25} t^{5t+12} N^{3t+4d}.$$

Further, if $S = S_\infty$, the bound in (2.2) can be replaced by (2.3) with

$$c_2 = 3^{r+26}(r+1)^{7r+19} d^{4r+2} n^{2(n+r+6)},$$

and this completes the proof of Theorem 1. ■

Proof of Theorem 2. Let $x_1, \ldots, x_m$ be a solution of (2.1), and let m' be the largest integer for which $x_{m'} \neq 0$. Then (2.1) implies

$$N_{\mathbb{M}/\mathbb{K}}(x_1\alpha_1 + \ldots + x_{m'}\alpha_{m'}) = \beta. \tag{6.25}$$

For $m' \geq 2$, the estimates occurring in Theorem 2 immediately follow from Theorem 1. If $m' = 1$, then equation (6.25) reduces to $x_1^n = \beta$ and the assertion follows. ■

Proof of Theorem 3. Let x, y be a solution of equation (3.1). This equation can be written as

$$N_{\mathbb{M}/\mathbb{Q}}(x - \alpha y) = b/a_0 =: \beta \quad \text{in } x, y \in \mathbb{Z}, \tag{6.26}$$

where a_0 is the coefficient of X^n in $F(X,Y)$ and α is a zero of $F(X,1)$ with $\mathbb{M} = \mathbb{Q}(\alpha)$. Then $h(\alpha) \leq (\sqrt{n+1}H)^{1/n}$ (see e.g. [5]) and $h(\beta) \leq |b| \cdot H$. Now estimate (3.2) follows from the second part of Theorem 2.

Denote by q the number of complex places on $\mathbb{M}$. Then combining the estimate $n < \log |D_{\mathbb{M}}|$ (see e.g. [8]) with an upper bound for $R_{\mathbb{M}} h_{\mathbb{M}}$ in terms of $|D_{\mathbb{M}}|$ and n (see [18]) we get

$$R_{\mathbb{M}} h_{\mathbb{M}} < ((n-1)!)^{-1} |D_{\mathbb{M}}|^{1/2} (\log |D_{\mathbb{M}}|)^{n-1}. \tag{6.27}$$

Further, as is known, we have $|D_{\mathbb{M}}| \le |D(F)|$, where $D(F)$ denotes the discriminant of F. Finally, it follows from arguments of Lewis and Mahler [19] that $|D(F)| \le n^{2n-1} H^{2n-2}$. Now, combining estimate (3.2) with these inequalities, we obtain (3.3). ∎

Proof of Theorem 4. Let $x, y, z_1, \ldots, z_s$ be a solution of (3.4). Put $z_i = nu_i + v_i$ with $u_i, v_i \in \mathbb{Z}$, $0 \le v_i < n$ for $i = 1, \ldots, s$. Further, let

$$x' = \frac{x}{p_1^{u_1} \ldots p_s^{u_s}}, \quad y' = \frac{y}{p_1^{u_1} \ldots p_s^{u_s}} \quad \text{and} \quad b' = b \frac{p_1^{v_1} \ldots p_s^{v_s}}{a_0}.$$

Using the above notation, denote by S the set of places on $\mathbb{Q}$ consisting of the ordinary absolute value and the finite places corresponding to the primes $p_1, \ldots, p_s$. Let O_S be the ring of S-integers in $\mathbb{Q}$. Then (3.4) takes the form

$$N_{\mathbb{M}/\mathbb{Q}}(x' - \alpha y') = b' \quad \text{in } x', y' \in O_S.$$

Denote by T the set of extensions to $\mathbb{M}$ of the places in S, and let R_T denote the T-regulator of $\mathbb{M}$. Then Theorem 2 implies that

$$\begin{aligned} &\max\{h(x'), h(y')\} \\ &\quad < \exp\{c_{24} P^N R_T (\log^* R_T)(\log^*(PR_T)/\log^* P)(R_{\mathbb{M}} + s h_{\mathbb{M}} + \log(HB'))\} \\ &\quad =: C_8, \end{aligned} \tag{6.28}$$

where $N = n(n-1)(n-2)$, $c_{24} = c_1(n, s+1, N)$ and $B' \le BHP^{sn}$.

By assumption, x or y is relatively prime to p_i for each i. Hence (6.28) gives $p_i^{u_i} \le C_8$ and so

$$\max\{|x|, |y|, p_1^{u_1} \ldots p_s^{u_s}\} < C_8^3.$$

By Lemma 3, we have $R_T \le R_{\mathbb{M}} h_{\mathbb{M}} (n \log^* P)^{ns}$. Thus we obtain

$$\begin{aligned} &\max\{|x|, |y|, p_1^{z_1} \ldots p_s^{z_s}\} \\ &\quad < C_8^{3n} P^{ns} \\ &\quad < \exp\{c_{25} P^N (\log^* P)^{ns+2} R_{\mathbb{M}} h_{\mathbb{M}} \\ &\qquad \times (\log^*(R_{\mathbb{M}} h_{\mathbb{M}}))^2 (R_{\mathbb{M}} + s h_{\mathbb{M}} + \log(HB))\} \end{aligned} \tag{6.29}$$

with $c_{25} = 1.8 n^{ns+4} (\log n)^2 s^3 c_{24}$, whence (3.5) follows. Finally, using the above upper estimates for $R_{\mathbb{M}} h_{\mathbb{M}}$, $|D_{\mathbb{M}}|$ and $|D(F)|$, we deduce from (6.29) the estimate (3.6) of Theorem 4. ∎

Proof of Theorem 5. We keep the notation of Section 4. Let a denote the product of the leading coefficients of the minimal defining polynomials of $\alpha_0 = 1, \alpha_1, \ldots, \alpha_m$ over $\mathbb{Z}$. Then $\alpha'_i := a\alpha_i \in O_{\mathbb{M}}$, $|a| \le A^{dm}$ and $h(\alpha'_i) \le A^{dm+1}$ for $i = 0, \ldots, m$. Let $\mathbf{x} = (x_0, \ldots, x_m) \in O_{\mathbb{K}}^{m+1} \setminus \{\mathbf{0}\}$ and write $l(\mathbf{x}) = x_0\alpha'_0 + \ldots + x_m\alpha'_m$. Denote by $\mathfrak{p}_1, \ldots, \mathfrak{p}_u$ the prime ideals in $O_{\mathbb{K}}$ corresponding to the finite places in S. Obviously, we have $u \le t_0$. Consider in $O_{\mathbb{K}}$ the following decomposition into ideals:

$$(N_{\mathbb{M}/\mathbb{K}}(l(\mathbf{x}))) = \mathfrak{a}\mathfrak{p}_1^{v_1} \ldots \mathfrak{p}_u^{v_u},$$

where $\mathfrak{a}$ is an integral ideal in $O_{\mathbb{K}}$ which is relatively prime to $\mathfrak{p}_1, \ldots, \mathfrak{p}_u$. We recall that $R_{\mathbb{K}}$, $h_{\mathbb{K}}$ and $r_{\mathbb{K}}$ denote the regulator, class number and unit rank of $\mathbb{K}$. Let π_j be a generator of the principal ideal $\mathfrak{p}_j^{h_{\mathbb{K}}}$ for $j = 1, \ldots, u$. In view of Lemma 2, π_j can be chosen so that

$$h(\pi_j) \le C_9 \exp\{h_{\mathbb{K}} \log^* P\}, \tag{6.30}$$

where $C_9 = \exp\{c_{10}(k, r_{\mathbb{K}})R_{\mathbb{K}}\}$. Denoting by w_j the quotient in the Euclidean division of v_j by $h_{\mathbb{K}}$, we obtain

$$N_{\mathbb{M}/\mathbb{K}}(l(\mathbf{x})) = \delta\pi_1^{w_1} \ldots \pi_u^{w_u}, \tag{6.31}$$

where $\delta \in O_{\mathbb{K}}$ and $\mathfrak{p}_j^{h_{\mathbb{K}}} \nmid \delta$, $j = 1, \ldots, u$.

For $j = 1, \ldots, u$, set $w_j = q_j n + z_j$ with rational integers q_j, z_j for which $0 \le z_j < n$. Further, denote by d_j the greatest non-negative integer for which $d_j \le q_j$ and $\pi_j^{d_j}$ divides x_i in $O_{\mathbb{K}}$ for $i = 0, \ldots, m$. By Lemma 2, there are a unit η and a β in $O_{\mathbb{K}}$ such that $\delta\pi_1^{z_1} \ldots \pi_u^{z_u} = \eta^n\beta$ and

$$h(\beta) \le |N_{\mathbb{K}/\mathbb{Q}}(\beta)|^{1/k} C_9^n. \tag{6.32}$$

Put $\varepsilon = \eta\pi_1^{d_1} \ldots \pi_u^{d_u}$, $\varrho = \pi_1^{e_1} \ldots \pi_u^{e_u}$ with $e_j = q_j - d_j$ and $\mathbf{x}' = (x'_0, \ldots, x'_m) = \varepsilon^{-1}\mathbf{x}$. Then ε is an S-unit in $O_{\mathbb{K}}$ and $\mathbf{x}' \in O_{\mathbb{K}}^{m+1} \setminus \{\mathbf{0}\}$. Further, (6.31) implies that

$$N_{\mathbb{M}/\mathbb{K}}(l(\varrho^{-1}\mathbf{x}')) = \beta. \tag{6.33}$$

It follows that

$$|N_{\mathbb{K}/\mathbb{Q}}(\beta)| = N_S(\beta) \prod_{j=1}^{u} |\beta|_{\mathfrak{p}_j}^{-1} \le N_S(\beta) P^{uknh_{\mathbb{K}}}.$$

Hence, in view of (6.32), we infer that

$$h(\beta) \le N_S(\beta)^{1/k} P^{unh_{\mathbb{K}}} C_9^n. \tag{6.34}$$

We have $\varrho^{-1}\mathbf{x}' \in O_S^{m+1} \setminus \{\mathbf{0}\}$. On applying now Theorem 2 to equation (6.33) and using Lemma 4 and (6.34), we get the estimate

$$\max_{0 \le i \le m} h(\varrho^{-1}x'_i) < \exp C_{11} \tag{6.35}$$

with $C_{11} = C_{10}(R_{\mathbb{M}} + t_0 h_{\mathbb{M}} + (dm + 1)\log A + \log N_S(\beta))$ and $C_{10} = 3c_1 P^N R_T (\log^* R_T)^2$.

Write $\gamma = \beta\varrho^n$. Then, by (6.33), $\gamma \in O_{\mathbb{K}}$ and $N_S(\gamma) = N_S(\beta)$. We recall that for each j, either $e_j = 0$ or there is an x_i' such that π_j does not divide x_i'. Hence it follows from (6.35) that $e_j \le 2C_{11}$, whence, by (6.30),

$$(6.36) \qquad h(\varrho) \le \exp\{2t_0 C_{11}(c_{10}(k, r_{\mathbb{K}})R_{\mathbb{K}} + h_{\mathbb{K}} \log^* P)\}.$$

Putting $X = \max_{0\le i\le m} \overline{|x_i'|}$, from (6.35), (6.36) and $\overline{|x_i'|} \le h(x_i')^k$ we get

$$X \le \exp\{(2t_0+1)kC_{11}(c_{10}(k, r_{\mathbb{K}})R_{\mathbb{K}} + h_{\mathbb{K}} \log^* P)\}$$

whence, using $c_{10}(k, r_{\mathbb{K}})R_{\mathbb{K}} + h_{\mathbb{K}} \log^* P \le 2^{k-4}k^{2(k-1)}R_{\mathbb{K}}h_{\mathbb{K}}(\log^* P)$, it follows that

$$(6.37) \qquad N_S(\gamma) \ge X^{1/C_{12}} A^{-(dm+1)} C_{13}$$

where $C_{12} = (t_0+1)2^{k-1}k^{2k-1}c_1 P^N(\log^* P)R_T(\log^* R_T)^2 R_{\mathbb{K}}h_{\mathbb{K}}$ and $C_{13} = \exp\{-(R_{\mathbb{M}} + t_0 h_{\mathbb{M}})\}$. Further, in view of (6.33),

$$(6.38) \qquad N_{\mathbb{M}/\mathbb{K}}(l(\mathbf{x}')) = \gamma.$$

Put $\Omega_0 = \Omega_{\mathbb{M}} \setminus \Omega_\infty$ and $T_0 = T \setminus \Omega_\infty$. By the product formula (5.1) in $\mathbb{M}$, we have

$$(6.39) \quad \prod_{w\in\Gamma} |l(\mathbf{x}')|_w = \prod_{w\in\Omega_\infty\setminus\Gamma_\infty} |l(\mathbf{x}')|_w^{-1} \prod_{w\in\Omega_0\setminus T_0} |l(\mathbf{x}')|_w^{-1} \prod_{w\in T_0\setminus\Gamma_0} |l(\mathbf{x}')|_w^{-1}.$$

We can bound $\prod_{w\in\Omega_\infty\setminus\Gamma_\infty} |l(\mathbf{x}')|_w$ from above by $\prod_{w\in\Omega_\infty\setminus\Gamma_\infty}(2m)^{d_w} \times A^{d(dm+1)}X^{d_w}$, where $d_w = 1$ or 2 according as w is a real or complex place. Further, using (5.3), (6.37) and the product formula in $\mathbb{K}$, from (6.38) we obtain

$$(6.40) \qquad \prod_{w\in\Omega_0\setminus T_0} |l(\mathbf{x}')|_w^{-1} = \prod_{\substack{\mathfrak{p}\text{ prime ideal in } O_{\mathbb{K}}\\ \mathfrak{p}\ne\mathfrak{p}_1,\ldots,\mathfrak{p}_u}} |\gamma|_{\mathfrak{p}}^{-1} = N_S(\gamma)$$
$$\ge X^{1/C_{12}} A^{-(dm+1)} C_{13}.$$

Finally, we have $|l(\mathbf{x}')|_w \le 1$ for each $w \in T_0\setminus\Gamma_0$. Hence, using (6.39), (6.40) and $\prod_{w\in\Gamma} |a|_w \le A^{d^2 m}$, we infer that

$$\prod_{w\in\Gamma} |l(\mathbf{x}')/a|_w \ge \kappa_1 X^{-d+r_1+2r_2+\tau_1}$$

with $\kappa_1 = (2m)^{-d+r_1+2r_2} A^{-(d^2+1)(dm+1)} C_{13}$ and $\tau_1 = 1/C_{12}$, which is just estimate (4.1) of our theorem.

The second part (when $\Gamma = \Gamma_\infty$) of Theorem 5 follows in a similar way from the bound (2.3) occurring in Theorem 1. ∎

Proof of Corollary 1. Let $(x_1, \ldots, x_m) \in \mathbb{Z}^m \setminus \{\mathbf{0}\}$. Denoting by $-y$ the nearest rational integer to $x_1\alpha_1 + \ldots + x_m\alpha_m$, we have

$$\|x_1\alpha_1 + \ldots + x_m\alpha_m\| = |y + x_1\alpha_1 + \ldots + x_m\alpha_m| > 0.$$

We may obviously assume that $\|x_1\alpha_1+\ldots+x_m\alpha_m\| < 1$. Then, noting that $|y| \le 1+mA^nX \le 2mA^nX$ and applying the second part of Theorem 5, we get

$$|y + x_1\alpha_1 + \ldots + x_m\alpha_m| > \kappa_3 X^{-(n-\sigma-\tau_3)/\sigma},$$

with the constants κ_3, τ_3 specified in our Corollary 1. ∎

Proof of Corollary 2. We follow the proof of Corollary 3 of [10]. Let a_0 denote the leading coefficient of the minimal defining polynomial of α over $\mathbb{Z}$. Then $a_0\alpha \in O_{\mathbb{K}}$. Consider $x_1\theta + x_2$ with $x_1 = a_0$, $x_2 = -a_0\alpha$. By Theorem 5 there exists an S-unit ε in $O_{\mathbb{K}}$ such that $\varepsilon^{-1}x_1$, $\varepsilon^{-1}x_2 \in O_{\mathbb{K}}$ and

$$(6.41)\qquad \prod_{v\in\Gamma\cup\Omega_\infty} |(\varepsilon^{-1}x_1)\theta - (\varepsilon^{-1}x_2)|_v > \kappa_1 X^{\tau_1}, \qquad X = \max_{i=1,2}\overline{|\varepsilon^{-1}x_i|}$$

with $m = 1$ in κ_1. However, we have $\prod_{v\in\Gamma\cup\Omega_\infty} |\varepsilon^{-1}|_v \le 1$. Further, it follows that

$$h(\alpha) = h\biggl(\frac{\varepsilon^{-1}x_2}{\varepsilon^{-1}x_1}\biggr) \le h(\varepsilon^{-1}x_2)h(\varepsilon^{-1}x_1) \le X^2$$

and $|a_0\theta - a_0\alpha|_v \le 2^{d_v}A^d h(\alpha)^{kd_v}$, where d_v is defined as in the proof of Theorem 5. Thus, we deduce from (6.41) that

$$(6.42)\qquad \prod_{v\in\Gamma} |\theta-\alpha|_v > \kappa_4 h(\alpha)^{k(-d+r_1+2r_2)+\tau_1/2} a_0^{-(r_1+2r_2)}$$

with $\kappa_4 = \kappa_1(2A^d)^{-d+r_1+2r_2}$ and if $\alpha \in O_{\mathbb{K}}$, (4.3) follows. If α is not integral, then $a_0 \le (h(\alpha))^k$ and (6.42) implies (4.2). ∎

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