Bounds for the solutions of unit equations

by

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1. Introduction. Many diophantine problems can be reduced to (ordinary) unit equations and $S$-unit equations in two unknowns (for references, see e.g. [15], [24], [11], [16], [25]). Several effective bounds have been established for the heights of the solutions of such equations (see e.g. [24], [11], [25], [3] and the references given there). Except in [3], their proofs involved Baker’s method and its $p$-adic analogue as well as certain quantitative results concerning independent units. The best known estimates for $S$-unit equations are due to Győry [13] and, for (ordinary) unit equations, to Schmidt [23], Sprindžuk [25] (with not completely explicit constants) and Győry [14] (with explicit constants). These led to a lot of applications.

The purpose of the present paper is to considerably improve (in completely explicit form) the above-mentioned estimates in terms of the cardinality of $S$ and of the parameters involved (degree, unit rank, regulator, class number) of the ground field. To obtain these improvements we use, among other things, some recent improvements of Waldschmidt [26] and Kunrui Yu [27] concerning linear forms in logarithms, some recent estimates of Brindza [5] and Hajdu [18] for fundamental systems of $S$-units, some upper and lower bounds for $S$-regulators (cf. Lemma 3 of this paper) and an idea of Schmidt [23]. Further, in our arguments we pay a particular attention to the dependence on the parameters in question. As a consequence of our result, we derive explicit bounds for the solutions of homogeneous linear equations of three terms in $S$-integers of bounded $S$-norm. These improve some earlier estimates of Győry [13], [14].

An application of our improvements is given in [17] to decomposable form equations (including Thue equations, norm form equations and discriminant form equations) in $S$-integers of a number field. Some other applications will be published in two further works.

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2. Bounds for the solutions of $S$-unit equations. We shall use throughout this paper the following standard notation. Let $\mathbb{K}$ be an algebraic number field of degree $d$ with regulator $R_{\mathbb{K}}$, class number $h_{\mathbb{K}}$ and unit rank $r$. Denote by $\mathcal{O}_{\mathbb{K}}$ the ring of integers of $\mathbb{K}$, and by $\mathcal{O}_{\mathbb{K}}^*$ the unit group of $\mathcal{O}_{\mathbb{K}}$. Let $S$ be a finite set of places on $\mathbb{K}$ containing the set of infinite places $S_\infty$. Denote by $s$ the cardinality of $S$, by $t$ the number of finite places in $S$, and by $P$ the largest of the rational primes lying below the finite places of $S$, with the convention that $P = 1$ if $S = S_\infty$, i.e. if $t = 0$. Further, denote by $\mathcal{O}_S$ the ring of $S$-integers, and by $\mathcal{O}_S^*$ the group of $S$-units in $\mathbb{K}$. Then $s - 1 = r + t$ is the rank of $\mathcal{O}_S^*$. The case $s = 1$ being trivial, we assume throughout the paper that $s \geq 2$. We denote by $R_S$ the $S$-regulator of $\mathbb{K}$ (for its definition, see Section 3). We note that for $S = S_\infty$ (i.e. $t = 0$), we have $\mathcal{O}_S = \mathcal{O}_{\mathbb{K}}$ and $R_S = R_{\mathbb{K}}$.

For any algebraic number $\alpha$, we denote by $h(\alpha)$ the (absolute) height of $\alpha$ (cf. Section 3). There exists a $\delta_{\mathbb{K}} > 0$, depending only on $\mathbb{K}$, such that $d \log h(\alpha) \geq \delta_{\mathbb{K}}$ for any $\alpha \in \mathbb{K} \setminus \{0\}$ which is not a root of unity (cf. Section 3).

Throughout this paper, we use the notation $\log^* a$ for $\max\{\log a, 1\}$.

Let $\alpha, \beta$ be non-zero elements of $\mathbb{K}$ with

$$\max\{h(\alpha), h(\beta)\} \leq H \quad (H \geq e).$$

Consider the $S$-unit equation

$$ax + \beta y = 1 \quad \text{in} \quad x, y \in \mathcal{O}_S^*.$$  

When $S = S_\infty$ (i.e. $t = 0$) then (1) is an (ordinary) unit equation.

**Theorem.** All solutions $x, y$ of (1) satisfy

$$\max\{h(x), h(y)\} < \exp\{c_1 P^d R_S (\log^* R_S) (\log^* (PR_S) / \log^* P) \log H\},$$

where

$$c_1 = c_1(d, s, \mathbb{K}) = 3^{25} (9d^2 / \delta_{\mathbb{K}})^{s+1} s^{5s+10}.$$

Further, if in particular $S = S_\infty$ (i.e. $t = 0$), then the bound in (2) can be replaced by

$$\exp\{c_2 R_{\mathbb{K}} (\log^* R_{\mathbb{K}}) \log H\}$$

where

$$c_2 = c_2(d, r, \mathbb{K}) = 3^{r+27} (r+1)^{5r+17} d^3 \delta_{\mathbb{K}}^{-(r+1)}.$$

**Remark 1.** It is clear that the factor $(\log^* (PR_S) / \log^* P)$ in (2) does not exceed $2 \log^* R_S$, and if $\log^* R_S \leq \log^* P$, then it is at most 2. Further, by Lemma 3 (cf. Section 3), we have

$$R_S \leq R_{\mathbb{K}} h_{\mathbb{K}} (d \log^* P)^t.$$
Remark 2. As is known, \( R_d h_K \) can be estimated from above in terms of \( d \) and \( D_K \), the discriminant of \( K \). Denote by \( q \) the number of complex places of \( K \), and put \( \Delta = (2/\pi)^q |D_K|^{1/2} \). If \( d \geq 2 \), then we have e.g. (cf. [21])

\[
R_d h_K \leq \Delta (\log \Delta)^{d-1-q}(d-1 + \log \Delta)^q/(d-1)!
\]

(5)

Our theorem provides a considerable improvement of earlier estimates of Kotov and Trelina [19], Győry [13], [14], Schmidt [23] and Sprindžuk [25] for \( S \)-unit equations.

For \( \alpha \in K \setminus \{0\} \), the ideal generated by \( \alpha \) can be uniquely written in the form \( \mathfrak{a}_1 \cdot \mathfrak{a}_2 \) where the ideal \( \mathfrak{a}_1 \) (resp. \( \mathfrak{a}_2 \)) is composed of prime ideals outside (resp. inside) \( S \). Then the \( S \)-norm of \( \alpha \), denoted by \( N_S(\alpha) \), is defined as \( N(\mathfrak{a}_1) \). In the particular case \( S = S_\infty \), we have \( N_{S_\infty}(\alpha) = |N_{K/Q}(\alpha)| \).

Further, \( N_S(\alpha) \) is a positive integer for every \( \alpha \in O_S \setminus \{0\} \).

In some applications, it is more convenient to consider the following equation instead of (1):

\[
\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0
\]

in \( x_i \in O_S \setminus \{0\} \) with \( N_S(x_i) \leq N \) for \( i = 1, 2, 3 \),

where \( \alpha_1, \alpha_2, \alpha_3 \in K \setminus \{0\} \) with \( \max_{1 \leq i \leq 3} h(\alpha_i) \leq H \) (\( H \geq e \)).

Let \( c_3 = c_3(d, r, K) = r^{r+1} \delta_K^{(r-1)/2} \) and let \( c_1 = c_1(d, s, K) \), \( c_2 = c_2(d, r, K) \) denote the numbers specified in the Theorem. Then we have

**Corollary.** For every solution \( x_1, x_2, x_3 \) of (6) there is an \( \varepsilon \in O^*_S \) such that

\[
\max_{1 \leq i \leq 3} h(\varepsilon x_i) < \exp \{3c_1 c_3 P^d R_S(\log^* R_S)/(\log^* (PR_S)/\log^* P) \}
\]

\[
\times (R_K + th_K \log^* P + \log(HN)) \).
\]

Further, if \( S = S_\infty \), then the bound in (7) can be replaced by

\[
\exp \{3c_2 c_3 R_K(\log^* R_K)(R_K + \log(HN))\}.
\]

Our Corollary considerably improves the earlier bounds of Győry [13], [14] concerning equation (6).

3. **Bounds for \( \mathcal{S} \)-units and \( \mathcal{S} \)-regulators.** Keeping the notations of Section 2, denote by \( M_K \) the set of places on \( K \). In every place \( v \) we choose a valuation \( |\cdot|_v \) in the following way: if \( v \) is infinite and corresponds to an embedding \( \sigma : K \rightarrow \mathbb{C} \) then we put, for every \( \alpha \in K \),

\[
|\alpha|_v = |\sigma(\alpha)|^{d_v},
\]

where \( d_v = 1 \) or \( 2 \) according as \( \sigma(K) \) is contained in \( \mathbb{R} \) or not; if \( v \) is a finite place corresponding to the prime ideal \( \mathfrak{p} \) in \( K \) then we put \( |0|_v = 0 \) and, for
\[ \alpha \in K \setminus \{0\}, \]
\[ |\alpha|_v = N(p)^{-\text{ord}_p(\alpha)}. \]

The (absolute) \textit{height} of an algebraic number \( \alpha \) contained in \( K \) is defined by
\[ h(\alpha) = \left( \prod_{v \in M_K} \max(1, |\alpha|_v) \right)^{1/d}. \]

This height is independent of the choice of \( K \). If the algebraic number \( \alpha \) is of degree \( n \) with minimal polynomial \( a_0(X - \alpha_1) \ldots (X - \alpha_n) \in \mathbb{Z}[X] \) over \( \mathbb{Z} \), then, by ([20], p. 54), we have
\[ h(\alpha) = \left( |a_0| \prod_{i=1}^{n} \max(1, |\alpha_i|) \right)^{1/n}. \]

(8)

There is a positive constant \( \delta_K \), depending only on \( K \), such that for every non-zero algebraic number \( \alpha \in K \) which is not a root of unity we have \( \log h(\alpha) \geq \delta_K/d \) (we recall that \( d \) denotes the degree of \( K \)). Further, if \( \alpha \) is not an algebraic integer then (8) implies that \( \log h(\alpha) \geq \log 2/d \). Hence we have \( \delta_K \leq \log 2 \).

It is easy to see that we can take
\[ \delta_K = \frac{\log 2}{r + 1} \quad \text{for } d = 1, 2, \]
where \( r \) denotes the unit rank of \( K \). Further, it follows from results of Blanksby and Montgomery [2] and of Dobrowolski [7], [8] that both
\[ \delta_K = \frac{1}{53d \log 6d} \quad \text{and} \quad \delta_K = \frac{1}{1201} \left( \frac{\log \log d}{\log d} \right)^3 \]
are appropriate choices for \( d \geq 3 \). For large \( d \), the factor 1/1201 can be replaced by a larger one (see e.g. [9]).

We recall that \( s \) denotes the cardinality of \( S \). For \( v \in S \), denote by \( |\cdot|_v \) the corresponding valuation normalized as above. Let \( v_1, \ldots, v_{s-1} \) be a subset of \( S \), and let \( \{ \varepsilon_1, \ldots, \varepsilon_{s-1} \} \) be a fundamental system of \( S \)-units in \( K \). Denote by \( R_S \) the absolute value of the determinant of the matrix \( (\log |\varepsilon_i|_{v_j})_{i,j=1,\ldots,s-1} \). It is easy to verify that \( R_S \) is a positive number which is independent of the choice of \( v_1, \ldots, v_{s-1} \) and of the fundamental system of \( S \)-units \( \{ \varepsilon_1, \ldots, \varepsilon_{s-1} \} \). \( R_S \) is called the \textit{S-regulator} of \( K \). If in particular \( S = S_\infty \), then we have \( R_S = R_K \).

There are several quantitative results in the literature for units and \( S \)-units of small height; for references, see e.g. [24], [5] and [18]. The following lemma is in fact due to Hajdu [18]. It is an extended version of an earlier

\[ (1) \text{ Added in proof. By a recent result of P. M. Voutier (see this issue), one can take here } 1/4 \text{ instead of } 1/1201. \]
theorem of Brindza [5]. For convenience of the reader, we give here a proof
for Lemma 1 with a slightly better value for \( c_4 \) than in [18].

Put
\[
c_4 = c_4(d, s) = ((s - 1)!)^2 / (2^{s-2}d^{s-1})
\]
and
\[
c_5 = c_5(d, s, \mathbb{K}) = c_4 \left( \frac{\delta_{\mathbb{K}}}{d} \right)^{2-s}, \quad c_6 = c_6(d, s, \mathbb{K}) = c_4 d^{s-1} \delta_{\mathbb{K}}^{-1}.
\]

**Lemma 1.** There exists in \( \mathbb{K} \) a fundamental system \( \{ \varepsilon_1, \ldots, \varepsilon_{s-1} \} \) of \( S \)-units with the following properties:

(i) \[
\prod_{i=1}^{s-1} \log h(\varepsilon_i) \leq c_4 R_S;
\]

(ii) \[
\log h(\varepsilon_i) \leq c_5 R_S, \quad i = 1, \ldots, s - 1;
\]

(iii) the absolute values of the entries of the inverse matrix of \( (\log |\varepsilon_i|_{v_j})_{i,j=1,\ldots,s-1} \) do not exceed \( c_6 \).

**Proof.** We shall combine some arguments from the proofs of [5] and [18]. For \( \alpha \in \mathbb{K} \setminus \{0\} \) put
\[
v(\alpha) = (\log |\alpha|_{v_1}, \ldots, \log |\alpha|_{v_{s-1}}).
\]
The lattice \( A \) in \( \mathbb{R}^{s-1} \) spanned by the vectors \( v(\eta) \) with \( \eta \in O_S^* \) has determinant \( R_S \).

The function \( F : \mathbb{R}^{s-1} \to \mathbb{R} \) defined by
\[
F(x) = |x_1| + \ldots + |x_{s-1}|
\]
for \( x = (x_1, \ldots, x_{s-1}) \in \mathbb{R}^{s-1} \) is a symmetric convex distance function (cf. [6], Ch. IV), i.e. it is non-negative, continuous, \( F(\alpha x) = \alpha F(x) \) (\( \alpha \geq 0 \) real) and \( F(x + y) \leq F(x) + F(y) \) for \( x, y \in \mathbb{R}^{s-1} \). Denote by \( V_F \) the volume of the bounded set \( \{ x \in \mathbb{R}^{s-1} \mid F(x) < 1 \} \). It is easy to check that \( V_F = 2^{s-1} / (s - 1)! \). By a theorem of Minkowski (cf. [6], Ch. VIII) the successive minima \( \lambda_1, \ldots, \lambda_{s-1} \) of \( A \) with respect to \( F \) have the property
\[
\lambda_1 \ldots \lambda_{s-1} \leq 2^{s-1} R_S / V_F = (s - 1)! R_S.
\]
Further, there are multiplicatively independent \( S \)-units \( \eta_1, \ldots, \eta_{s-1} \) for which
\[
F(v(\eta_i)) = \lambda_i, \quad i = 1, \ldots, s - 1.
\]
It follows (cf. [6], p. 135, Lemma 8) that there exists a fundamental system \( \{ \varepsilon_1, \ldots, \varepsilon_{s-1} \} \) of \( S \)-units such that
\[
F(v(\varepsilon_i)) \leq \max\{1, i/2\} F(v(\eta_i)), \quad i = 1, \ldots, s - 1.
\]
However, for every $\eta \in O_S$, we have $\prod_{v \in S} |\eta|_v = 1$, hence
\[
\log h(\eta) = \frac{1}{d} \sum_{v \in S} \max\{0, \log |\eta|_v\} = \frac{1}{2d} \sum_{v \in S} |\log |\eta|_v|,
\]
which implies that
\[
\frac{1}{2d} F(\nu(\eta)) \leq \log h(\eta) \leq \frac{1}{d} F(\nu(\eta)).
\]
Hence, by (12), (11), (10) and (9), we have
\[
\prod_{i=1}^{s-1} \log h(\varepsilon_i) \leq \frac{1}{d^{s-1}} \prod_{i=1}^{s-1} F(\nu(\varepsilon_i)) \leq \frac{(s-1)!}{2e^{2d}d^{s-1}} \prod_{i=1}^{s-1} F(\nu(\varepsilon_i))
\]
\[
\leq ((s-1)!^2 R_S/(2e^{2d}d^{s-1}),
\]
which proves (i).
(ii) follows immediately from (i) and $\log h(\varepsilon_i) \geq \delta_K/d$ for $i = 1, \ldots, s-1$.

To prove (iii), let $E = (\log |\varepsilon_i|_{v_j})_{i,j=1,\ldots,s-1}$ and $e_{ij} = \det(E_{ij})/\det(E)$, where $E_{ij}$ denotes the matrix obtained from $E$ by omitting the $i$th row and $j$th column. It follows from (13) and Hadamard’s inequality that
\[
|\det(E_{ij})| \leq \prod_{p=1, p \neq i}^{s-1} \prod_{q=1, q \neq j}^{s-1} (\log |\varepsilon_p|_{v_q})^2 \leq \prod_{p=1}^{s-1} F(\nu(\varepsilon_p)) \leq c_4 R_S/F(\nu(\varepsilon_i)).
\]
Together with (12), $|\det(E)| = R_S$ and $\log h(\varepsilon_i) \geq \delta_K/d$ this implies $|e_{ij}| \leq c_4 \delta_K^{-1} d^{s-1}$, which completes the proof. ■

The next lemma has various versions in the literature (for references, see e.g. [15], [24], [10], [18]). Our lemma is an explicit version of Lemma 10 of [10].

Let $c_3 = c_3(d, r, K)$ denote the constant specified in the Corollary.

**LEMMA 2.** For every $\alpha \in O_S \setminus \{0\}$ and every integer $n \geq 1$ there exists an $S$-unit $\varepsilon$ such that
\[
h(\varepsilon^n \alpha) \leq NS(\alpha)^{1/d} \exp\{n(c_3 R_K + th_K \log^* P)\}.
\]

**Proof.** First consider the case when $S = S_{\infty}$. So let $\alpha \in O_K \setminus \{0\}$ and put $M = |N_{K/Q}(\alpha)|$. Let $S_{\infty} = \{v_1, \ldots, v_{r+1}\}$ and $L(\alpha) = \max_{1 \leq i \leq r} |\log |\alpha|_{v_i}|$. Then there are multiplicatively independent units $\eta_1, \ldots, \eta_r$ in $O_K$ such that $L(\eta_1) \ldots L(\eta_r) \leq R_K$ (cf. [14]). On the other hand, we have $L(\eta_j) \geq (d/r) \log h(\eta_j) \geq \delta_K/r$, whence $L(\eta_j) \leq r^{r-1} \delta_K^{-(r-1)} R_K$ for each $j$.

Consider the system of linear equations
\[
\sum_{j=1}^{r} X_j \log |\eta_j|_{v_i} = -\log(M^{-d_{v_i}/d}|\alpha|_{v_i}), \quad i = 1, \ldots, r+1,
\]
in $X_1, \ldots, X_r$. It has a unique solution $x_1, \ldots, x_r$ in $\mathbb{R}$. For $1 \leq j \leq r$, there exist $b_j \in \mathbb{Z}$ and $g_j \in \mathbb{R}$ with $|g_j| \leq n/2$ such that $x_j = nb_j + g_j$. Putting $\eta_1^{b_1} \cdots \eta_r^{b_r} = \varepsilon$, we infer that

$$
\sum_{j=1}^{r} b_j \log |\eta_j|_{v_i} \leq \frac{n r}{2} \max_{1 \leq j \leq r} |\log |\eta_j|_{v_i}| \leq \frac{n r}{2} \cdot r \max_{1 \leq j \leq r} L(\eta_j) \leq n c_3 R_K, \quad i = 1, \ldots, r + 1,
$$

which implies (14).

The general case of our lemma follows from the case $S = S_{\infty}$ as in the proof of Lemma 10 of [10].

Denote by $p_1, \ldots, p_t$ the prime ideals in $K$ corresponding to the finite places in $S$. We recall that $P$ denotes the largest of the rational primes lying below these prime ideals.

The following lemma is an improvement of some estimates of Pethő [22] and Hajdu [18] for $R_S$. It should, however, be remarked that Pethő’s estimate was established in a more general situation, for some $S$-orders instead of $O_S$.

**Lemma 3.** If $t > 0$, then

$$
R_S \leq R_K h_K \prod_{i=1}^{t} \log N(p_i) \leq R_K h_K (d \log^* P)^t
$$

and

$$
R_S \geq R_K \prod_{i=1}^{t} \log N(p_i) \geq c_7 (\log 2)(\log^* P),
$$

where $c_7 = 0.2052$.

**Proof.** $O_S^* / O_K^*$ is a free abelian group of rank $t$ which is isomorphic to the multiplicative group of principal ideals in $K$ generated by the elements of $O_S^*$. This latter group is a subgroup of finite index, say $i_S$, of the multiplicative group generated by $p_1, \ldots, p_t$ and we have $i_S \leq h_K$. Hence, as is known (see e.g. [4], pp. 85 and 125), this subgroup has a basis of the form

$$(\varepsilon_i) = p_1^{a_{1i}} p_{r+1}^{a_{1i+1}} \cdots p_t^{a_{1i}}, \quad i = 1, \ldots, t,$$

with rational integers $a_{ij}$ such that $a_{ii} > 0$ for $i = 1, \ldots, t$ and that $a_{11} \ldots a_{tt} = i_S$. It now follows that if $\{\varepsilon_{t+1}, \ldots, \varepsilon_{t+r}\}$ is a fundamental system of units in $O_K$ then $\{\varepsilon_1, \ldots, \varepsilon_{t}, \ldots, \varepsilon_{t+r}\}$ is a fundamental system of $S$-units in $K$. Consequently, it is easy to see that

$$
R_S = |\det (\log |\varepsilon_i|_{v_j})_{i,j=1,\ldots,r+t}| = R_K i_S \prod_{i=1}^{t} \log N(p_i),
$$

where $c_7 = 0.2052$. 


which gives (16). Inequalities (17) follow from (18) and the estimate $R_k \geq c_7$ of Friedman [12].

We remark that, in our Theorem and its Corollary, the improvements of the previous bounds in terms of $R_k$, $h_k$ and $P$ are mainly due to the use of fundamental systems of $S$-units, $S$-regulators as well as Lemmas 1 to 3.

4. Estimates for linear forms in logarithms. In our proofs, we shall use the best known estimates, due to Waldschmidt [26] and Kunrui Yu [27] respectively, for linear forms in logarithms in the complex and in the $p$-adic case. We shall formulate them in a more convenient form for our purpose. These estimates enable us to considerably improve the previous bounds for the solutions of equation (1) in terms of $d, r$ and $s$.

Let $\alpha_1, \ldots, \alpha_n (n \geq 2)$ be non-zero algebraic numbers and let $K = \mathbb{Q}(\alpha_1, \ldots, \alpha_n)$. Put $[K : \mathbb{Q}] = d$. Let $A_1, \ldots, A_n$ be real numbers such that

$$\log A_i \geq \max \left\{ \log h(\alpha_i), \frac{|\log \alpha_i|}{3.3d}, \frac{1}{d} \right\}, \quad i = 1, \ldots, n,$$

where $\log$ denotes the principal value of the logarithm. Let $b_1, \ldots, b_n$ be rational integers and put $B = \max\{|b_1|, \ldots, |b_n|, 3\}$. Further, set

$$A = \alpha_1^{b_1} \ldots \alpha_n^{b_n} - 1.$$

In Proposition 1, it will be convenient to make the following technical assumptions:

$$B \geq \log A_n \exp\{4(n+1)(7 + 3\log(n+1))\}$$

and

$$7 + 3\log(n+1) \geq \log d.$$

Proposition 1 is a consequence of Corollary 10.1 of Waldschmidt [26].

PROPOSITION 1 (M. Waldschmidt [26]). If $A \neq 0$, $b_n = 1$ and (20), (21) hold, then

$$|A| \geq \exp \left\{ -c_8(n)d^{n+2}\log A_1 \ldots \log A_n \log \left( \frac{2nB}{\log A_n} \right) \right\},$$

where $c_8(n) = 1500 \cdot 38^{n+1}(n+1)^{3n+9}$.

We remark that a recent explicit estimate of Baker and Wüstholz [1] for linear forms in logarithms would give here a smaller value for $c_8(n)$ in terms of $n$. However, the lower bound in (22) is better in terms of $A_n$, which is essential for our present applications.

Proof of Proposition 1. We denote by $\log$ the principal value of the logarithm. Setting $\alpha_0 = -1$, there is a $b_0 \in \mathbb{Z}$ such that $|b_0| \leq
\[|b_1| + \ldots + |b_{n-1}| + 2 \leq nB\] and that
\[\log(\alpha_1^{b_1} \ldots \alpha_n^{b_n}) = \sum_{j=1}^{n} b_j \log \alpha_j + b_0 \log \alpha_0 := \Omega,\]
where \(b_n = 1\). It suffices to deal with the case when \(|A| \leq 1/3\). Since \(|\log z| \leq 2|z - 1|\) for any \(z \in \mathbb{C}\) with \(|z - 1| \leq 1/3\), we get
\[\log(|\Lambda|) \geq \frac{1}{2} \log(2nB),\]
Together with (23) this implies (22).

In Proposition 2, let \(v = v_p\) be a finite place on \(K\), corresponding to the prime ideal \(p\) of \(K\). Let \(p\) denote the rational prime lying below \(p\), and denote by \(|\cdot|_v\) the non-archimedean valuation normalized as in Section 3. Instead of (19), assume now that \(A_1, \ldots, A_n\) are real numbers such that
\[\log A_i \geq \max\{\log h(\alpha_i)|\log \alpha_i|/(10d), \log p\}, \quad i = 1, \ldots, n.\]

The following proposition is a simple consequence of the main result of Kunrui Yu [27].

**Proposition 2** (Kunrui Yu [27]). Let
\[\Phi = c_9(n)(d/\sqrt{\log p})^{2(n+1)}p^d \log A_1 \ldots \log A_n \log(10nd \log A),\]
where \(c_9(n) = 22000(9.5(n + 1))^{2(n+1)}\) and \(A = \max\{A_1, \ldots, A_n, e\}\). If \(A \neq 0\) then
\[|A|_v \geq \exp\{-d(\log p) \Phi \log(dB)\}.\]
Further, if \(b_n = 1\) and \(A_n \geq A_i\) for \(i = 1, \ldots, n - 1\), then \(A\) can be replaced by \(\max\{A_1, \ldots, A_{n-1}, e\}\) and for any \(\delta \) with \(0 < \delta \leq 1\), we have
\[|A|_v \geq \exp\{-d(\log p) \max\{\Phi \log(\delta^{-1} \Phi / \log A_n), \delta B\}\}.\]

**Proof.** This is a reformulation of the result presented in the introduction of Kunrui Yu [27].

**Remark 6.** We remark that, in Propositions 1 and 2, the condition \(K = \mathbb{Q}(\alpha_1, \ldots, \alpha_n)\) can be removed. It is enough to assume that \(K\) is an algebraic number field of degree \(d\) which contains \(\alpha_1, \ldots, \alpha_n\). This observation will be needed in Section 5.
5. Proofs of the Theorem and the Corollary

Proof of the Theorem. Let \( x, y \) be an arbitrary but fixed solution of
\[
\alpha x + \beta y = 1 \quad \text{in } x, y \in O_S^*.
\]
We assume that \( h(x) \geq h(y) \). Let \( \{\varepsilon_1, \ldots, \varepsilon_{s-1}\} \) be a fundamental system of \( S \)-units in \( K \) with the properties specified in Lemma 1. Then we can write
\[
y = \zeta \varepsilon_1^{b_1} \cdots \varepsilon_{s-1}^{b_{s-1}}
\]
with a root of unity \( \zeta \) in \( K \) and with rational integers \( b_1, \ldots, b_{s-1} \). Put \( B = \max\{|b_1|, \ldots, |b_{s-1}|, 3\} \) and \( S = \{v_1, \ldots, v_s\} \). Then (25) implies
\[
\log |y|_{v_j} = \sum_{i=1}^{s-1} b_i \log |\varepsilon_i|_{v_j}, \quad j = 1, \ldots, s - 1,
\]
whence, by (iii) of Lemma 1 and (12), we get
\[
B \leq c_6 \sum_{j=1}^{s-1} \log |y|_{v_j} \leq 2dc_6 \log h(y) \leq 2dc_6 \log h(x)
\]
with the \( c_6 = c_6(d, s, K) \) specified in Lemma 1.

Let \( v \in S \) for which \( |x|_v \) is minimal. Setting \( \alpha_s = \zeta \beta \) and \( b_s = 1 \), we deduce from (1) that
\[
|\alpha x|_v = |\varepsilon_1^{b_1} \cdots \varepsilon_{s-1}^{b_{s-1}} \alpha_s^{b_s} - 1|_v.
\]
We shall derive a lower bound for \( |\alpha x|_v \).

First assume that \( v \) is infinite. In order to apply Proposition 1, put
\[
\log A_i = \delta_k^{-1} \log h(\varepsilon_i), \quad i = 1, \ldots, s - 1,
\]
\[
\log A_s = \delta_k^{-1} \log H.
\]
It is easy to check that \( 7 + 3 \log(s + 1) \geq \log d \). Further, we may assume that
\[
B \geq \log A_s \exp\{4(s + 1)(7 + 3 \log(s + 1))\}.
\]
Indeed, (1) implies that
\[
h(x) \leq 2H^2 h(y).
\]
Further, it follows from (25) and (ii) of Lemma 1 that
\[
h(y) \leq \prod_{i=1}^{s-1} h(\varepsilon_i)^{|b_i|} \leq \exp\{(s - 1)c_5 R_S B\}.
\]
Hence, if (29) does not hold, we get at once a bound for \( h(x) \) which is better than that in the Theorem.
We have \(|\cdot|_v = |\sigma(\cdot)|^{d_v}\) for some \(\sigma : K \rightarrow \mathbb{C}\). Applying \(\sigma\) to equation (1) and then omitting \(\sigma\) everywhere, we may assume that \(|\cdot|_v = |\cdot|^{d_v}\). On applying now Proposition 1 to (27) and using (i) of Lemma 1, we derive that

\[
|\alpha x|_v \geq \exp \left\{ - c_{10} R_S \log H \log \left( \frac{c_{11} B}{\log H} \right) \right\},
\]

where \(c_{10} = d_v c_8 (s) c_4 d^{s+2} \delta_{K}^{-s}\) and \(c_{11} = 2 s \delta_{K}\).

Since \(|x|_v\) is minimal, we have

\[
h(x) = h(1/x) \leq |x|^{-(s-1)/d}.
\]

Hence it follows from (32), (26) and \(|\alpha|_v \leq H^d\) that

\[
\log h(x) \leq \frac{2(s-1)}{d} c_{10} R_S \log \left( \frac{c_{12} \log h(x)}{\log H} \right),
\]

where \(c_{12} = 2 d c_6 c_{11}\). This gives (2)

\[
h(x) \leq \exp \{ c_{13} R_S (\log^* R_S) \log H \}
\]

with

\[
c_{13} = 3^{s+26} d^3 \delta_{K}^{-s} s^{5s+12}.
\]

We remark that in the particular case \(S = S_{\infty}\), i.e. when \(t = 0\), (34) implies the second part of the Theorem.

Next assume that \(v\) is finite. To apply Proposition 2, we put now

\[
\log A_i = \delta_{K}^{-1} \log h(\varepsilon_i) + \log^* P, \quad i = 1, \ldots, s - 1,
\]

\[
\log A_s = \delta_{K}^{-1} \log H + \log^* P.
\]

Using (i) of Lemma 1, we get

\[
\log A_1 \ldots \log A_{s-1} \leq \prod_{i=1}^{s-1} (\delta_{K}^{-1} \log h(\varepsilon_i)) \left( \sum_{j=0}^{s-1} \binom{s-1}{j} (d \log^* P)^{j} - (d \log^* P)^{s-1} \right) + (\log^* P)^{s-1} \leq (\log^* P)^{s-2} (c_{14} R_S + \log^* P)
\]

with \(c_{14} = (s/d)(s-1)!^2 \delta_{K}^{-s} \delta_{K}^{-s+1}\). Together with the second inequality of Lemma 3 this gives

\[
\log A_1 \ldots \log A_{s-1} \leq 2c_{14} R_S (\log^* P)^{s-2}.
\]

(2) In certain applications (e.g. in case of practical solutions of \(S\)-unit equations), it can be more useful to work with our upper bounds of \(B\), provided by (26), (34) and (43).
We distinguish two cases. First assume that \( \log H < c_5 R_S \). Then, by Lemmas 1 and 3, we have
\[
\log A := \max_{1 \leq i \leq s} \log A_i \leq c_{15} R_S
\]
with \( c_{15} = c_5 \delta_{K}^{-1} + (c_7 \log 2)^{-1} \). We now apply to (27) the first part of Proposition 2. Putting
\[
\Phi = c_{16} \frac{P^d}{(\log^* P)^{s+1}} \log A_1 \ldots \log A_s \log(10sd \log A)
\]
with \( c_{16} = c_9(s)(d^2/\log 2)^s + 1 \), we infer that
\[
|ax|_v \geq \exp\left\{-d(\log^* P)\Phi \log(dB)\right\},
\]
whence, by (33), (26) and \( |a|_v \leq H^d \),
\[
\log h(x) \leq 2(s - 1)(\log^* P)\Phi \log(c_{17} \log h(x))
\]
follows with \( c_{17} = 2d^2c_6 \). Together with (36), (37) and \( \log H < c_5 R_S \) this gives
\[
\log h(x) \leq \exp\left\{c_{18} P^d R_S(\log^* R_S)(\log^* (PR_S)/\log^* P) \log H\right\},
\]
where
\[
c_{18} = 3^{26}(18d^2/\delta_{K})^{s+1}s^{4s+7}.
\]

Next assume that \( \log H \geq c_5 R_S \). Then, by Lemmas 1 and 3, we have \( A_s \geq A_i \) for \( i = 1, \ldots, s - 1 \) and
\[
\log A := \max_{1 \leq i \leq s-1} \log A_i \leq c_{15} R_S.
\]
Consider now the above defined \( \Phi \) with this value of \( \log A \). First we give an upper bound for \( h(x) \) in terms of \( \Phi \).

If \( B < \Phi(\log^* P)/(c_5 R_S) \) then (30), (31) and (35) imply that
\[
\log h(x) \leq 2H^2 \exp\{(s - 1)\Phi \log^* P\} < \exp\{s \Phi \log^* P\}.
\]

Assume now that \( B \geq \Phi(\log^* P)/(c_5 R_S) \). We apply the second part of Proposition 2 to (27). Putting \( \delta = \Phi(\log^* P)/(Bc_5 R_S) \) we obtain
\[
|ax|_v \geq \exp\left\{-d(\log^* P)\Phi \log\left(\frac{Bc_5 R_S}{\log^* P \log A_s}\right)\right\}.
\]

Hence, proceeding again as above, we deduce that
\[
\log h(x) \leq 2(s - 1)(\Phi/\log A_s) \log(c_{19} R_S \log h(x))/\log^* P \log A_s
\]
with \( c_{19} = 2dc_6c_5 \). From this we infer as above that
\[
\log h(x) \leq \exp\{c_{20} \Phi(\log^* P) \log^* (PR_S)\},
\]
where \( c_{20} = 19(s - 1) \log(c_{16}) \).
The right hand side of (42) is greater than that of (41). Lemma 3, (35) and \( \log H \geq c_5 R_S \) imply that \( \log A_s < c_{21} \log H \) with \( c_{21} = (c_5 c_7 \log 2)^{-1} + \delta_{K}^{-1} \). Hence, estimating from above \( \Phi \), we obtain in both cases that

\[
(43) \quad h(x) \leq \exp\{c_{18} P^d R_S (\log^* R_S) (\log^* (PR_S) / \log^* P) \log H \},
\]

with the constant \( c_{18} \) defined above. However, it is easy to verify that both \( c_{13} \) in (34) and \( c_{18} \) in (39) and (43) are less than \( c_1 = c_1(d, s, K) \) specified in the Theorem. This completes the proof of our assertion. 

**Proof of the Corollary.** Let \( x_1, x_2, x_3 \) be a solution of (6). Then, by Lemma 2, there are \( \varepsilon_i \in O_S^* \) such that

\[
(44) \quad h(\varepsilon_i x_i) \leq N^{1/d} \exp\{c_3 R_K + th_K \log^* P\}
\]

with the constant \( c_3 \) specified in Lemma 2. Put

\[
\alpha = \frac{\alpha_1(\varepsilon_1 x_1)}{\alpha_3(\varepsilon_3 x_3)}, \quad \beta = \frac{\alpha_2(\varepsilon_2 x_2)}{\alpha_3(\varepsilon_3 x_3)}.
\]

Then \( x = -\varepsilon_3 / \varepsilon_1, y = -\varepsilon_3 / \varepsilon_2 \) is a solution of equation (1). We have

\[
\max\{h(\alpha), h(\beta)\} \leq \exp\{2c_3 (R_K + th_K \log^* P + \log(HN))\}.
\]

Now our Theorem provides an explicit upper bound for \( \max\{h(x), h(y)\} \). Together with (44), this implies (7) with the choice \( \varepsilon = -\varepsilon_3 \).

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**References**


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