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Sets of integers and quasi-integers with pairwise common divisor

by

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0. Introduction. Consider the set $\mathbb{N}_s(n) = \{u \in \mathbb{N} : (u, \prod_{i=1}^{s-1} p_i) = 1\} \cap \langle 1, n\rangle$ of positive integers between 1 and n, which are not divisible by the first $s-1$ primes $p_1, \ldots, p_{s-1}$.

Erdős introduced in [4] (and also in [5]–[7], [9]) the quantity $f(n,k,s)$ as the largest integer ϱ for which an $A \subset \mathbb{N}_s(n)$, $|A| = \varrho$, exists with no $k+1$ numbers being coprimes. Certainly the set

(1) $\quad \mathbb{E}(n,k,s) = \{u \in \mathbb{N}_s(n) : u = p_{s+i}v \text{ for some } i = 0, 1, \ldots, k-1\}$

does not have $k+1$ coprimes.

The following conjecture was disproved in [1]:

CONJECTURE 1 (Erdős [4]). $f(n,k,1) = |\mathbb{E}(n,k,1)|$ *for all* $n, k \in \mathbb{N}$.

This disproves of course also the

GENERAL CONJECTURE (Erdős [7]). *For all* $n, k, s \in \mathbb{N}$,

$$f(n,k,s) = |\mathbb{E}(n,k,s)|. \tag{2}$$

However, in [2] we proved (2) for every k, s and for large n (relative to k, s).

In the present paper we are concerned with the case $k = 1$, which in [1] and [2] we called

CONJECTURE 2. $f(n,1,s) = |\mathbb{E}(n,1,s)|$ *for all* $n, s \in \mathbb{N}$.

Erdős mentioned in [7] that he did not even succeed in settling this special case of the General Conjecture. Whereas in [1] we proved this by a completely different approach for $n \geq (p_{s+1} - p_s)^{-1} \prod_{i=1}^{s+1} p_i$, we establish it now for all n (Theorem 2).

We generalize and analyze Conjecture 2 first for quasi-primes in order to understand how the validity of Conjecture 2 depends on the distribution of the quasi-primes and primes. Our main result is a simply structured sufficient condition on this distribution (Theorem 1). Using sharp estimates on the prime number distribution by Rosser and Schoenfeld [14] we show that this condition holds for $\mathbb{Q} = \{p_s, p_{s+1}, \ldots\}$, $s \geq 1$, as set of quasi-primes and thus Theorem 2 follows.

1. Basic definitions for natural numbers and quasi-numbers. Whenever possible we keep the notation of [2]. $\mathbb{N}$ denotes the set of positive integers and $\mathbb{P} = \{p_1, p_2, \ldots\} = \{2, 3, 5, \ldots\}$ denotes the set of all primes. $\mathbb{N}^*$ is the set of squarefree numbers.

For two numbers $u, v \in \mathbb{N}$ we write $u \mid v$ (resp. $u \nmid v$) iff u divides v (resp. u does not divide v), $[u, v]$ stands for the smallest common multiple of u and v, (u, v) is the largest common divisor of u and v, and we say that u and v have a common divisor if $(u, v) > 1$. $\langle u, v\rangle$ denotes the interval $\{x \in \mathbb{N} : u \leq x \leq v\}$.

For any set $A \subset \mathbb{N}$ let

$$A(n) = A \cap \langle 1, n\rangle \tag{1.1}$$

and $|A|$ be the cardinality of A. The set of multiples of A is

$$M(A) = \{m \in \mathbb{N} : a \mid m \text{ for some } a \in A\}. \tag{1.2}$$

For a set $\{a\}$ with one element we also write $M(a)$ instead of $M(\{a\})$. For $u \in \mathbb{N}$, $p^+(u)$ denotes the largest prime in its prime number representation

$$u = \prod_{i=1}^{\infty} p_i^{\alpha_i}, \quad \sum_{i=1}^{\infty} \alpha_i < \infty. \tag{1.3}$$

We also need the function π, where for $y \in \mathbb{N}$,

$$\pi(y) = |\mathbb{P}(y)|, \tag{1.4}$$

and the set Φ, where

$$\Phi(u, y) = \{x \in \mathbb{N}(u) : (x, p) = 1 \text{ for all } p < y\}. \tag{1.5}$$

We note that $1 \in \Phi(u, y)$ for all $u \geq y$, $u \geq 1$.

Clearly, by (1.3), $u \in \mathbb{N}$ corresponds to a multiset $(\alpha_1, \alpha_2, \ldots)$. Therefore, instead of saying that $A \subset \mathbb{N}(z)$ has pairwise (nontrivial) common divisors, we adopt the following shorter multiset terminology.

Definition 1. $A \subset \mathbb{N}(z)$, $z \geq 1$, is said to be *intersecting* iff for all $a, b \in A$, we have $a = \prod_{i=1}^{\infty} p_i^{\alpha_i}$ and $b = \prod_{i=1}^{\infty} p_i^{\beta_i}$ with $\alpha_j \beta_j \neq 0$ for some j.

In order to better understand how properties depend on the multiset structure and how on the distribution of primes it is very useful to introduce quasi-(natural) numbers and quasi-primes. Results can then also be applied to a subset of primes if it is viewed as the set of quasi-primes.

A set $\mathbb{Q} = \{1 < r_1 < r_2 < \ldots\}$ of positive real numbers with $\lim_{i\to\infty} r_i = \infty$ is called a (complete) set of *quasi prime numbers*, if every number in

$$\mathbb{X} = \Big\{x \in \mathbb{R}^+ : x = \prod_{i=1}^{\infty} r_i^{\alpha_i},\ \alpha_i \in \{0,1,2,\ldots\},\ \sum_{i=1}^{\infty} \alpha_i < \infty\Big\} \tag{1.6}$$

has a unique representation. (See also Remark 1 after Theorem 1.)

The set $\mathbb{X}$ is the set of *quasi-numbers* corresponding to the set of quasi-primes $\mathbb{Q}$.

We can now replace $\mathbb{P}, \mathbb{N}$ by $\mathbb{Q}, \mathbb{X}$ in all concepts of this section up to Definition 1 and thus for any $u, v \in \mathbb{X}$, the notations $u \mid v$, $u \nmid v$, (u, v), $[u, v]$, $\langle u, v\rangle$ $(= \{x \in \mathbb{X} : u \le x \le v\})$; for any $A \subset \mathbb{X}$, $A(z)$, $M(A)$ $(= \{m \in \mathbb{X} : a \mid m$ for some $a \in A\})$; and "intersecting" are well defined. So are also the function π and the sets $\Phi(u, y)$ for $u \ge y, u \ge 1$.

We study $\mathcal{I}(z)$, the family of all intersecting $A \subset \mathbb{X}(z)$, and

$$f(z) = \max_{A\in\mathcal{I}(z)} |A|, \qquad z \in \mathbb{X}. \tag{1.7}$$

The subfamily $\mathcal{O}(z)$ of $\mathcal{I}(z)$ consists of the optimal sets, that is,

$$\mathcal{O}(z) = \{A \in \mathcal{I}(z) : |A| = f(z)\}. \tag{1.8}$$

A key role is played by the following configuration.

Definition 2. $A \subset \mathbb{X}(z)$ is called a *star* if

$$A = M(\{r\}) \cap \mathbb{X}(z) \quad \text{ for some } r \in \mathbb{Q}.$$

2. Auxiliary results concerning left compressed sets, "upsets" and "downsets". There is not only one way to define "left pushing" of subsets of $\mathbb{X}$. Here the following is most convenient.

For any $i, j \in \mathbb{N}$, $j < i$, we define the "left pushing" operation $L_{i,j}$ on subsets of $\mathbb{X}$. For $A \subset \mathbb{X}$ let $A_1 = \{a \in A : a = a_1 r_i^{\alpha},\ \alpha \ge 1,\ (a_1, r_i r_j) = 1,$ $a_1 r_j^{\alpha} \notin A\}$ and $L_{i,j}(A) = (A \setminus A_1) \cup A_1^*$, where $A_1^* = \{a = a_1 r_j^{\alpha} : (a_1, r_i r_j) = 1$ and $a_1 r_j^{\alpha} \in A_1\}$. Clearly $|L_{i,j}(A) \cap \mathbb{X}(z)| \ge |A(z)|$ for every $z \in \mathbb{R}^+$. It is easy to show that the operation $L_{i,j}$ preserves the "intersecting" property.

By finitely many (resp. countably many) "left pushing" operations $L_{i,j}$ one can transform every $A \subset \mathbb{X}(z)$, $z \in \mathbb{R}^+$ (resp. $A \subset \mathbb{X}$) into a "left compressed" set A', where the concept of left compressedness is defined as follows:

Definition 3. $A \subset X$ is said to be *left compressed* if

$$L_{i,j}(A) = A \quad \text{ for all } i, j \text{ with } i > j.$$

We note that there are left compressed sets A' and A'' which are obtained by left pushing from the same set A.

For any $z \in \mathbb{X}$ let $\mathcal{C}(z) = \{A \in \mathcal{I}(z) : A \text{ is left compressed}\}$.

Lemma 1. *For all* $z \in \mathbb{X}$,

$$f(z) = \max_{A \in \mathcal{C}(z)} |A|.$$

Clearly, any $A \in \mathcal{O}(z)$ is an "upset":

$$A = M(A) \cap \mathbb{X}(z), \tag{2.1}$$

and it is also a "downset" in the following sense:

(2.2) for $a \in A$ with $a = r_{i_1}^{\alpha_1} \ldots r_{i_t}^{\alpha_t}$ and $\alpha_i \geq 1$ also $a' = r_{i_1} \ldots r_{i_t} \in A$.

For every $B \subset \mathbb{X}$ we introduce the unique primitive subset $P(B)$ which has the properties

$$b_1, b_2 \in P(B) \quad \text{implies} \quad b_1 \nmid b_2 \text{ and } B \subset M(P(B)). \tag{2.3}$$

We know from (2.2) that for any $A \in \mathcal{O}(z)$, $P(A)$ consists only of squarefree quasi-numbers and that by (2.1),

$$A = M(P(A)) \cap \mathbb{X}(z). \tag{2.4}$$

From Lemma 1 we know that $\mathcal{O}(z) \cap \mathcal{C}(z) \neq \emptyset$.

Let now $A \in \mathcal{O}(z) \cap \mathcal{C}(z)$ and $P(A) = \{a_1, \ldots, a_m\}$, where the a_i's are written in lexicographic order. The set of multiples of $P(A)$ in $\mathbb{X}(z)$ can be written as a union of disjoint sets $B^i(z)$:

$$M(P(A)) \cap \mathbb{X}(z) = \dot{\bigcup} B^i(z), \tag{2.5}$$

$$B^i(z) = \{x \in M(P(A)) \cap X(z) : a_i \mid x,\ a_j \nmid x \text{ for } j = 1, \ldots, i-1\}. \tag{2.6}$$

We can say more about $B^i(z)$ if we use the factorization of the squarefree quasi-numbers a_i.

Lemma 2. *Let* $a_i = r_{j_1}, \ldots, r_{j_l}$ *with* $r_{j_1} < \ldots < r_{j_l}$. *Then*

$$B^i(z) = \Big\{x \in \mathbb{X}(z) : x = r_{j_1}^{\alpha_1} \ldots r_{j_l}^{\alpha_l} T,\ \alpha_i \geq 1,\ \Big(T, \prod_{r_i \leq r_{j_l}} r_i\Big) = 1\Big\}.$$

Proof. This immediately follows from the facts that A is left compressed, an "upset" and a "downset".

Finally, a result for stars. Keep in mind that they contain a single prime and that Lemma 1 holds.

Lemma 3. *For any* $B \subset \mathcal{I}(z)$ *and* $B' \subset \mathbb{X}(z)$ *which is left compressed and obtained from* B *by left pushing we have*: *B is a star if and only if B' is.*

3. The main result

THEOREM 1. *Suppose the quasi-primes* $\mathbb{Q}$ *satisfy the following condition*: *for all* $u \in \mathbb{R}^+$ *and for all* r_l, $l \geq 2$,

$$2|\Phi(u, r_l)| \leq |\Phi(ur_l, r_l)|. \tag{a}$$

Then, *for all* $z \in \mathbb{R}^+$, *every optimal* $A \in \mathcal{O}(z)$ *is a star. In particular*,

$$f(z) = |M(r_1) \cap X(z)| \quad \textit{for all } z \in \mathbb{X}.$$

Remarks. 1. This result and also Lemma 4 below immediately extend to the case where quasi-primes are defined without the requirement of the uniqueness of the representations in (1.6), if multiplicities of representations are taken into consideration. $\mathbb{X}$ is thus just a free, discrete commutative semigroup in $\mathbb{R}^+_{\geq 1}$.

2. Without the uniqueness requirement we are led to a new problem by not counting multiplicities.

3. However, without the assumption $\lim_{i\to\infty} r_i = \infty$ or without the assumption of discreteness the quasi-primes have a cluster point ϱ and one can produce infinitely many infinite, intersecting sets in $\mathbb{X}(\varrho^3 + \varepsilon)$ which are not stars.

4. In Section 5 we discuss the case of finitely many quasi-primes.

Proof of Theorem 1. Let $A \in \mathcal{O}(z)$ and let $P(A) = \{a_1, \ldots, a_m\}$ be the primitive subset of A which generates A. Under condition (a), the theorem is equivalent to the statement: for all $z \in \mathbb{X}$, we have $m = 1$ and $a_1 = r_l$ for some quasi-prime r_l.

Suppose, to the contrary, that for some $z \in \mathbb{X}$ there exists $A \in \mathcal{O}(z)$ which is not a star, i.e. if $P(A) = \{a_1, \ldots, a_m\}$ is the primitive generating subset of A, then $m > 1$ and hence every element $a_i \in P(A)$ is a product of at least two different quasi-primes.

According to Lemma 1 we can assume that $A \in \mathcal{O}(z) \cap \mathcal{C}(z)$ and $P(A) = \{a_1, \ldots, a_m\}$, where a_i's are written in lexicographic order, $m > 1$ and

$$p^+(a_m) = r_t, \quad t \geq 2.$$

Write $P(A)$ in the form

$$P(A) = S_1 \cup \ldots \cup S_t, \quad t \geq 2,\ S_t \neq \emptyset,$$

where

$$S_i = \{a \in P(A) : p^+(a) = r_i\}.$$

Since $A \in \mathcal{O}(z) \cap \mathcal{C}(z)$, we have

$$A = M(P(A)) \cap X(z) = \dot{\bigcup_{1 \leq j \leq t}} B(S_j),$$

where $B(S_j) = \dot{\bigcup}_{a_i \in S_j} B^i(z)$ and $B^i(z)$ are described in Lemma 2.

Now we consider $S_t = \{a_l, a_{l+1}, \dots, a_m\}$ for some $l \le m$, and let $S_t = S_t^1 \dot\cup S_t^2$, where

$$S_t^1 = \{a_i \in S_t : r_{t-1} \,|\, a_i\}, \qquad S_t^2 = S_t \setminus S_t^1.$$

We have

$$B(S_t) = B(S_t^1) \dot\cup B(S_t^2), \tag{3.1}$$

where

$$B(S_t^j) = \bigcup_{a_i \in B_t^j} B^{(i)}(z), \qquad j = 1,2.$$

Let $\widetilde{S}_t = \{a_l/r_t, a_{l+1}/r_t, \dots, a_m/r_t\}$ and similarly $\widetilde{S}_t^j = \{a_i/r_t : a_i \in S_t^j\}$, $j = 1,2$. It is clear that $a_i/r_t > 1$ for all $a_i \in S_t$.

Obviously $\widetilde{S}_t^1 \in \mathcal{I}(z)$, because all elements of $\widetilde{S}_t^1$ have common factor r_{t-1}. Let us show that $\widetilde{S}_t^2 \in \mathcal{I}(z)$ as well. Suppose, to the contrary, there exist $b_1, b_2 \in \widetilde{S}_t^2$ with $(b_1, b_2) = 1$. We have $b_1 r_t, b_2 r_t \in S_t^2 \subset A$ and $(b_1 r_t, r_{t-1}) = 1$, $(b_2 r_t, r_{t-1}) = 1$. Since $A \in \mathcal{C}(z)$ and $r_{t-1} \nmid b_1 b_2$ (see definition of S_t^2), we conclude that $r_{t-1} b_1 \in A$ as well. Hence $r_{t-1} b_1, r_t b_2 \in A$ and at the same time $(r_{t-1} b_1, r_t b_2) = 1$, which is a contradiction. So, we have $\widetilde{S}_t^j \in \mathcal{I}(z)$, $j = 1,2$, and hence

$$A_j = M((P(A) \setminus S_t) \cup \widetilde{S}_t^i) \cap \mathbb{X}(z) \in \mathcal{I}(z), \qquad j = 1,2.$$

We now prove that either $|A_1| > |A|$ or $|A_2| > |A|$, and this will lead to a contradiction.

From (3.1) we know that $\max\{|B(S_t^1)|, |B(S_t^2)|\} \ge \frac{1}{2}|B(S_t)|$. Let us assume, say

$$|B(S_t^2)| \ge \tfrac{1}{2}|B(S_t)|, \tag{3.2}$$

and let us show that

$$|A_2| > |A| \tag{3.3}$$

(if $|B(S_t^1)| \ge \frac{1}{2}|B(S_t)|$ the situation is symmetrically the same).

Let $b \in \widetilde{S}_t^2$ and $b = r_{i_1} r_{i_2} \dots r_{i_s}$ with $r_{i_1} < r_{i_2} < \dots < r_{i_s} < r_t$. We know that

$$a_i = b r_t = r_{i_1} \dots r_{i_s} r_t \in S_t^2 \quad \text{for some} \ \ i \le m,$$

and that (see Lemma 2), the contribution of $M(a_i)$ in $B(S_t)$ (and as well in A) are the elements in the form

$$B^i(z) = \Big\{x \in \mathbb{X}(z) : x = r_{i_1}^{\alpha_1} \dots r_{i_s}^{\alpha_s} r_t^{\alpha_t} T, \ \text{where} \ \alpha_i \ge 1 \ \text{and} \ \Big(T, \prod_{i \le t} r_i\Big) = 1\Big\}.$$

We write $B^i(z)$ in the following form:

$$B^i(z) = \dot{\bigcup_{(\alpha_1,\dots,\alpha_s),\, \alpha_i \ge 1}} D(\alpha_1,\dots,\alpha_s), \tag{3.4}$$

where

$$\begin{aligned} D(\alpha_1,\dots,\alpha_s) \\ &= \Big\{x \in \mathbb{X}(z) : x = r_{i_1}^{\alpha_1}\dots r_{i_s}^{\alpha_s} r_t T_1, \Big(T_1, \prod_{i\le t-1} r_i\Big) = 1\Big\}. \end{aligned} \tag{3.5}$$

Now we look at the contribution of $M(b)$ in $A_2 = M((P(A)\setminus S_t)\cup \widetilde{S}_t^2)\cap\mathbb{X}(z)$, namely at those elements in A_2 (denoted by $B(b)$) which are divisible by b, but not divisible by any element from $(P(A)\setminus S_t)\cup(\widetilde{S}_t^2\setminus b)$.

Since $A \subset \mathcal{C}(z)$ and r_t is the largest quasi-prime in $P(A)$, we conclude that

$$\begin{aligned} B(b) &\supseteq B^*(b) \\ &= \Big\{x \in \mathbb{X}(z) : x = r_{i_1}^{\alpha_1}\dots r_{i_s}^{\alpha_s}\widetilde{T},\ \alpha_i \ge 1,\ \text{where } \Big(\widetilde{T}, \prod_{i\le t-1} r_i\Big) = 1\Big\}, \end{aligned}$$

and we can write

$$B^*(b) = \dot{\bigcup_{(\alpha_1,\dots,\alpha_s),\alpha_i \ge 1}} \widetilde{D}(\alpha_1,\dots,\alpha_s), \tag{3.6}$$

where

$$\begin{aligned} \widetilde{D}(\alpha_1,\dots,\alpha_s) \\ &= \Big\{x \in \mathbb{X}(z) : x = r_{i_1}^{\alpha_1}\dots r_{i_s}^{\alpha_s}\widetilde{T},\ \alpha_i \ge 1, \Big(\widetilde{T}, \prod_{i\le t-1} r_i\Big) = 1\Big\}. \end{aligned} \tag{3.7}$$

Hence

$$|B(b)| \ge |B^*(b)| = \sum_{(\alpha_1,\dots,\alpha_s),\, \alpha_i\ge 1} |\widetilde{D}(\alpha_1,\dots,\alpha_s)|. \tag{3.8}$$

First we prove that $|A_2| \ge |A|$. In the light of (3.2) and (3.4)–(3.8), for this it is sufficient to show that

$$|\widetilde{D}(\alpha_1,\dots,\alpha_s)| \ge 2|D(\alpha_1,\dots,\alpha_s)|, \tag{3.9}$$

for all $(\alpha_1,\dots,\alpha_s), \alpha_i \ge 1$. However, this is exactly the condition (a) in Theorem 1 for $u = z/(r_{i_1}^{\alpha_1}\dots r_{i_s}^{\alpha_s} r_t)$ and $l = t$. Hence $|A_2| \ge |A|$.

To prove (3.3), that is, $|A_2| > |A|$, it is sufficient to show the existence of $(\alpha_1,\dots,\alpha_s)$, $\alpha_i \ge 1$, for which strict inequality holds in (3.9). For this we take $\beta \in \mathbb{N}$ and $(\alpha_1,\dots,\alpha_s) = (\beta,1,1,\dots,1)$ such that

$$z/r_t < r_{i_1}^{\beta} r_{i_2}\dots r_{i_s} \le z.$$

This is always possible, because

$$r_{i_1} r_{i_2} \dots r_{i_s} r_t \le z \quad \text{implies} \quad r_{i_1} \dots r_{i_s} \le z/r_t \text{ and } r_{i_1} < \dots < r_{i_s} < r_t.$$

We have $|\widetilde{D}(\beta, 1, \dots, 1)| = 1$ and $|D(\beta, 1, \dots, 1)| = 0$. Hence $|A_2| > |A|$, which is a contradiction, since $A_2 \in \mathcal{I}(z)$. This completes the proof.

Lemma 4. *Sufficient for condition* (a) *in Theorem* 1 *to hold is the condition*

$$\text{(b)} \qquad 2\pi(v) \le \pi(r_2 v) \quad \textit{for all } v \in \mathbb{R}^+.$$

Proof. Under condition (b) it is sufficient to prove for every $u \in \mathbb{R}^+$ and r_l $(l \ge 2)$ that $|\Phi(u, r_l)| \le |\Phi_1(ur_l, r_l)|$, where $\Phi_1(ur_l, r_l) = \{x \in \Phi(ur_l, r_l) : u < x \le ur_l\}$.

We avoid the trivial cases $u < 1$, for which $\Phi(u, r_l) = \emptyset$, and $1 \le u < r_l$, for which $\Phi(u, r_l) = \{1\}$ and $r_l \in \Phi_1(ur_l, r_l)$. Hence, we assume $u \ge r_l$.

Let $F(u, r_l) = \{a \in \Phi(u, r_l), a \ne 1 : ap^+(a) \le u\} \cup \{1\}$. It is clear that for any $b \in \Phi(u, r_l), b \ne 1$, we have $b/p^+(b) \in F(u, r_l)$ and that

$$|\Phi(u, r_l)| = 1 + \sum_{a \in F(u, r_l)} |\tau(a)|, \tag{3.10}$$

where $\tau(a) = \{r \in Q : r_l \le p^+(a) \le r \le u/a\}$ and integer 1 in (3.10) stands to account for the element $1 \in \Phi(u, r_l)$.

On the other hand, we have

$$|\Phi_1(ur_l, r_l)| \ge \sum_{a \in F(u, r_l)} |\tau_1(a)|, \tag{3.11}$$

where

$$\tau_1(a) = \{r \in Q : u/a < r \le ur_l/a\}.$$

We have

$$|\tau(a)| \le \pi\left(\frac{u}{a}\right) - l + 1 \le \pi\left(\frac{u}{a}\right) - 1 \quad (l \ge 2)$$

and by condition (b),

$$|\tau_1(a)| = \pi\left(\frac{ur_l}{a}\right) - \pi\left(\frac{u}{a}\right) \ge \pi\left(\frac{u}{a}\right). \tag{3.12}$$

Hence $|\tau_1(a)| > |\tau(a)|$ for all $a \in F(u, r_l)$ and, since $F(u, r_l) \ne \emptyset$ $(u \ge r_l)$, from (3.10)–(3.12) we get $|\Phi_1(ur_l, r_l)| \ge |\Phi(u, r_l)|$.

4. Proof of Erdős' Conjecture 2. For a positive integer s let $\mathbb{N}_s = \{u \in \mathbb{N} : (u, \prod_{i=1}^{s-1} p_i) = 1\}$ and let $\mathbb{N}_s(n) = \mathbb{N}_s \cap \langle 1, n \rangle$.

Erdős introduced in [4] (and also in [5]–[7], [9]) the quantity $f(n, k, s)$ as the largest integer ϱ for which an $A \subset \mathbb{N}_s(n)$, $|A| = \varrho$, exists with no $k+1$ numbers being coprimes.

Certainly the set

$$\mathbb{E}(n,k,s)=\{u\in\mathbb{N}_s(n):u=p_{s+i}v \text{ for some } i=0,1,\ldots,k-1\} \tag{4.1}$$

does not have $k+1$ coprimes.

The case $s=1$, in which $\mathbb{N}_1(n)=\langle 1,n\rangle$, is of particular interest.

The following conjecture was disproved in [1]:

CONJECTURE 1 (Erdős [4]).

$$f(n,k,1)=|\mathbb{E}(n,k,1)| \quad \textit{for all } n,k\in\mathbb{N}. \tag{4.2}$$

This disproves of course also the

GENERAL CONJECTURE (Erdős [7]). *For all* $n,k,s\in\mathbb{N}$,

$$f(n,k,s)=|\mathbb{E}(n,k,s)|. \tag{4.3}$$

However, in [2] we proved (4.3) for every k,s and (relative to k,s) large n. For further related work we refer to [8]–[10].

Erdős mentions in [7] that he did not succeed in settling even the case $k=1$. This special case of the General Conjecture was called in [1] and [2]

CONJECTURE 2. $f(n,1,s)=|\mathbb{E}(n,1,s)|$ *for all* $n,s\in\mathbb{N}$.

Notice that

$$\mathbb{E}(n,1,s)=\{u\in\mathbb{N}_s(n):p_s\mid u\}, \quad \text{i.e. } \mathbb{E}(n,1,s) \text{ is a star.} \tag{4.4}$$

In the language of quasi-primes we can define

$$\mathbb{Q}=\{r_1,r_2,\ldots,r_l,\ldots\}=\{p_s,p_{s+1},\ldots,p_{s+l-1},\ldots\} \tag{4.5}$$

and the corresponding quasi-integers $\mathbb{X}$.

Now, Conjecture 2 is equivalent to

$$f(n,1,s)=|M(p_s)\cap\mathbb{X}(n)| \quad \text{for all } n,s\in\mathbb{N}. \tag{4.6}$$

Notice that $\mathbb{X}(n)$ is the set of those natural numbers not larger than n which factor into primes not smaller than p_s. Clearly, condition (1.6) for a quasi-prime is satisfied.

THEOREM 2. (i) *Conjecture* 2 *is true.*

(ii) *For all* $s,n\in\mathbb{N}$, *every optimal configuration is a star.*

(iii) *The optimal configuration is unique if and only if*

$$|M(p_s)\cap\mathbb{N}_s(n)|>|M(p_{s+1})\cap\mathbb{N}_s(n)|,$$

which is equivalent to the inequality

$$\left|\Phi\left(\frac{n}{p_s},p_s\right)\right|>\left|\Phi\left(\frac{n}{p_{s+1}},p_s\right)\right|.$$

Remark 5. We believe that also for $k=2,3$,

$$f(n,k,s)=|\mathbb{E}(n,k,s)| \quad \text{for all } n,s\in\mathbb{N}.$$

For $k = 4$ our counterexample in [1] applies. Moreover, we believe that every optimal configuration in the case $k = 2$ is the union of two stars. In the case $k = 3$ it is not always true, as shown by the following

EXAMPLE. Let $s \in \mathbb{N}$ be such that $p_s p_{s+7} > p_{s+5} p_{s+6}$ (take for instance the primes from the mentioned counterexample) and let $p_{s+5} p_{s+6} \le n < p_s p_{s+7}$. We verify that

$$|\mathbb{E}(n, 2, s)| = |M(p_s, p_{s+1}, p_{s+2}) \cap \mathbb{N}(n)| = 21.$$

On the other hand, the set $A = \{p_i p_j,\ s \le i < j \le s+6\}$ has no 4 coprime elements and is not a union of stars, but again $|A| = 21$.

Proof of Theorem 2. We prove (ii). Since $M(p_s) \cap \mathbb{N}_s(n)$ is not smaller than any competing star, this implies (i) and (iii). In the light of Theorem 1 and Lemma 4, it is sufficient to show that

$$2\pi(v) \le \pi(p_{s+1} v) \quad \text{for all } v \in \mathbb{R}^+. \tag{4.7}$$

Since for $v < p_s$, $\pi(v) = 0$, we can assume $v \ge p_s$. Now, (4.7) is equivalent to

$$2(\Pi(v) - s + 1) \le \Pi(p_{s+1} v) - s + 1, \tag{4.8}$$

where $\Pi(\cdot)$ is the usual counting function of primes. To show (4.8) it is sufficient to prove that for all $v \in \mathbb{R}^+$,

$$2\Pi(v) \le \Pi(3v). \tag{4.9}$$

It suffices to show (4.9) only for $v \in \mathbb{P}$.

We use the very sharp estimates on the distribution of primes due to Rosser and Schoenfeld [14]:

$$\frac{v}{\log v - 1/2} < \Pi(v) < \frac{v}{\log v - 3/2} \quad \text{for every } v \ge 67. \tag{4.10}$$

From (4.10) we get $2\Pi(v) < \Pi(3v)$ for all $v > 298$. The cases $v < 298$, $v \in \mathbb{P}$, are verified by inspection. We just mention that for $v \in \{3, 5, 7, 13, 19\}$ one has even the equality $2\Pi(v) = \Pi(3v)$.

5. Examples of sets of quasi-primes for which almost all optimal intersecting sets of quasi-numbers are not stars. Suppose we are given only a *finite number* of quasi-primes:

$$1 < r_1 < \ldots < r_m, \quad m \ge 3,$$

satisfying (1.6). The sets $\mathbb{X}, \mathbb{X}^*, \mathbb{X}(z), \mathcal{I}(z), \mathcal{O}(z)$ are defined as in Section 1. Here $\mathbb{X}^*$ has exactly 2^m elements. We are again interested in the quantity

$$f(z) = \max_{A \in \mathcal{I}(z)} |A|, \quad z \in \mathbb{X}.$$

For all $y \in \mathbb{X}^*$ with $y = r_1^{\alpha_1} \ldots r_m^{\alpha_m}$ and $\alpha_i \in \{0,1\}$, let $w(y) = \sum_{i=1}^m \alpha_i$ and let

$$T(y) = \{x \in \mathbb{X}, x = r_1^{\beta_1} \ldots r_m^{\beta_m} : \beta_i \geq 1 \text{ iff } \alpha_i = 1\}.$$

We distinguish two cases.

Case I: $m = 2m_1 + 1$. Define $\mathbb{X}_1^* = \{x \in \mathbb{X}^* : w(x) \geq m_1 + 1\}$.

Proposition 1. *Let $m = 2m_1 + 1$ be odd. There exists a constant $z_0 = z(r_1, \ldots, r_m)$ such that for all $z > z_0$, $|\mathcal{O}(z)| = 1$ and $A \in \mathcal{O}(z)$ has the form*

$$A = M(\mathbb{X}_1^*) \cap X(z) = \bigcup_{y \in \mathbb{X}_1^*} T(y) \cap \mathbb{X}(z).$$

Proof. Suppose $B \in \mathcal{O}(z)$. Since by optimality B is a "downset" and an "upset", we have

$$B = \bigcup_{y \in Y} T(y) \cap X(z) \quad \text{ for some } Y \subset \mathbb{X}^*.$$

It is clear that $|Y| \leq 2^{m-1}$, because by the intersecting property $y \in Y$ implies $\overline{y} = r_1 \ldots r_m / y \notin Y$.

Write $Y = Y_1 \dot{\cup} Y_2$, where

$$Y_1 = \{y \in Y : w(y) \leq m_1\} \quad \text{and} \quad Y_2 = \{y \in Y : w(y) \geq m_1 + 1\}.$$

Our aim is to prove that for large enough z one always has $Y_1 = \emptyset$, whence the proposition follows. Since $\mathbb{X}^*$ is finite, it is sufficient to show that for all $y \in \mathbb{X}^*$ with $w(y) \leq m_1$,

$$|T(y) \cap \mathbb{X}(z)| < |T(\overline{y}) \cap \mathbb{X}(z)| \quad \text{ if } z > z(y). \tag{5.1}$$

Let $y = r_1^{\alpha_1} \ldots r_m^{\alpha_m}$ with $\alpha_i \in \{0,1\}$, and let $\mathcal{I}(y) \subset \{1, 2, \ldots, m\}$, $|\mathcal{I}(y)| = w(y)$, be the positions with $\alpha_i = 1$. We introduce

$$c_i = \log r_i \quad \text{ for } i = 1, \ldots, m. \tag{5.2}$$

Then it is easy to see that $|T(y) \cap \mathbb{X}(z)|$ is the number of solutions of

$$\sum_{i \in \mathcal{I}(y)} c_i \gamma_i \leq \log z \quad \text{ in } \gamma_i \in \mathbb{N}$$

and $|T(\overline{y}) \cap \mathbb{X}(z)|$ is the number of solutions of

$$\sum_{i \in \mathcal{I}(\overline{y})} c_i \delta_i \leq \log z \quad \text{ in } \delta_i \in \mathbb{N}.$$

We verify that

$$|T(y) \cap \mathbb{X}(z)| \sim c_* (\log z)^{w(y)}, \quad \text{ where } c_* = \frac{1}{\prod_{i \in \mathcal{I}(y)} c_i \cdot (w(y))!}$$

and

$$|T(\overline{y}) \cap \mathbb{X}(z)| \sim c_{**}(\log z)^{m-w(y)}, \qquad \text{where } c_{**} = \frac{1}{\prod_{i\in\mathcal{I}(\overline{y})} c_i \cdot (m-w(y))!}.$$

Since $w(y) \le m_1$ and $m - w(y) \ge m_1 + 1$, there exists a $z(y)$ for which (5.1) is satisfied.

Case II: $m = 2m_1$. Let $\mathbb{X}_1^* = \{x \in \mathbb{X}^* : w(x) \ge m_1 + 1\}$ and $\mathbb{X}_0^* = \{x \in \mathbb{X}^* : w(x) = m_1\}$. For every $y \in \mathbb{X}_0^*$ let

$$g(y) = \prod_{i\in\mathcal{I}(y)} c_i \quad \text{with } c_i \text{ defined as in (5.2).}$$

Finally, define $\widetilde{\mathbb{X}}_0^* = \{y \in \mathbb{X}_1^* : g(y) \le g(\overline{y})\}$. If $g(y) = g(\overline{y})$ we take as an element of $\widetilde{\mathbb{X}}_0^*$ one of them, so $|\mathbb{X}_0^*| = \binom{2m_1}{m_1}/2$.

Using the same approach as in the proof of Proposition 1 we get

Proposition 2. *Let $m = 2m_1$ be even. There exists a constant $z_0 = z(r_1,\ldots,r_m)$ such that for all $z > z_0$ an optimal set $A \in \mathcal{O}(z)$ is*

$$A = M(\mathbb{X}_1^* \cup \widetilde{\mathbb{X}}_0^*) \cap \mathbb{X}(z) = \bigcup_{x\in\mathbb{X}_1^*\cup\widetilde{\mathbb{X}}_0^*} T(y) \cap \mathbb{X}(z)$$

and, if $g(y) \ne g(\overline{y})$ for all $y \in X_0^*$, then the optimal set is unique.

From these propositions it follows that for *finite* sets Q of quasi-primes, for all sufficiently large z, the optimal intersecting sets are *not stars*.

By choosing Q's consisting of infinitely many quasi-primes which are sufficiently far from each other, say $r_{i+1} > \exp(r_i)$ (details are omitted), one can make sure that again for all sufficiently large z, the optimal intersecting sets are *never stars*.

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References

[1] R. Ahlswede and L. H. Khachatrian, *On extremal sets without coprimes*, Acta Arith. 66 (1994), 89–99.

[2] —, —, *Maximal sets of numbers not containing $k+1$ pairwise coprime integers*, ibid. 72 (1995), 77–100.

[3] N. G. de Bruijn, *On the number of uncancelled elements in the sieve of Eratosthenes*, Indag. Math. 12 (1950), 247–256.

[4] P. Erdős, *Remarks in number theory, IV*, Mat. Lapok 13 (1962), 228–255.

[5] —, *Extremal problems in number theory*, in: Theory of Numbers, Proc. Sympos. Pure Math. 8, Amer. Math. Soc., Providence, R.I., 1965, 181–189.

[6] —, *Problems and results on combinatorial number theory*, Chapt. 12 in: A Survey of Combinatorial Theory, J. N. Srivastava *et al.* (eds.), North-Holland, 1973.

[7] P. Erdős, *A survey of problems in combinatorial number theory*, Ann. Discrete Math. 6 (1980), 89–115.

[8] P. Erdős and A. Sárközy, *On sets of coprime integers in intervals*, Hardy–Ramanujan J. 16 (1993), 1–20.

[9] P. Erdős, A. Sárközy and E. Szemerédi, *On some extremal properties of sequences of integers*, Ann. Univ. Sci. Budapest. Eötvös 12 (1969), 131–135.

[10] —, —, —, *On some extremal properties of sequences of intergers*, *II*, Publ. Math. Debrecen 27 (1980), 117–125.

[11] R. Freud, *Paul Erdős*, *80—A personal account*, Period. Math. Hungar. 26 (1993), 87–93.

[12] H. Halberstam and K. F. Roth, *Sequences*, Oxford University Press, 1966; Springer, 1983.

[13] R. R. Hall and G. Tenenbaum, *Divisors*, Cambridge Tracts in Math. 90, 1988.

[14] J. B. Rosser and L. Schoenfeld, *Approximate formulas for some functions of prime numbers*, Illinois J. Math. 6 (1962), 64–89.

[15] C. Szabó and G. Tóth, *Maximal sequences not containing* 4 *pairwise coprime integers*, Mat. Lapok 32 (1985), 253–257 (in Hungarian).

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