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Partitions with numbers in their gaps

by

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1. Partitions with gaps. Bijections between various restricted partitions of integers have been extensively studied (see [2], [3]). In this paper we introduce a generalization of partitions, which are really a kind of restricted composition [3], and obtain bijections between certain classes of them and classes of ordinary partitions. Our generalization arises naturally in connection with solutions of q-difference equations and continued fractions. In fact, continued fractions provide an easy framework for analyzing these generalized partitions.

Throughout this paper we write an ordinary partition as

$$n_1 + \ldots + n_k, \quad n_1 \ge \ldots \ge n_k.$$

Define $n_0 = \infty$ and $n_{k+1} = 0$.

We consider the class $C(S, m)$ of partitions with parts n_i taken from a set $S = \{a_1, a_2, \ldots\}$, where $a_1 < a_2 < \ldots$ We require that if $n_i = a_r$ and $n_{i+1} = a_s$ are two consecutive parts, then $r - s \ge m$. For such a partition we define the *gapspace* g_i between n_i and n_{i+1} to be $r - s - m$. By convention we put $g_0 = \infty$ and $g_k = t - m$, where $n_k = a_t$.

Before proceeding we give some examples of gapspaces.

Examples. 1. Consider partitions into parts which differ by at least 3; thus $S = \mathbb{Z}^+$ and $m = 3$. One such partition is $15 + 10 + 2$.

The gapspace between 10 and 2 is $(10 - 2) - 3 = 5$ and the gapspace between 15 and 10 is $(15 - 10) - 3 = 2$.

2. Consider partitions into non-consecutive odd integers; thus $a_r = 2r - 1$ and $m = 2$. One partition of this type is:

$$63 + 51 + 17 + 13 + 7 + 1.$$

The gapspaces between the parts are:

$$\infty,\ 4,\ 15,\ 0,\ 1,\ 1,\ -1.$$

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We now introduce the class of generalized partitions to be studied in this paper. We start with partitions from the class $C(S,m)$. Note that such a partition can be written uniquely in the form

$$a_{n_1} + a_{n_2} + \ldots + a_{n_k}, \qquad n_1 >_m n_2 >_m \ldots >_m n_k,$$

where $x >_m y$ means that $x \geq y + m$. We now fix a positive integer j and insert up to g_l parts j in the gap between a_{n_l} and $a_{n_{l+1}}$. The resulting compositions are the objects of our study. Recall that two compositions are counted as distinct if their sequences of parts are distinct. Define $C(S,m,j)$ to be the class of compositions thus obtained. We refer to these compositions in picturesque language as "partitions with numbers in their gaps". We give some examples of partitions with ones in the gaps.

Examples. 1. We first consider $C(\mathbb{Z}^+,2,1)$, i.e. partitions into natural numbers with minimal difference two with 1's in the gaps. We start with a partition into parts with minimal difference 2:

$$21 + 19 + 12 + 7 + 1.$$

Then up to g_l ones are inserted into the lth gap; thus

$$21 + 19 + 1 + 1 + 1 + 12 + 7 + 1$$

and

$$21 + 19 + 12 + 1 + 1 + 1 + 7 + 1$$

and

$$21 + 19 + 12 + 1 + 1 + 7 + 1 + 1$$

are counted as distinct partitions of 63. Another partition satisfying the conditions for this class of partitions is:

$$\underbrace{1 + \ldots + 1}_{i \text{ ones}} + 21 + 19 + 12 + 7 + 1.$$

Recall that $g_0 = \infty$, thus we may insert i ones for any non-negative integer i.

It is easily found that the partitions of 3 of this type are:

$$3$$
$$1 + 2$$
$$1 + 1 + 1.$$

The partitions of 4 of this type are:

$$4$$
$$3 + 1$$
$$1 + 3$$
$$1 + 1 + 2$$
$$1 + 1 + 1 + 1,$$

and the partitions of 5 are:

$$\begin{gathered} 5 \\ 4+1 \\ 1+4 \\ 1+3+1 \\ 1+1+3 \\ 1+1+1+2 \\ 1+1+1+1+1. \end{gathered}$$

Calling the number of partitions of n of this type $P^*(n)$, it follows that $P^*(n) = 3, 5$, and 7, for $n = 3, 4$, 5.

2. Consider partitions into distinct non-consecutive odd numbers excluding 3 with ones in the gaps. Thus $m = 2$ and

$$a_r = \begin{cases} 1 & \text{for } r = 1, \\ 2r+1 & \text{for } r > 1. \end{cases}$$

To obtain the partitions of 11 in this class we first begin by forming partitions of numbers less than or equal to 11 into distinct non-consecutive odd numbers excluding 3. These partitions are:

$$\begin{gathered} 11 \\ 9+1 \\ 9 \\ 7+1 \\ 7 \\ 5+1 \\ 5 \\ 1. \end{gathered}$$

To these partitions we now insert up to g_l ones into the lth gap so that the sum of the resulting partition is 11. This process yields:

$$\begin{gathered} 11 \\ 9+1+1 \\ 1+9+1 \\ 1+1+9 \\ 1+1+1+7+1 \\ 1+1+1+1+7 \\ 1+1+1+1+1+1+5 \\ 1+1+1+1+1+1+1+1+1+1+1. \end{gathered}$$

Some of these partitions arise non-uniquely from our initial list of partitions. For example the partition $9+1+1$ arises from both the partition $9+1$ and the partition 9, in the former case by inserting one 1 between the 9 and the 1, while in the latter case by inserting two 1's in the gap after the 9 ($g_1 = 2$). Thus there are eight partitions of 11 in this example. The non-uniqueness problem could have been avoided by first listing just those partitions into parts greater than 1 and then inserting 1's according to the rule. The definition of gapspace assures that this would give the complete list of partitions of 11 we are considering.

We are now in a position to state some theorems. It may have been noticed in the first example that P^* coincided with the ordinary partition function for the values considered. This is a consequence of the first theorem.

THEOREM 1. *The number of partitions of* n *into parts with minimal difference two with ones in the gaps is equal to* $p(n)$.

A similar theorem giving a bijection with another well studied class of partitions is:

THEOREM 2. *The number of partitions of* n *into distinct non-consecutive odd parts* ($a_i = 2i-1$, $m = 2$) *with ones in the gaps is equal to the number of partitions of* n *into distinct parts.*

Corresponding to our second example we have

THEOREM 3. *The number of partitions of* n *into distinct non-consecutive odd parts excluding* 3 *with ones in the gaps is equal to the number of partitions of* $n+1$ *into an odd number of distinct parts.*

In the theorems so far we had $m = 2$. The following two theorems are for $m = 1$.

THEOREM 4. *The number of partitions of* n *into distinct even parts with ones in the gaps is equal to the number of partitions of* n *into parts satisfying* $n_1 \geq n_2 > n_3 \geq n_4 > \ldots$

THEOREM 5. *The number of partitions of* n *into distinct odd parts with ones in the gaps is equal to the number of partitions of* n *into parts satisfying* $n_1 > n_2 \geq n_3 > n_4 \geq \ldots$

It is well known that the partitions in these last two theorems are equinumerous with partitions into parts $\equiv 1,2,5,6,8,9,11,12,14,15,18,19 \pmod{20}$ and $\equiv 1,3,4,5,7,9,11,13,15,16,17,19 \pmod{20}$ respectively. See [6] and [7].

2. Proof of the theorems. In this section we give proofs of the last five theorems. In fact, they all are combinatorial interpretations of specializations of the $m = 2$ case of the canonical q-difference equation for the general ${}_m\Phi_{m-1}$. The following proposition is also used; it leads to partitions with parts in the gaps. It is interesting to note that partitions with numbers in their gaps give combinatorial interpretations of the partial quotients of two classes of continued fractions.

PROPOSITION 1. *Let P_n and P'_n denote the n-th numerators of the convergents of the continued fractions*

$$\frac{P_n}{Q_n} = 1 + q^j + \frac{q^{a_2} - q^j}{1 + q^j +}\,\frac{q^{a_3} - q^j}{1 + q^j + \dots}\,\frac{q^{a_{n+1}} - q^j}{1 + q^j}$$

and

$$\frac{P'_n}{Q'_n} = 1 + q^j + q^{a_1} + \frac{-q^j}{1 + q^j + q^{a_2} + \dots}\,\frac{-q^j}{1 + q^j + q^{a_{n+1}}}.$$

Let $S = \{a_2, a_3, \dots, a_{n+1}\}$ and $S' = S \cup \{a_1\}$. Then

1. *P_n is the generating function for partitions from the set $C(S, 2, j)$, where the gapspace before the largest part a_k is $n - k + 1$.*
2. *P'_n is the generating function for partitions from the set $C(S', 1, j)$, where the gapspace before the largest part a_k is $n - k + 1$.*

Proof. The standard recurrences for P_n and P'_n are

$$P_n = P_{n-1} + q^j(P_{n-1} - P_{n-2}) + q^{a_{n+1}} P_{n-2} \tag{1}$$

and

$$P'_n = P'_{n-1} + q^{a_{n+1}} P'_{n-1} + q^j(P'_{n-1} - P'_{n-2}). \tag{2}$$

New partitions enumerated by P_n are built up in the exponents of q by adjoining j or a_{n+1} to the left of certain of those previously enumerated. The proof proceeds by induction. Let A_n and A'_n be the generating functions for partitions of the types $C(S, 2, j)$ and $C(S', 1, j)$ respectively. It is easy to see that $P_n = A_n$ and $P'_n = A'_n$ for $n = 0, 1$. Assume $P_i = A_i$ and $P'_i = A'_i$ for $0 \le i < n$. Now partitions enumerated by A_n and A'_n fall into two classes:

1. those not containing the part a_{n+1}, and
2. those containing the part a_{n+1}.

This leads to two cases:

Case 1. Consider any partition π enumerated by A_n not containing the part a_{n+1}. Let a_k be the part in π with the largest subscript; so $k \le n$. Now the gapspace before a_k is $n-k+1$ by the definition of A_n. Suppose this gap is not full. Then by the induction hypothesis, π is enumerated by P_{n-1}. On the other hand, if the gap is full, then the number of j's preceding a_k is exactly $n - k + 1$, and π is not enumerated by the term P_{n-1} in (1), again by induction and the definition of A_n. Then removing one of the j's leaves a partition π' with $n - k$ j's preceding a_k. By induction π' is enumerated by P_{n-1}. Moreover, it has a full gap in $A_{n-1} = P_{n-1}$ and so as above is not enumerated by P_{n-2}. Hence π is enumerated by the term $q^j(P_{n-1} - P_{n-2})$ in (1). This case applies *mutatis mutandis* to the corresponding case for A'_n and P'_n.

Case 2a. Consider any partition π enumerated by A_n containing the part a_{n+1}. Since $m = 2$, the part with the next largest subscript is a_k with $k < n$. Then the gapspace between a_{n+1} and a_k is $n-k-1$. Hence removing a_{n+1} from π leaves a partition π' beginning with at most $n-k-1$ j's followed by a_k, $k < n$. By induction π' is enumerated by P_{n-2}. Hence π is enumerated by the term $q^{a_{n+1}} P_{n-2}$.

Case 2b. Consider any partition π enumerated by A'_n containing the part a_{n+1}. Since $m = 1$, the part with the next largest subscript is a_k with $k \le n$. Then the gapspace between a_{n+1} and a_k is $n-k$. Hence removing a_{n+1} from π leaves a partition π' beginning with at most $n-k$ j's followed by a_k, $k \le n$. By induction π' is enumerated by P'_{n-1}. Hence π is enumerated by the term $q^{a_{n+1}} P'_{n-1}$. This completes the proof.

Lemma 1. *Suppose*

$$c_m y_m = b_m y_{m+1} + a_m y_{m+2} \quad \textit{for } m \ge 1. \tag{3}$$

Let

$$\frac{P_m}{Q_m} = b_1 + \frac{a_1 c_2}{b_2 +}\,\frac{a_2 c_3}{b_3 + \dots}\,\frac{a_m c_{m+2}}{b_{m+2}},$$

so that

$$P_k = b_{k+1} P_{k-1} + a_k c_{k+1} P_{k-2} \quad \textit{for } k \ge 1. \tag{4}$$

Then

$$c_1 c_2 \dots c_n y_1 = P_{n-1} y_{n+1} + a_n P_{n-2} y_{n+2} \quad \textit{for } n \ge 1. \tag{5}$$

Proof. When $n = 1$ the lemma asserts that $c_1 y_1 = P_0 y_2 + a_1 P_{-1} y_3 = b_1 y_2 + a_1 y_3$, which is (3) with $m = 1$. Assume the result true for $n = k$. Then the result for $n = k+1$ follows easily from the induction hypothesis, (3) and (4).

The $c_m = 1$ case of this lemma is given in [8].

Proof of Theorems 1–5. As noted in [4] the sequences (see [5] for notation)

$$\begin{gathered} y_m = {}_2\Phi_1\left(\begin{matrix} -aq, -bq \\ cq^2 \end{matrix}; xq^{m-1}\right), \qquad a_m = abxq^{m+1} - cq, \\ b_m = 1 + cq + (a+b)xq^m, \qquad c_m = 1 - xq^{m-1} \end{gathered} \tag{6}$$

and the continued fraction

$$(7)\quad 1+cq+(a+b)xq+\frac{(abxq^2-cq)(1-xq)}{1+cq+(a+b)xq^2+}\underset{\cdots}{}\frac{(abxq^{n+1}-cq)(1-xq^n)}{1+cq+(a+b)xq^{n+1}}$$

satisfy the conditions of Lemma 1. Putting $a=b=0$ and $c=1$ in (6) gives

$$y_m=\sum_{n\geq 0}\frac{q^{(m-1)n}x^n}{(q^2)_n(q)_n},$$

$$a_m=-q,\quad b_m=1+q,\quad c_m=1-xq^{m-1}.$$

The continued fraction becomes

$$1+q+\frac{xq^2-q}{1+q+}\,\frac{xq^3-q}{1+q+}\underset{\cdots}{}\frac{xq^{n+1}-q}{1+q}.$$

Let P_n be the numerator convergent of this continued fraction. Equation (5) becomes

$$(8)\qquad (xq)_{n-1}(1-x)y_1=P_{n-1}y_{n+1}-qP_{n-2}y_{n+2}.$$

Formally $\lim_{n\to\infty}y_n=1$. It is clear from the proof of Proposition 1 that as $n\to\infty$, P_n formally tends to a limit which we call P. (The coefficient of x^mq^n is the number of partitions of n from $C(\mathbb{Z}^+,2,1)$ containing m parts greater than or equal to 2.) Thus letting $n\to\infty$ in (8) gives

$$(9)\qquad (xq)_\infty(1-x)y_1=(1-q)P.$$

Notice that

$$\lim_{x\to 1}c_1y_1=\lim_{x\to 1}(1-x)\sum_{n\geq 0}\frac{x^n}{(q^2)_n(q)_n}=\frac{1}{(q^2)_\infty(q)_\infty}.$$

Letting $x\to 1$ in (9) gives

$$(q)_\infty\frac{1}{(q^2)_\infty(q)_\infty}=(1-q)P,$$

where now P is the generating function for $C(\mathbb{Z}^+,2,1)$. Thus $P=1/(q)_\infty$. It is well known that $1/(q)_\infty$ is the generating function for ordinary partitions and so Theorem 1 is proved.

Theorem 2 is similar. Here $a=c=1$ and $b=-1$. Equations (6) become

$$y_m=\sum_{n\geq 0}\frac{(-q)_n}{(q)_n}q^{(m-1)n}x^n,$$

$$a_m=-xq^{m+1}-q,\quad b_m=1+q,\quad c_m=1-xq^{m-1}.$$

The continued fraction becomes

$$1+q+\frac{x^2q^3-q}{1+q+}\,\frac{x^2q^5-q}{1+q+}\underset{\cdots}{}\frac{x^2q^{2n+1}-q}{1+q}.$$

Equation (5) becomes

$$(1+xq^{m-1})y_m = (1+q)y_{m+1} - (q+xq^{m+1})y_{m+2}.$$

Similarly,

$$\lim_{x\to 1} c_1 y_1 = \frac{(-q)_\infty}{(q^2)_\infty}.$$

The same reasoning as before shows that the limiting numerator convergent of the continued fraction

$$1+q+\frac{q^3-q}{1+q+}\,\frac{q^5-q}{1+q+}\,\ldots$$

is $(-q)_\infty$. Proposition 1 and the well known fact that this product is the generating function for partitions into distinct parts gives Theorem 2.

To get Theorem 3, notice that by Theorem 3 of [4] the fact that the limiting numerator of the last continued fraction is $(-q)_\infty$ implies that the limiting denominator is

$$\begin{aligned}\frac{(q)_\infty}{1-q}\,{}_2\Phi_1\left(\begin{matrix}-q,\, q\\ q^2\end{matrix};q\right) &= \frac{(q)_\infty}{1-q}\sum_{n\ge 0}\frac{(-q)_n}{(q^2)_n}q^n = \frac{(q)_\infty}{2q}\sum_{n\ge 0}\frac{(-1)_{n+1}}{(q^2)_{n+1}}q^{n+1}\\ &= \frac{(q)_\infty}{2q}\left[\frac{(-q)_\infty}{(q)_\infty}-1\right] = \frac{1}{2q}[(-q)_\infty-(q)_\infty].\end{aligned}$$

On the other hand, the limiting denominator is equal to the limiting numerator of

$$1+q+\frac{q^5-q}{1+q+}\,\frac{q^7-q}{1+q+}\,\ldots.$$

By Proposition 1, Theorem 3 is proved.

To get Theorems 4 and 5, put $q\to q^k$, $b\to q^{l-k}/x$, $c\to q^{j-k}$ and let $a,x\to 0$. Then equations (6) become

$$y_m = \sum_{n\ge 0}\frac{q^{k\binom{n}{2}+ln}q^{k(m-1)n}}{(q^{k+j};q^k)_n(q^k;q^k)_n},$$

$$a_m = -q^j,\quad b_m = 1+q^j+q^{k(m-1)+l},\quad c_m = 1.$$

The continued fraction is then

$$1+q^j+q^l+\frac{-q^j}{1+q^j+q^{k+l}+}\,\frac{-q^j}{1+q^j+q^{2k+l}+}\,\ldots.$$

Equation (5) becomes

$$y_1 = P_{n-1}y_{n+1} - q^j P_{n-2}y_{n+2}. \tag{10}$$

By Proposition 1, as $n \to \infty$, $P_n \to P$, where P is the generating function for partitions from $C(S, 1, j)$ and $S = \{kn + l : n \geq 0\}$. So as $n \to \infty$, we get $y_1 = (1 - q^j)P$ from (10), which implies

$$P = \sum_{n \geq 0} \frac{q^{k\binom{n}{2}+ln}}{(q^j; q^k)_{n+1}(q^k; q^k)_n}.$$

When $k = 2$, $j = 1$ and $l = 2, 1$ respectively, the sum on the right generates partitions of the type required in Theorems 4 and 5. This is not too hard to see directly, and is proved in [7].

In a future paper we shall give extensions of these theorems which also include not only the celebrated partition theorem of I. Schur (see [2], [9]) but also the generalization of it due to Alladi and Gordon [1].

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