On a form of the Erdős–Turán inequality

by

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1. Introduction. Let $\mathcal{P} = \{x_1, \ldots, x_N\}$ be a set of points in $\mathbb{R}$, and define the $\mathbb{Z}$-periodic set

$\mathcal{P}^* = \{x + m : x \in \mathcal{P}, m \in \mathbb{Z}\}$.

The discrepancy $D(\mathcal{P})$ gives a measure of how evenly (or unevenly) distributed $\mathcal{P}$ is in $\mathbb{R}/\mathbb{Z}$. There are a number of ways to define the discrepancy (see, for instance, [2, 8, 13]); a common form is as follows: Let $s, t$ be real numbers which satisfy $s < t < s + 1$, and let $\chi_{s,t}(x)$ denote the characteristic function of the interval $[s, t]$. Then we define

$D(\mathcal{P}) = \sup_{s < t < s + 1} \left| \sum_{x \in \mathcal{P}^*} \chi_{s,t}(x) - N(t - s) \right|.$

In 1948, P. Erdős and P. Turán [4] established a quantitative connection between $D(\mathcal{P})$ and the exponential sums $\left| \sum_{n=1}^{N} e(mx_n) \right|$, where $m$ is a nonzero integer and $e(\theta) = e^{2\pi i \theta}$. Specifically, they showed that there exist absolute constants $C_1$ and $C_2$ such that

$D(\mathcal{P}) \leq C_1 NM^{-1} + C_2 \sum_{m=1}^{M} m^{-1} \left| \sum_{n=1}^{N} e(mx_n) \right|$

holds for all integers $M \geq 1$. Explicit values for $C_1$ and $C_2$ are given in ([8], pp. 112–114) and ([12], Theorem 20).

The notion of the discrepancy of a point set has been generalized to a wide variety of settings. Bounds in the style of (1.2) have been given in several cases (see [3, 5, 11]), and are typically referred to as “Erdős–Turán” inequalities. Here we establish such an inequality for points distributed on the unit torus $\mathbb{R}^k/\mathbb{Z}^k$, where $k \geq 2$. In a manner analogous to the one-dimensional case, we let $\mathcal{P} = \{x_1, \ldots, x_N\}$ be a set of points in $\mathbb{R}^k$ and then
define
\[ \mathcal{P}^* = \{ \mathbf{x} + \mathbf{m} : \mathbf{x} \in \mathcal{P}, \mathbf{m} \in \mathbb{Z}^k \}. \]

For \( r > 0 \) and \( \mathbf{c} \in \mathbb{R} \), let \( B_k(r, \mathbf{c}) \) denote the closed ball of radius \( r \) centered at \( \mathbf{c} \) given by
\[ B_k(r, \mathbf{c}) = \{ \mathbf{x} \in \mathbb{R}^k : |\mathbf{x} - \mathbf{c}| \leq r \}, \]
where \(| \cdot |\) denotes the usual Euclidean metric on \( \mathbb{R}^k \).

For each such \( r \) and \( \mathbf{c} \), define
\[ \Delta[\mathcal{P}; B_k(r, \mathbf{c})] = Z[\mathcal{P}^*; B_k(r, \mathbf{c})] - N \mu(B_k(r, \mathbf{c})), \]
where \( Z[Q; A] \) denotes the number of points of a discrete set \( Q \subset \mathbb{R}^k \) which fall in a compact set \( A \subset \mathbb{R}^k \), and \( \mu \) is the usual Euclidean volume. For each \( r > 0 \) we then define the discrepancy \( D_r(\mathcal{P}) \) by
\[ D_r(\mathcal{P}) = \sup_{\mathbf{c} \in \mathbb{R}^k} |\Delta[\mathcal{P}; B_k(r, \mathbf{c})]|. \]

By applying an observation of H. L. Montgomery (see [1], Section 2.3) together with functions constructed by J. Vaaler and the author [7], we establish the following bound:

**Theorem 1.** Let \( r > 0 \) and \( \mathcal{P} = \{ \mathbf{x}_1, \ldots, \mathbf{x}_N \} \) be a subset of \( \mathbb{R}^k \). Then
\[ D_r(\mathcal{P}) \leq NA_k(r, s) + \sum_{|\mathbf{m}| < s} \left\{ A_k(r, s) + (r/|\mathbf{m}|)^{k/2} |J_{k/2}(2\pi r|\mathbf{m}|)| \right\} \left| \sum_{n=1}^N e(\mathbf{m} \cdot \mathbf{x}_n) \right| \]
for all \( s > 0 \), where
\[ A_k(r, s) = \omega_k s^{-1} r^{k-1} \left\{ \pi rs(J_{k-2}/2(\pi rs)^2) + J_{k/2}(\pi rs)^2 \right\} - (k - 1)J_{k-2}/2(\pi rs)J_{k/2}(\pi rs), \]
\( J_{\nu}(x) \) is the \( \nu \)-th order Bessel function and \( \omega_k = 4\pi^{(k-2)/2} \Gamma(k/2)^{-1} \).

In 1969, W. Schmidt [9] showed that the discrepancy cannot be uniformly small. Suppose that \( \varepsilon > 0 \) and that \( \delta \) satisfies \( N \delta^k \geq 1 \). Schmidt proved that there exists a ball \( B_k(r, \mathbf{c}) \) with \( r \leq \delta \) such that
\[ |\Delta[\mathcal{P}; B_k(r, \mathbf{c})]| > c_1(k, \varepsilon)(N\delta^k)^{(k-1)/2k-\varepsilon}. \]

On the other hand, J. Beck ([2], Theorem 14) has shown that there exists an infinite sequence \( \mathbf{x}_1, \mathbf{x}_2, \ldots \) such that for all \( N \geq 2 \) and any ball \( B_k(r, \mathbf{c}) \) with \( r \leq 1 \) and \( N \delta^k \geq 1 \), we have
\[ |\Delta[\{ \mathbf{x}_1, \ldots, \mathbf{x}_N \}; B_k(r, \mathbf{c})]| \leq c_2(k)(N\delta^k)^{(k-1)/2k}(\log N)^{3/2}. \]

In view of Schmidt’s lower bound, we see that (1.6) must be close to best possible. Applying Theorem 1 and a simple averaging procedure, we may establish the existence of a set of \( p \) points in \( \mathbb{R}^k \) (here \( p \) is a prime) which
has modest discrepancy. (A similar application is given in [8], pp. 154–157.) Although our result is not as sharp as Beck’s, our proof is much simpler and we do not have the requirement that \( r \leq 1 \). For \( h \in \mathbb{Z}^k \), let \( \mathcal{P}_h \) be the collection of \( p \) points of the form \((n/p)h, n = 1, \ldots, p\).

**Theorem 2.** Let \( p \) be a prime number, and suppose that \( r \geq p^{-1/k} \). Then there exists a lattice point \( h \in \mathbb{Z}^k \) such that \(|h| < p\) and

\[
D_r(\mathcal{P}_h) \leq c_3(k)(pr^k)^{(k-1)/k}.
\]

**Notation.** We use the definition for Fourier transforms and series of Stein and Weiss [10]. For \( x \in \mathbb{R}^k \) and \( r > 0 \), \( \chi_r(x) \) denotes the characteristic function of \( B_k(r, 0) \). To simplify expressions, we adopt the convention that \( m \) appearing in a sum will always be a point in \( \mathbb{Z}^k \). Finally, \( \sum' \) means that the term in the sum corresponding to \( m = 0 \) is omitted.

2. Proof of theorems

**Proof of Theorem 1.** We require two auxiliary functions. Combining the results in ([7], Theorem 3) and a \( k \)-dimensional form of the Paley–Wiener theorem ( [10], Chapter III, Theorem 4.9), we see that for \( r > 0 \) and \( s > 0 \) there exist functions \( \mathcal{F}_k(x; r, s) \) and \( \mathcal{G}_k(x; r, s) \) that satisfy

\[
\begin{align*}
\mathcal{F}_k(x; r, s) &\leq \chi_r(x) \leq \mathcal{G}_k(x; r, s) \quad \text{for all } x \in \mathbb{R}^k, \\
\mathcal{F}_k(t; r, s) &\mathcal{G}_k(t; r, s) = 0 \quad \text{for all } |t| \geq s, \\
\int_{\mathbb{R}^k} (\mathcal{G}_k(x; r, s) - \mathcal{F}_k(x; r, s)) \, dx &= A_k(r, s),
\end{align*}
\]

where \( A_k(r, s) \) is defined in the statement of the theorem. From (1.3) we see that for \( c \in \mathbb{R}^k \),

\[
\Delta[\mathcal{B}_k(r, c)] = \sum_{n=1}^{N} \sum_{m} \chi_r(x_n - c + m) - N\mu(B_k(r, 0)).
\]

Now suppose that for a given \( r \) and \( c \) we have \( \Delta[\mathcal{B}_k(r, c)] \geq 0 \). Then by (2.1), the Poisson summation formula and the triangle inequality we have

\[
\begin{align*}
\Delta[\mathcal{B}_k(r, c)] &\leq \sum_{n=1}^{N} \sum_{m} \mathcal{G}_k(x_n - c + m; r, s) - N\mu(B(r, 0)) \\
&= \sum_{n=1}^{N} \sum_{m} \mathcal{G}_k(m; r, s) \delta_m((x_n - c)) - N\chi_r(0)
\end{align*}
\]
Thus it follows from \((2.4)\) and \((2.5)\) that

\[ \sum_{|m| < s} \hat{G}_k(m; r, s)e(-m \cdot c) \sum_{n=1}^{N} e(m \cdot x_n) - N \hat{\chi}_r(0) \]

\[ \leq N(\hat{G}_k(0; r, s) - \hat{\chi}_r(0)) + \sum_{|m| < s} |\hat{G}_k(m; r, s)| \left| \sum_{n=1}^{N} e(m \cdot x_n) \right|. \]

From \((2.1)\) and \((2.3)\) we know that

\[ \hat{G}_k(0; r, s) - \hat{\chi}_r(0) = \int_{\mathbb{R}^k} (\hat{G}_k(x; r, s) - \chi_r(x)) \, dx \leq A_k(r, s). \]

A general expression for \(\hat{G}_k(t; r, s)\) seems difficult to find. However, we can obtain an estimate that will do for our purposes. First note that

\[ \hat{\chi}_r(t) = \int_{\mathbb{R}^k} \chi_r(x)e(-t \cdot x) \, dx = (r/|t|)^{k/2}J_{k/2}(2\pi r|t|). \]

Applying the triangle inequality together with \((2.5)\) and the above identity, we see that

\[ |\hat{G}_k(t; r, s)| \]

\[ \leq \left| \int_{\mathbb{R}^k} (\hat{G}_k(x; r, s) - \chi_r(x))e(-t \cdot x) \, dx \right| + \left| \int_{\mathbb{R}^k} \chi_r(x)e(-t \cdot x) \, dx \right| \]

\[ \leq (\hat{G}_k(0; r, s) - \hat{\chi}_r(0)) + |\hat{\chi}_r(t)| \]

\[ \leq A_k(r, s) + (r/|t|)^{k/2}|J_{k/2}(2\pi r|t|)|. \]

Thus it follows from \((2.4)\) and \((2.5)\) that

\[ \Delta[\mathcal{P}; B_k(r, c)] \]

\[ \leq NA_k(r, s) + \sum_{|m| < s} \left\{ A_k(r, s) + (r/|m|)^{k/2}|J_{k/2}(2\pi r|m|)| \right\} \left| \sum_{n=1}^{N} e(m \cdot x_n) \right|. \]

If it should happen that for a given \(r\) and \(c\) we have \(\Delta[\mathcal{P}; B_k(r, c)] < 0\), then following the preceding analysis using \(\mathcal{F}_k(x; r, s)\) in place of \(\hat{G}_k(x; r, s)\) yields inequality \((2.6)\) with a minus sign attached to the left-hand term. Combining these bounds verifies \((1.5)\) and completes the proof.

**Proof of Theorem 2.** We begin by using some estimates to simplify the bound given in Theorem 1. If \(rs \geq 1\), then \(A_k(r, s) \ll s^{-1}r^{k-1}\) (see [7], Theorem 1). Combining this with the bound \(|J_\nu(x)| \leq 1\) for \(\nu > 0\) and \(x > 0\) reduces \((1.5)\) to

\[ D_r(\mathcal{P}) \llk Ns^{-1}r^{k-1} + \sum_{|m| < s} \{s^{-1}r^{k-1} + (r/|m|)^{k/2}\} \left| \sum_{n=1}^{N} e(m \cdot x_n) \right|. \]
For a prime $p$, if $\mathbf{h}$ and $\mathbf{m}$ are lattice points, then
\[
\sum_{n=1}^{p} e((n/p) \mathbf{h} \cdot \mathbf{m}) = \begin{cases} 
0 & \text{if } \mathbf{h} \cdot \mathbf{m} \not\equiv 0 \pmod{p}, \\
p & \text{if } \mathbf{h} \cdot \mathbf{m} \equiv 0 \pmod{p}.
\end{cases}
\]
Therefore we have
\[
(2.7) \quad D_r(\mathcal{P}_\mathbf{h}) \ll_k p s^{-1} r^{k-1} + \sum_{\substack{|\mathbf{m}| \leq s \\ \mathbf{h} \cdot \mathbf{m} \equiv 0}} \{s^{-1} r^{k-1} + (r/|\mathbf{m}|)^{k/2}\} p.
\]
The sum above is difficult to handle alone, but the problem simplifies if we average over all lattice points $\mathbf{h}$ such that $|\mathbf{h}| \leq p$. On doing so, the right side of (2.7) is equal to
\[
(2.8) \quad ps^{-1} r^{k-1} + p(\mathbb{Z}^k; B_k(p, 0))^{-1} \sum_{|\mathbf{h}| \leq p} \sum_{\substack{|\mathbf{m}| < s \\ \mathbf{h} \cdot \mathbf{m} \equiv 0}} \{s^{-1} r^{k-1} + (r/|\mathbf{m}|)^{k/2}\} \sum_{|\mathbf{m}| < s} 1.
\]
For each $\mathbf{m}$ in the outer sum on the right side of (2.8), there is at least one nonzero component $m_g$. We fix such an $\mathbf{m}$, and consider the inner sum. For a given $\mathbf{h}$, once the components $h_1, \ldots, h_{g-1}, h_{g+1}, \ldots, h_k$ are set, there is only one choice for $h_g$ (mod $p$) for which $\mathbf{h} \cdot \mathbf{m} \equiv 0 \pmod{p}$. Since $|\mathbf{h}| \leq p$, there are at most three possible choices for $h_g$. Furthermore, the other components $h_1, \ldots, h_{g-1}, h_{g+1}, \ldots, h_k$ must satisfy $|h_j| \leq p$ for each appropriate $j$. Thus for $\mathbf{m} \neq 0$,
\[
(2.9) \quad \sum_{\substack{|\mathbf{h}| \leq p \\ \mathbf{h} \cdot \mathbf{m} \equiv 0}} 1 \ll_k p^{k-1}.
\]
We also note that for each $i \geq 2$, there exist constants $c_4(i)$ and $c_5(i)$ such that
\[
(2.10) \quad c_4(i) p^i \leq Z[\mathbb{Z}^i; B_i(p, 0)] \leq c_5(i) p^i.
\]
(See [6], Theorem 339, for a discussion of the case $i = 2$.) As we are averaging, we see that (2.7)–(2.10) imply there exists a lattice point $\mathbf{h}$ with $|\mathbf{h}| \leq p$ such that
\[
D_r(\mathcal{P}_\mathbf{h}) \ll_k ps^{-1} r^{k-1} + \sum_{|\mathbf{m}| < s} \{s^{-1} r^{k-1} + (r/|\mathbf{m}|)^{k/2}\}.
\]
Applying the inequality
\[
\sum_{|\mathbf{m}| < s} |\mathbf{m}|^{-k/2} \ll s^{k/2}
\]
and assuming that \( rs \geq 1 \) (required for our bound on \( A_k(r,s) \) to be valid), we find that
\[
D_r(P_h) \ll_k ps^{-1}r^{k-1} + (sr)^{k-1}.
\]
The expression on the right above is minimized upon setting \( s = p^{1/k} \), which yields
\[
D_r(P_h) \ll_k (pr^k)^{(k-1)/k},
\]
and completes the proof.

References