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On the diophantine equation $\frac{x^m - 1}{x - 1} = y^n$

by

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1. Introduction. Let $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Q}$ be the sets of integers, positive integers and rational numbers respectively. The solutions (x, y, m, n) of the equation

$$(1)\qquad \frac{x^m - 1}{x - 1} = y^n, \quad x, y, m, n \in \mathbb{N},\ x > 1,\ y > 1,\ m > 2,\ n > 1,$$

were investigated in many papers. In this respect, Ljunggren [6] proved that equation (1) has only the solutions $(x, y, m, n) = (3, 11, 5, 2)$ and $(7, 20, 4, 2)$ with $n = 2$. Shorey and Tijdeman [10, Theorem 12·5] showed that equation (1) has only finitely many solutions (x, y, m, n) if at least one of the following conditions holds: (i) x is fixed, (ii) m is fixed, (iii) n has a fixed prime factor, (iv) y has a fixed prime factor. Moreover, all the solutions can be effectively determined. In general, Shorey and Tijdeman conjectured that equation (1) has only finitely many solutions (x, y, m, n). This problem has not been resolved yet.

Recently, Le [4] proved that if (x, y, m, n) is a solution of (1) such that x is a prime power and $y \equiv 1 \pmod{x}$, then $x^m < C$, where C is an effectively computable absolute constant. In this paper we improve this result as follows:

Theorem 1. *The equation* (1) *has no solution* (x, y, m, n) *with* $y \equiv 1$ $(\mathrm{mod}\ x)$.

For fixed $a, b \in \mathbb{N}$ with $\gcd(a, b) = 1$, we denote by $\mathrm{ord}_b\, a$ the least positive integer value of t for which $a^t \equiv 1 \pmod{b}$. In [1], Edgar showed that if (x, y, m, n) is a solution of (1), then $m = \mathrm{ord}_y\, x$. Simultaneously, he asked if

$$(2)\qquad n = \mathrm{ord}_x\, y.$$

Edgar offered \$50.00 for settling this problem. On using Theorem 1, we can completely solve this problem as follows:

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THEOREM 2. *Every solution of* (1) *satisfies* (2).

Let $\mathbb{P}$ be the set of odd primes. Edgar asked if there exists a solution (x, y, m, n) of (1) which satisfies $x, y \in \mathbb{P}$, $m \geq 5$ and $(x, y, m, n) \neq (3, 11, 5, 2)$ (see [2, Problem D10]). Shi [8] posed a similar problem for $x = 3$. In this paper we prove the following result.

THEOREM 3. *If* (x, y, m, n) *is a solution of* (1) *with* $x \in \mathbb{P}$, *then* $\gcd(x(x-1), n) = 1$. *Moreover, the equation* (1) *has only the solution* $(x, y, m, n) = (3, 11, 5, 2)$ *with* $x = 3$.

Let $a, m \in \mathbb{N}$ with $1 \leq a \leq 9$ and $m > 1$, and let $N_a(m) = a + 10a + \ldots + 10^{m-1}a$. Obláth [7] proved that if $a > 1$, then $N_a(m)$ is never a perfect power. Shorey and Tijdeman [9, Theorem 6] proved that $N_1(m)$ is not a pth power for any $p \in \mathbb{P}$ with $p \leq 19$. In this respect, we have:

THEOREM 4. $N_1(m)$ *is never a perfect power.*

All the above-mentioned results are in tune with the conjecture of Shorey and Tijdeman.

2. Proofs of Theorems 1 and 2

Proof of Theorem 1. Let (x, y, m, n) be a solution of (1) with $y \equiv 1 \pmod{x}$. By [6], the only solutions of (1) with $2 \mid n$ are $(x, y, m, n) = (3, 11, 5, 2)$ and $(7, 20, 4, 2)$. We may assume that $2 \nmid n$. Since $n > 1$, n must has an odd prime factor p. Then $(x, y^{n/p}, m, p)$ is a solution of (1) satisfying $y^{n/p} \equiv 1 \pmod{x}$ and p is an odd prime. Therefore, it suffices to consider the case that n is an odd prime.

If $n \mid x$ and $n^\alpha \parallel x$, then from $y^n - 1 = x(x^{m-2} + \ldots + x + 1)$ we get $n^\alpha \parallel y^n - 1$, hence $n^{\alpha-1} \parallel y - 1$, which contradicts $y \equiv 1 \pmod{x}$. So we have $n \nmid x$.

If $x - 1$ is an nth power, then $x - 1 = y_1^n$, and $x^m - 1 = (y_1 y)^n$ for some $y_1 \in \mathbb{N}$. This is impossible by the proof of [10, Theorem 12.3]. Thus $(x-1)^{1/n} \notin \mathbb{Q}$.

Let $\theta = (x-1)^{1/n}$, and let $K = \mathbb{Q}(\theta)$. Then K is an algebraic number field of degree n. Let O_K and U_K be the algebraic integer ring and the unit group of K respectively. For $\alpha_1, \ldots, \alpha_r \in O_K$, let $[\alpha_1, \ldots, \alpha_r]$ be the ideal of K generated by $\alpha_1, \ldots, \alpha_r$, and let $N([\alpha_1, \ldots, \alpha_r])$ denote the norm of $[\alpha_1, \ldots, \alpha_r]$. Since $\theta = (x-1)^{1/n}$, we have

$$[x] = [1 + \theta^n] = [1 + \theta]\left[\frac{1 + \theta^n}{1 + \theta}\right]. \tag{3}$$

On the other hand, by (1), we get

$$[x]^m = [1 + y^n\theta^n] = [1 + y\theta]\left[\frac{1 + y^n\theta^n}{1 + y\theta}\right]. \tag{4}$$

Since $y \equiv 1 \pmod{x}$, we have

$$[1+\theta] \mid [1+y\theta]. \tag{5}$$

Further, since $n \nmid x$, the ideals $[1+y\theta]$ and $[(1+y^n\theta^n)/(1+y\theta)]$ are coprime. Therefore, by (4) and (5), we get

$$[1+\theta]^m \mid [1+y\theta]. \tag{6}$$

Notice that $N([1+\theta]) = x$ and $N([1+y\theta]) = x^m$. We find from (6) that $[1+y\theta] = [1+\theta]^m$. So we have

$$1+y\theta = (1+\theta)^m\varepsilon, \quad \varepsilon \in U_K,\ N_{K/\mathbb{Q}}(\varepsilon) = 1, \tag{7}$$

where $N_{K/\mathbb{Q}}(\varepsilon)$ is the norm of ε. By the proof of [5, Theorem], (7) is impossible. Thus, the equation (1) has no solution (x,y,m,n) with $y \equiv 1 \pmod{x}$. The theorem is proved.

Proof of Theorem 2. Let $k = \mathrm{ord}_x y$. If (x,y,m,n) is a solution of (1), then we have $y^n \equiv 1 \pmod{x}$ and $n \equiv 0 \pmod{k}$. Let $n' = n/k$ and $y' = y^k$. If $n' > 1$, then (x,y',m,n') is a solution of (1) with $y' \equiv 1 \pmod{x}$. By Theorem 1, this is impossible. Thus $n' = 1$ and the theorem is proved.

3. Proofs of Theorems 3 and 4.

Theorems 3 and 4 are based on the following result from [5]:

LEMMA. *Equation* (1) *has no solution* (x,y,m,n) *with* $\gcd(x\varphi(x),n) = 1$, *where* $\varphi(x)$ *is Euler's totient function of* x.

Proof of Theorem 3. Since $\varphi(x) = x-1$ for any $x \in \mathbb{P}$, the first part of Theorem 3 is an immediate consequence of the Lemma.

By the Lemma with $x = 3$, we get either $2 \mid n$ or $3 \mid n$. By [6], the equation (1) has only the solution $(x,y,m,n) = (3,11,5,2)$ with $x = 3$ and $2 \mid n$. In [3], Inkeri showed that equation (1) has only the solution $(x,y,m,n) = (18,7,3,3)$ with $1 < x < 70$ and $3 \mid n$. Thus the theorem is proved.

Proof of Theorem 4. If $N_1(m)$ is a perfect power, then equation (1) has a solution (x,y,m,n) with $x = 10$. By the Lemma, we get either $2 \mid n$ or $5 \mid n$. By [9, Theorem 6], this is impossible. The proof is complete.

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