ACTA ARITHMETICA
LXXIV.1 (1996)

On a form of the Erdős–Turán inequality

by

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1. Introduction. Let $\mathcal{P} = \{x_1, \ldots, x_N\}$ be a set of points in $\mathbb{R}$, and define the $\mathbb{Z}$-periodic set

$$\mathcal{P}^* = \{x + m : x \in \mathcal{P},\ m \in \mathbb{Z}\}.$$

The *discrepancy* $D(\mathcal{P})$ gives a measure of how evenly (or unevenly) distributed $\mathcal{P}$ is in $\mathbb{R}/\mathbb{Z}$. There are a number of ways to define the discrepancy (see, for instance, [2, 8, 13]); a common form is as follows: Let s, t be real numbers which satisfy $s < t < s+1$, and let $\chi_{s,t}(x)$ denote the characteristic function of the interval $[s, t]$. Then we define

$$D(\mathcal{P}) = \sup_{s<t<s+1} \Big| \sum_{x \in \mathcal{P}^*} \chi_{s,t}(x) - N(t-s) \Big|. \tag{1.1}$$

In 1948, P. Erdős and P. Turán [4] established a quantitative connection between $D(\mathcal{P})$ and the exponential sums

$$\Big| \sum_{n=1}^{N} e(mx_n) \Big|,$$

where m is a nonzero integer and $e(\theta) = e^{2\pi i\theta}$. Specifically, they showed that there exist absolute constants C_1 and C_2 such that

$$D(\mathcal{P}) \le C_1 N M^{-1} + C_2 \sum_{m=1}^{M} m^{-1} \Big| \sum_{n=1}^{N} e(mx_n) \Big| \tag{1.2}$$

holds for all integers $M \ge 1$. Explicit values for C_1 and C_2 are given in ([8], pp. 112–114) and ([12], Theorem 20).

The notion of the discrepancy of a point set has been generalized to a wide variety of settings. Bounds in the style of (1.2) have been given in several cases (see [3, 5, 11]), and are typically referred to as "Erdős–Turán" inequalities. Here we establish such an inequality for points distributed on the unit torus $\mathbb{R}^k/\mathbb{Z}^k$, where $k \ge 2$. In a manner analogous to the one-dimensional case, we let $\mathcal{P} = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$ be a set of points in $\mathbb{R}^k$ and then define

$$\mathcal{P}^* = \{\mathbf{x}+\mathbf{m} : \mathbf{x}\in\mathcal{P},\ \mathbf{m}\in\mathbb{Z}^k\}.$$

For $r>0$ and $\mathbf{c}\in\mathbb{R}$, let $B_k(r,\mathbf{c})$ denote the closed ball of radius r centered at $\mathbf{c}$ given by

$$B_k(r,\mathbf{c}) = \{\mathbf{x}\in\mathbb{R}^k : |\mathbf{x}-\mathbf{c}|\le r\},$$

where $|\cdot|$ denotes the usual Euclidean metric on $\mathbb{R}^k$. For each such r and $\mathbf{c}$, define

$$\varDelta[\mathcal{P}; B_k(r,\mathbf{c})] = Z[\mathcal{P}^*; B_k(r,\mathbf{c})] - N\mu(B_k(r,\mathbf{c})), \tag{1.3}$$

where $Z[Q;A]$ denotes the number of points of a discrete set $Q\subset\mathbb{R}^k$ which fall in a compact set $A\subset\mathbb{R}^k$, and μ is the usual Euclidean volume. For each $r>0$ we then define the discrepancy $D_r(\mathcal{P})$ by

$$D_r(\mathcal{P}) = \sup_{\mathbf{c}\in\mathbb{R}^k} |\varDelta[\mathcal{P}; B_k(r,\mathbf{c})]|. \tag{1.4}$$

By applying an observation of H. L. Montgomery (see [1], Section 2.3) together with functions constructed by J. Vaaler and the author [7], we establish the following bound:

Theorem 1. *Let $r>0$ and $\mathcal{P}=\{\mathbf{x}_1,\ldots,\mathbf{x}_N\}$ be a subset of $\mathbb{R}^k$. Then*

$$\begin{aligned} &D_r(\mathcal{P}) \\ &\le NA_k(r,s) + \sideset{}{'}\sum_{|\mathbf{m}|<s} \{A_k(r,s) + (r/|\mathbf{m}|)^{k/2}|J_{k/2}(2\pi r|\mathbf{m}|)|\}\Big|\sum_{n=1}^{N} e(\mathbf{m}\cdot\mathbf{x}_n)\Big| \end{aligned} \tag{1.5}$$

for all $s>0$, where

$$\begin{aligned} A_k(r,s) = \omega_k s^{-1} r^{k-1}\{&\pi rs(J_{(k-2)/2}(\pi rs)^2 + J_{k/2}(\pi rs)^2) \\ &- (k-1)J_{(k-2)/2}(\pi rs)J_{k/2}(\pi rs)\}^{-1}, \end{aligned}$$

$J_\nu(x)$ is the ν-th order Bessel function and $\omega_k = 4\pi^{(k-2)/2}\Gamma(k/2)^{-1}$.

In 1969, W. Schmidt [9] showed that the discrepancy cannot be uniformly small. Suppose that $\varepsilon>0$ and that δ satisfies $N\delta^k\ge 1$. Schmidt proved that there exists a ball $B_k(r,\mathbf{c})$ with $r\le\delta$ such that

$$|\varDelta[\mathcal{P}; B_k(r,\mathbf{c})]| > c_1(k,\varepsilon)(N\delta^k)^{(k-1)/2k-\varepsilon}.$$

On the other hand, J. Beck ([2], Theorem 14) has shown that there exists an infinite sequence $\mathbf{x}_1,\mathbf{x}_2,\ldots$ such that for all $N\ge 2$ and any ball $B_k(r,\mathbf{c})$ with $r\le 1$ and $Nr^k\ge 1$, we have

$$|\varDelta[\{\mathbf{x}_1,\ldots,\mathbf{x}_N\}; B_k(r,\mathbf{c})]| \le c_2(k)(Nr^k)^{(k-1)/2k}(\log N)^{3/2}. \tag{1.6}$$

In view of Schmidt's lower bound, we see that (1.6) must be close to best possible. Applying Theorem 1 and a simple averaging procedure, we may establish the existence of a set of p points in $\mathbb{R}^k$ (here p is a prime) which has modest discrepancy. (A similar application is given in [8], pp. 154–157.) Although our result is not as sharp as Beck's, our proof is much simpler and we do not have the requirement that $r \le 1$. For $\mathbf{h} \in \mathbb{Z}^k$, let $\mathcal{P}_{\mathbf{h}}$ be the collection of p points of the form $(n/p)\mathbf{h}$, $n = 1, \ldots, p$.

THEOREM 2. *Let p be a prime number, and suppose that $r \ge p^{-1/k}$. Then there exists a lattice point $\mathbf{h} \in \mathbb{Z}^k$ such that $|\mathbf{h}| < p$ and*

$$D_r(\mathcal{P}_{\mathbf{h}}) \le c_3(k)(pr^k)^{(k-1)/k}. \tag{1.7}$$

Notation. We use the definition for Fourier transforms and series of Stein and Weiss [10]. For $\mathbf{x} \in \mathbb{R}^k$ and $r > 0$, $\chi_r(\mathbf{x})$ denotes the characteristic function of $B_k(r, \mathbf{0})$. To simplify expressions, we adopt the convention that $\mathbf{m}$ appearing in a sum will always be a point in $\mathbb{Z}^k$. Finally, $\sum'$ means that the term in the sum corresponding to $\mathbf{m} = \mathbf{0}$ is omitted.

2. Proof of theorems

Proof of Theorem 1. We require two auxiliary functions. Combining the results in ([7], Theorem 3) and a k-dimensional form of the Paley–Wiener theorem ([10], Chapter III, Theorem 4.9), we see that for $r > 0$ and $s > 0$ there exist functions $\mathcal{F}_k(\mathbf{x}; r, s)$ and $\mathcal{G}_k(\mathbf{x}; r, s)$ that satisfy

$$\mathcal{F}_k(\mathbf{x}; r, s) \le \chi_r(\mathbf{x}) \le \mathcal{G}_k(\mathbf{x}; r, s) \quad \text{for all } \mathbf{x} \in \mathbb{R}^k, \tag{2.1}$$

$$\widehat{\mathcal{F}}_k(\mathbf{t}; r, s) = \widehat{\mathcal{G}}_k(\mathbf{t}; r, s) = 0 \quad \text{for all } |\mathbf{t}| \ge s, \tag{2.2}$$

$$\int\limits_{\mathbb{R}^k} (\mathcal{G}_k(\mathbf{x}; r, s) - \mathcal{F}_k(\mathbf{x}; r, s))\, d\mathbf{x} = A_k(r, s), \tag{2.3}$$

where $A_k(r, s)$ is defined in the statement of the theorem. From (1.3) we see that for $\mathbf{c} \in \mathbb{R}^k$,

$$\Delta[\mathcal{P}; B_k(r, \mathbf{c})] = \sum_{n=1}^{N} \sum_{\mathbf{m}} \chi_r(\mathbf{x}_n - \mathbf{c} + \mathbf{m}) - N\mu(B_k(r, \mathbf{0})).$$

Now suppose that for a given r and $\mathbf{c}$ we have $\Delta[\mathcal{P}; B_k(r, \mathbf{c})] \ge 0$. Then by (2.1), the Poisson summation formula and the triangle inequality we have

$$\begin{aligned} \Delta[\mathcal{P}; B_k(r, \mathbf{c})] &\le \sum_{n=1}^{N} \sum_{\mathbf{m}} \mathcal{G}_k(\mathbf{x}_n - \mathbf{c} + \mathbf{m}; r, s) - N\mu(B(r, \mathbf{0})) \\ &= \sum_{n=1}^{N} \sum_{\mathbf{m}} \widehat{\mathcal{G}}_k(\mathbf{m}; r, s) e(\mathbf{m} \cdot (\mathbf{x}_n - \mathbf{c})) - N\widehat{\chi}_r(\mathbf{0}) \end{aligned} \tag{2.4}$$

$$= \sum_{|\mathbf{m}|<s} \widehat{\mathcal{G}}_k(\mathbf{m}; r, s) e(-\mathbf{m}\cdot\mathbf{c}) \sum_{n=1}^{N} e(\mathbf{m}\cdot\mathbf{x}_n) - N\widehat{\chi}_r(\mathbf{0})$$

$$\le N(\widehat{\mathcal{G}}_k(\mathbf{0}; r, s) - \widehat{\chi}_r(\mathbf{0})) + \sum_{|\mathbf{m}|<s}{}' |\widehat{\mathcal{G}}_k(\mathbf{m}; r, s)| \Big| \sum_{n=1}^{N} e(\mathbf{m}\cdot\mathbf{x}_n)\Big|.$$

From (2.1) and (2.3) we know that

$$\widehat{\mathcal{G}}_k(\mathbf{0}; r, s) - \widehat{\chi}_r(\mathbf{0}) = \int_{\mathbb{R}^k} (\mathcal{G}_k(\mathbf{x}; r, s) - \chi_r(\mathbf{x}))\, d\mathbf{x} \le A_k(r, s). \tag{2.5}$$

A general expression for $\widehat{\mathcal{G}}_k(\mathbf{t}; r, s)$ seems difficult to find. However, we can obtain an estimate that will do for our purposes. First note that

$$\widehat{\chi}_r(\mathbf{t}) = \int_{\mathbb{R}^k} \chi_r(\mathbf{x}) e(-\mathbf{t}\cdot\mathbf{x})\, d\mathbf{x} = (r/|\mathbf{t}|)^{k/2} J_{k/2}(2\pi r|\mathbf{t}|).$$

Applying the triangle inequality together with (2.5) and the above identity, we see that

$$\begin{aligned} &|\widehat{\mathcal{G}}_k(\mathbf{t}; r, s)| \\ &\quad\le \Big|\int_{\mathbb{R}^k} (\mathcal{G}_k(\mathbf{x}; r, s) - \chi_r(\mathbf{x})) e(-\mathbf{t}\cdot\mathbf{x})\, d\mathbf{x}\Big| + \Big|\int_{\mathbb{R}^k} \chi_r(\mathbf{x}) e(-\mathbf{t}\cdot\mathbf{x})\, d\mathbf{x}\Big| \\ &\quad\le (\widehat{\mathcal{G}}_k(\mathbf{0}; r, s) - \widehat{\chi}_r(\mathbf{0})) + |\widehat{\chi}_r(\mathbf{t})| \\ &\quad\le A_k(r, s) + (r/|\mathbf{t}|)^{k/2} |J_{k/2}(2\pi r|\mathbf{t}|)|. \end{aligned}$$

Thus it follows from (2.4) and (2.5) that

$$\begin{aligned} &\Delta[\mathcal{P}; B_k(r, \mathbf{c})] \\ &\le NA_k(r, s) + \sum_{|\mathbf{m}|<s}{}' \{A_k(r, s) + (r/|\mathbf{m}|)^{k/2} |J_{k/2}(2\pi r|\mathbf{m}|)|\} \Big| \sum_{n=1}^{N} e(\mathbf{m}\cdot\mathbf{x}_n)\Big|. \end{aligned} \tag{2.6}$$

If it should happen that for a given r and $\mathbf{c}$ we have $\Delta[\mathcal{P}; B_k(r, \mathbf{c})] < 0$, then following the preceding analysis using $\mathcal{F}_k(\mathbf{x}; r, s)$ in place of $\mathcal{G}_k(\mathbf{x}; r, s)$ yields inequality (2.6) with a minus sign attached to the left-hand term. Combining these bounds verifies (1.5) and completes the proof.

Proof of Theorem 2. We begin by using some estimates to simplify the bound given in Theorem 1. If $rs \ge 1$, then $A_k(r, s) \ll_k s^{-1} r^{k-1}$ (see [7], Theorem 1). Combining this with the bound $|J_\nu(x)| \le 1$ for $\nu > 0$ and $x > 0$ reduces (1.5) to

$$D_r(\mathcal{P}) \ll_k N s^{-1} r^{k-1} + \sum_{|\mathbf{m}|<s}{}' \{s^{-1} r^{k-1} + (r/|\mathbf{m}|)^{k/2}\} \Big| \sum_{n=1}^{N} e(\mathbf{m}\cdot\mathbf{x}_n)\Big|.$$

For a prime p, if $\mathbf{h}$ and $\mathbf{m}$ are lattice points, then

$$\sum_{n=1}^{p} e((n/p)\mathbf{h}\cdot\mathbf{m}) = \begin{cases} 0 & \text{if } \mathbf{h}\cdot\mathbf{m} \not\equiv 0 \pmod p, \\ p & \text{if } \mathbf{h}\cdot\mathbf{m} \equiv 0 \pmod p. \end{cases}$$

Therefore we have

$$D_r(\mathcal{P}_{\mathbf{h}}) \ll_k ps^{-1}r^{k-1} + \sideset{}{'}\sum_{\substack{|\mathbf{m}|<s \\ \mathbf{h}\cdot\mathbf{m}\equiv 0}} \{s^{-1}r^{k-1} + (r/|\mathbf{m}|)^{k/2}\}p. \tag{2.7}$$

The sum above is difficult to handle alone, but the problem simplifies if we average over all lattice points $\mathbf{h}$ such that $|\mathbf{h}| \le p$. On doing so, the right side of (2.7) is equal to

$$ps^{-1}r^{k-1} + p(Z[\mathbb{Z}^k; B_k(p,\mathbf{0})])^{-1} \sum_{|\mathbf{h}|\le p} \sideset{}{'}\sum_{\substack{|\mathbf{m}|<s \\ \mathbf{h}\cdot\mathbf{m}\equiv 0}} \{s^{-1}r^{k-1} + (r/|\mathbf{m}|)^{k/2}\} \tag{2.8}$$

$$= ps^{-1}r^{k-1} + p(Z[\mathbb{Z}^k; B_k(p,\mathbf{0})])^{-1} \sideset{}{'}\sum_{|\mathbf{m}|<s} \{s^{-1}r^{k-1} + (r/|\mathbf{m}|)^{k/2}\} \sum_{\substack{|\mathbf{h}|\le p \\ \mathbf{h}\cdot\mathbf{m}\equiv 0}} 1.$$

For each $\mathbf{m}$ in the outer sum on the right side of (2.8), there is at least one nonzero component m_g. We fix such an $\mathbf{m}$, and consider the inner sum. For a given $\mathbf{h}$, once the components $h_1, \ldots, h_{g-1}, h_{g+1}, \ldots, h_k$ are set, there is only one choice for $h_g \pmod p$ for which $\mathbf{h}\cdot\mathbf{m} \equiv 0 \pmod p$. Since $|\mathbf{h}| \le p$, there are at most three possible choices for h_g. Furthermore, the other components $h_1, \ldots, h_{g-1}, h_{g+1}, \ldots, h_k$ must satisfy $|h_j| \le p$ for each appropriate j. Thus for $\mathbf{m} \ne \mathbf{0}$,

$$\sum_{\substack{|\mathbf{h}|\le p \\ \mathbf{h}\cdot\mathbf{m}\equiv 0}} 1 \ll_k p^{k-1}. \tag{2.9}$$

We also note that for each $i \ge 2$, there exist constants $c_4(i)$ and $c_5(i)$ such that

$$c_4(i)p^i \le Z[\mathbb{Z}^i; B_i(p,\mathbf{0})] \le c_5(i)p^i. \tag{2.10}$$

(See [6], Theorem 339, for a discussion of the case $i = 2$.) As we are averaging, we see that (2.7)–(2.10) imply there exists a lattice point $\mathbf{h}$ with $|\mathbf{h}| \le p$ such that

$$D_r(\mathcal{P}_{\mathbf{h}}) \ll_k ps^{-1}r^{k-1} + \sideset{}{'}\sum_{|\mathbf{m}|<s} \{s^{-1}r^{k-1} + (r/|\mathbf{m}|)^{k/2}\}.$$

Applying the inequality

$$\sideset{}{'}\sum_{|\mathbf{m}|<s} |\mathbf{m}|^{-k/2} \ll_k s^{k/2}$$

and assuming that $rs \geq 1$ (required for our bound on $A_k(r,s)$ to be valid), we find that

$$D_r(\mathcal{P}_{\mathbf{h}}) \ll_k ps^{-1}r^{k-1} + (sr)^{k-1}.$$

The expression on the right above is minimized upon setting $s = p^{1/k}$, which yields

$$D_r(\mathcal{P}_{\mathbf{h}}) \ll_k (pr^k)^{(k-1)/k},$$

and completes the proof.

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Received on 2.11.1994 (2688)