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On the Piatetski-Shapiro–Vinogradov Theorem

by

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1. Introduction. In 1937, I. M. Vinogradov proved the famous three primes theorem. It states that for every sufficiently large odd integer N,

$$\sum_{N=p_1+p_2+p_3} 1 = \frac{1}{2}(1+o(1))C(N)\frac{N^2}{\log^3 N}, \tag{1}$$

where

$$C(N) = \prod_{p\mid N}\left(1-\frac{1}{(p-1)^2}\right)\prod_{p\nmid N}\left(1+\frac{1}{(p-1)^3}\right). \tag{2}$$

Afterwards people have been looking for thin subsets of primes for which the three primes theorem still holds. In 1986, Wirsing [13] showed that there exists such a set S with the property that

$$\sum_{\substack{p\le x\\ p\in S}} 1 \ll (x\log x)^{1/3}.$$

It is also interesting to find more familiar thin sets of primes which serve this purpose. An example is the set of Piatetski-Shapiro primes of type γ which are of the form $[n^{1/\gamma}]$. We denote this set by P_γ.

For the counting function of P_γ, Piatetski-Shapiro [11] first showed that for $11/12 < \gamma \le 1$ (the case $\gamma > 1$ is trivial),

$$P_\gamma(x) = \sum_{\substack{p\le x\\ p=[n^{1/\gamma}]}} 1 = (1+o(1))\frac{x^\gamma}{\log x}. \tag{3}$$

Heath-Brown [4] extended the range to $662/755 < \gamma \le 1$. Further improvements were made by Kolesnik [8], Liu and Rivat [9].

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Jia Chaohua [7] was the first to apply the sieve method to the investigation of Piatetski-Shapiro primes and proved that, for $17/20 < \gamma \le 1$ and sufficiently large x,

$$P_\gamma(x) \ge \frac{\varrho_0 x^\gamma}{\log x}, \tag{4}$$

where ϱ_0 is a definite positive constant. Using the sieve method of Harman [3], Jia Chaohua [6] extended the range in (4) to $11/13 < \gamma \le 1$.

Now we come back to the subject of this paper. In 1992, Balog and Friedlander [1] proved the Piatetski-Shapiro–Vinogradov theorem: *If* γ_1, γ_2, γ_3 *are fixed subject to* $0 < \gamma_i \le 1$ *and* $9(1-\gamma_3) < 1$, $9(1-\gamma_2)+6(1-\gamma_3) < 1$, $9(1-\gamma_1)+6(1-\gamma_2)+6(1-\gamma_3) < 1$, *then for every sufficiently large odd integer* N,

$$\begin{aligned} T(N) &= \sum_{\substack{N=p_1+p_2+p_3\\ p_i\in P_{\gamma_i}}} 1 \\ &= (1+o(1))\frac{\gamma_1\gamma_2\gamma_3\Gamma(\gamma_1)\Gamma(\gamma_2)\Gamma(\gamma_3)}{\Gamma(\gamma_1+\gamma_2+\gamma_3)}\cdot\frac{C(N)N^{\gamma_1+\gamma_2+\gamma_3-1}}{\log^3 N}. \end{aligned} \tag{5}$$

From this theorem, there are two interesting corollaries:

COROLLARY 1. *For any fixed* $20/21 < \gamma \le 1$, *every sufficiently large odd integer* N *can be written as a sum of three Piatetski-Shapiro primes of type* γ.

COROLLARY 2. *For any fixed* $8/9 < \gamma \le 1$, *every sufficiently large odd integer* N *can be written as a sum of one Piatetski-Shapiro prime of type* γ *and two primes.*

In his doctoral thesis, J. Rivat extended the range $20/21 < \gamma \le 1$ in Corollary 1 to $188/199 < \gamma \le 1$.

In this paper, we shall apply the sieve method combined with the circle method to this problem and prove:

THEOREM. *If* γ *is fixed with* $15/16 < \gamma \le 1$, *then for every sufficiently large odd integer* N,

$$T_1(N) = \sum_{\substack{N=p_1+p_2+p_3\\ p_i\in P_\gamma}} 1 \ge \frac{\varrho_0 C(N) N^{3\gamma-1}}{\log^3 N}, \tag{6}$$

where ϱ_0 *is a definite positive constant.*

Our Theorem improves Corollary 1 of Balog and Friedlander.

Throughout this paper, we always assume that N is a sufficiently large odd integer and ε is a sufficiently small positive constant. Assume that c, c_1, c_2 are positive constants which have different values at different places.

$m \sim M$ means that there are positive constants c_1 and c_2 such that $c_1 M < m \le c_2 M$. We also assume that γ is fixed with $15/16 < \gamma \le 20/21$ and that

$$N(d) = [-d^\gamma] - [-(d+1)^\gamma]. \tag{7}$$

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2. Some preliminary lemmas. In the following, we assume that

$$H = N^{1-\gamma+\Delta+8\varepsilon}. \tag{8}$$

By the discussion in [1], the asymptotic formula that for $0 \le \Delta \le 1-\gamma$,

$$\sum_{N/10<p\le N} N(p)e(\alpha p) = \gamma \sum_{N/10<p\le N} p^{\gamma-1}e(\alpha p) + O(N^{\gamma-\Delta-5\varepsilon}), \tag{9}$$

depends on the fact that for $J \le H$ and $0 \le u \le 1$,

$$\min\left(1, \frac{N^{1-\gamma}}{J}\right) \sum_{h\sim J} \Big| \sum_{n\sim N} \Lambda(n) e(\alpha n + h(n+u)^\gamma) \Big| \ll N^{1-\Delta-6\varepsilon}. \tag{10}$$

Lemma 1. *Assume that* $N^{1-\gamma+2\Delta+30\varepsilon} \ll M \ll N^{5\gamma-4-6\Delta-120\varepsilon}$ *and that* $a(m), b(k) = O(1)$. *Then for* $J \le H$ *and* $0 \le u \le 1$, *we have*

$$\min\left(1, \frac{N^{1-\gamma}}{J}\right) \sum_{h\sim J} \Big| \sum_{m\sim M} \sum_{km\sim N} a(m)b(k)e(\alpha km + h(km+u)^\gamma) \Big| \ll N^{1-\Delta-10\varepsilon}.$$

This is Proposition 2 of [1].

Lemma 2. *Assume that* $M \ll N^{4\gamma-3-5\Delta-50\varepsilon}$, $a(m) = O(1)$ *and*

$$6(1-\gamma) + \frac{19}{3}\Delta < 1. \tag{11}$$

Then for $J \le H$ *and* $0 \le u \le 1$, *we have*

$$\min\left(1, \frac{N^{1-\gamma}}{J}\right) \sum_{h\sim J} \Big| \sum_{m\sim M} a(m) \sum_{km\sim N} e(\alpha km + h(km+u)^\gamma) \Big| \ll N^{1-\Delta-10\varepsilon}.$$

This is Proposition 3 of [1].

Lemma 3. *Assume that* $a(m), b(k) = O(1)$ *and that for* $V \ll M \ll N/V$,

$$\sum_{m\sim M} \sum_{km\sim N} a(m)b(k)F(km) \ll \Phi N^{-10\varepsilon},$$

and for $M \ll V^2$,

$$\sum_{m\sim M} a(m) \sum_{km\sim N} F(km) \ll \Phi N^{-10\varepsilon}.$$

Then

$$\sum_{n\sim N} \varLambda(n)F(n) \ll \varPhi N^{-6\varepsilon}.$$

This can be deduced from Vaughan's identity.

Lemma 4. *Under the above assumptions, we have*

$$\sum_{N/10<p\le N} N(p)e(\alpha p) = \gamma \sum_{N/10<p\le N} p^{\gamma-1}e(\alpha p) + O(N^{3\gamma/2-1/2-5\varepsilon}). \tag{12}$$

Proof. In order to prove (12), we should prove (10) with $\varDelta = \frac{1}{2}(1-\gamma)$. Now,

$$\min\left(1, \frac{N^{1-\gamma}}{J}\right) \sum_{h\sim J} \Big| \sum_{n\sim N} \varLambda(n)e(\alpha n + h(n+u)^{\gamma})\Big|$$
$$= \min\left(1, \frac{N^{1-\gamma}}{J}\right) \sum_{h\sim J} \varepsilon(h,\alpha,u) \sum_{n\sim N} \varLambda(n)e(\alpha n + h(n+u)^{\gamma}).$$

Let $V = N^{2(1-\gamma)+30\varepsilon}$ and

$$F(n) = \min\left(1, \frac{N^{1-\gamma}}{J}\right) \sum_{h\sim J} \varepsilon(h,\alpha,u)e(\alpha n + h(n+u)^{\gamma}).$$

Note that (11) is satisfied for $0 \le \varDelta \le 1-\gamma$. By Lemmas 1–3, we can get

$$\sum_{n\sim N} \varLambda(n)F(n) \ll N^{3\gamma/2-1/2-6\varepsilon}.$$

So Lemma 4 follows.

Lemma 5. *Assume that* $|\alpha - a/q| < 1/q^2$, $(a,q)=1$. *Then*

$$\sum_{p\sim N} e(\alpha p) \ll \left(\frac{N}{\sqrt{q}} + \sqrt{Nq} + N^{4/5}\right)\log^5 N.$$

We refer to Section 25 of [2].

Lemma 6. *Let*

$$d_r(n) = \sum_{n=n_1\ldots n_r} 1$$

and k *be a positive integer. Then*

$$\sum_{n\le x} d_r^k(n) \ll x(\log x)^{r^k-1}.$$

See Theorem 2 of [12].

LEMMA 7. *Let*

$$C(q,m)=\sum_{\substack{r=1\\(r,q)=1}}^{q} e\Big(\frac{rm}{q}\Big). \tag{13}$$

Then $C(q,m)$ is a multiplicative function of q and

$$C(q,m)=\mu\Big(\frac{q}{(q,m)}\Big)\frac{\varphi(q)}{\varphi\big(\frac{q}{(m,q)}\big)}. \tag{14}$$

See Lemma 1.2 of [10].

We define $w(u)$ as the continuous solution of the equations

$$\begin{aligned} w(u)&=1/u, \qquad 1\le u\le 2,\\ (uw(u))'&=w(u-1), \qquad u>2. \end{aligned} \tag{15}$$

$w(u)$ is called *Buchstab's function*; it plays an important role in finding asymptotic formulas in the sieve method. In particular,

$$w(u)=\begin{cases} \dfrac{1+\log(u-1)}{u}, & 2\le u\le 3; \quad (16)\\[2mm] \dfrac{1+\log(u-1)}{u}+\dfrac{1}{u}\displaystyle\int_2^{u-1}\frac{\log(t-1)}{t}\,dt, & 3\le u\le 4. \quad (17)\end{cases}$$

LEMMA 8. *We have the following bounds*:

(i) $w(u)\ge 0.5607$ *for* $u\ge 2.47$;
(ii) $w(u)\le 0.5644$ *for* $u\ge 3$;
(iii) $w(u)\le 0.5672$ *for* $u\ge 1.7631$;
(iv) $w(u)\ge 0.5$ *for* $u\ge 1$.

Proof. (i) First assume $3\le u\le 4$. By (15), we have

$$w'(u)=\frac{w(u-1)-w(u)}{u}=\frac{1}{u^2}(uw(u-1)-uw(u)).$$

Now we investigate the behaviour of the function

$$h(u)=uw(u-1)-uw(u). \tag{18}$$

We have

$$\begin{aligned} h'(u)&=((u-1)w(u-1))'-(uw(u))'+w'(u-1)\\ &=w(u-2)-w(u-1)+\frac{w(u-2)-w(u-1)}{u-1}\\ &=\frac{u(1-(u-2)\log(u-2))}{(u-2)(u-1)^2}. \end{aligned} \tag{19}$$

There is exactly one u_0 satisfying

$$1-(u_0-2)\log(u_0-2)=0.$$

Calculation shows $3.7632\le u_0\le 3.7633$.

We have $h'(u) > 0$ to the left of u_0, and $h'(u) < 0$ to the right of u_0. Consequently, if $u_0 \le u \le 4$, then $h(u) \ge h(4) = 4(w(3) - w(4)) > 0$. Note that $h(3) = 3(w(2) - w(3)) < 0$ and that $h(u)$ is an increasing function in the interval $[3, u_0]$. So, $h(u)$ has exactly one zero v_0 in the interval $[3, 4]$. Calculation shows $3.469 \le v_0 \le 3.47$. We have $h(u) < 0$ to the left of v_0, and $h(u) > 0$ to the right of v_0. The same holds for $w'(u)$. Hence, $w(u) \ge w(v_0)$.

We note that for $3 \le u \le 4$,

$$\left(\frac{1 + \log(u-1)}{u}\right)' = \frac{1 - (u-1)\log(u-1)}{(u-1)u^2} < 0.$$

The fact that for $2 \le t \le 3$,

$$\left(\frac{\log(t-1)}{t}\right)' = \frac{t - (t-1)\log(t-1)}{(t-1)t^2} \ge 0,$$

implies that for $3 \le u \le 4$,

$$\begin{aligned}\left(\frac{1}{u}\int_2^{u-1}\frac{\log(t-1)}{t}dt\right)' &= \frac{1}{u^2}\left(\frac{u\log(u-2)}{u-1} - \int_2^{u-1}\frac{\log(t-1)}{t}\,dt\right)\\ &\ge \frac{1}{u^2}\left(\frac{u\log(u-2)}{u-1} - \frac{(u-3)\log(u-2)}{u-1}\right) \ge 0.\end{aligned}$$

Therefore we have

$$w(v_0) \ge \frac{1 + \log(2.47)}{3.47} + \frac{1}{3.469}\int_2^{2.469}\frac{\log(t-1)}{t}\,dt \ge 0.5607.$$

From the above discussion, we get $w(u) \ge 0.5607$ for $3 \le u \le 4$.

Then we employ induction. Suppose that $w(u) \ge 0.5607$ for $3 \le k \le u \le k+1$. For $k+1 \le u \le k+2$, we have

$$uw(u) = (k+1)w(k+1) + \int_k^{u-1} w(t)\,dt \ge 0.5607u.$$

By induction, we conclude that $w(u) \ge 0.5607$ for $u \ge 3$.

Now we turn to the case $2 \le u \le 3$. Then

$$w'(u) = \frac{1 - (u-1)\log(u-1)}{(u-1)u^2}$$

has exactly one zero z_0 with $2.7632 \le z_0 \le 2.7633$.

We have $w'(u) > 0$ to the left of z_0, and $w'(u) < 0$ to the right of z_0. Therefore $w(u) \ge \min(w(2.47), w(3)) \ge 0.5607$ for $2.47 \le u \le 3$, and $w(u) \le w(z_0) \le (1 + \log(1.7633))/2.7632 \le 0.5672$ for $2 \le u \le 3$.

(ii) The discussion in (i) implies that $w(u) \le \max(w(3), w(4)) \le 0.5644$ for $3 \le u \le 4$. By induction, it follows that $w(u) \le 0.5644$ for $u \ge 3$.

(iii) The discussion in (i) shows that $w(u) \le 0.5672$ for $2 \le u \le 3$. For $1.7631 \le u \le 2$, we have $w(u) = 1/u \le 0.5672$.

(iv) It is easy to see that $w(u) = 1/u \ge 0.5$ for $1 \le u \le 2$. By induction, we get the same conclusion for all $u \ge 1$.

The proof of Lemma 8 is complete.

LEMMA 9. *Assume that* $\mathcal{E} = \{n : x < n \le 2x\}$ *and that* $z \le x$. *Let*

$$P(z) = \prod_{p<z} p.$$

Then for sufficiently large x *and* z, *we have*

$$S(\mathcal{E}, z) = \sum_{\substack{x<n\le 2x\\ (n,P(z))=1}} 1 = \left(w\left(\frac{\log x}{\log z}\right) + O(\varepsilon)\right)\frac{x}{\log z}.$$

We refer to Lemma 5 of [6]. When $(2x)^{1/2} < z \le x$, this is the prime number theorem.

3. Mean value formulas in the sieve method

LEMMA 10. *Assume that* $M, K \ll N^{5/16}$ *and that* $a(m)$, $b(k) = O(1)$. *Let*

$$I(N) = \sum_{\substack{N=n_1+n_2+n_3\\ N/10<n_1,n_2,n_3\le N}} \frac{\gamma^3(n_1n_2n_3)^{\gamma-1}}{\log n_2 \log n_3} \tag{20}$$

and

$$\omega(r) = \prod_{\substack{p\mid r\\ p\mid N}} \left(1+\frac{1}{p-1}\right) \prod_{\substack{p\mid r\\ p\nmid N}} \left(1-\frac{1}{(p-1)^2}\right). \tag{21}$$

Then

$$\sum_{m\sim M}\sum_{k\sim K} a(m)b(k)\Bigg(\sum_{\substack{N=mkl+p_2+p_3\\ N/10<mkl\le N\\ N/10<p_2,\,p_3\le N}} N(mkl)N(p_2)N(p_3) - \frac{\omega(mk)}{mk} I(N)\Bigg) = O\left(\frac{N^{3\gamma-1}}{\log^{20} N}\right).$$

P r o o f. We have

$$\Sigma_1 = \sum_{m\sim M}\sum_{k\sim K} a(m)b(k) \sum_{\substack{N=mkl+p_2+p_3\\ N/10<mkl\le N\\ N/10<p_2,\,p_3\le N}} N(mkl)N(p_2)N(p_3)$$

$$= \int_0^1 \sum_{\substack{N/10<mkl\le N\\ m\sim M,\, k\sim K}} a(m)b(k)N(mkl)e(\alpha mkl) \times \Big(\sum_{N/10<p\le N} N(p)e(\alpha p)\Big)^2 e(-\alpha N)\, d\alpha.$$

Let

$$g(\alpha) = \sum_{\substack{N/10<mkl\le N\\ m\sim M,\, k\sim K}} a(m)b(k)N(mkl)e(\alpha mkl),$$

$$f(\alpha) = \gamma \sum_{\substack{N/10<mkl\le N\\ m\sim M,\, k\sim K}} a(m)b(k)(mkl)^{\gamma-1}e(\alpha mkl).$$

By the discussion in [1], the asymptotic formula

$$g(\alpha) = f(\alpha) + O(N^{2\gamma-1-5\varepsilon}) \tag{22}$$

depends on the fact that for $J \le H_1 = N^{2(1-\gamma)+8\varepsilon}$ and $0 \le u \le 1$,

$$\Sigma_2 = \min\Big(1, \frac{N^{1-\gamma}}{J}\Big) \times \sum_{h\sim J} \Big| \sum_{m\sim M} \sum_{k\sim K} \sum_{mkl\sim N} a(m)b(k)e(\alpha mkl + h(mkl+u)^{\gamma})\Big| \ll N^{\gamma-6\varepsilon}. \tag{23}$$

If either M or K is larger than $N^{3/16}$, then by Lemma 1 with $\Delta = 1-\gamma$, we obtain $\Sigma_2 \ll N^{\gamma-10\varepsilon}$. If $M, K \le N^{3/16}$, then $KM \ll N^{6/16} \ll N^{9\gamma-8-50\varepsilon}$. By Lemma 2 with $\Delta = 1-\gamma$, we also get $\Sigma_2 \ll N^{\gamma-10\varepsilon}$. Hence, (22) holds.

Let

$$D(\alpha) = \sum_{N/10<p\le N} N(p)e(\alpha p), \qquad S(\alpha) = \gamma \sum_{N/10<p\le N} p^{\gamma-1}e(\alpha p).$$

From (12) and (22), it follows that

$$\begin{aligned} g(\alpha)D^2(\alpha) - f(\alpha)S^2(\alpha) &= (g(\alpha)-f(\alpha))D^2(\alpha) + f(\alpha)(D(\alpha)-S(\alpha))D(\alpha)\\ &\quad + f(\alpha)S(\alpha)(D(\alpha)-S(\alpha))\\ &\ll N^{2\gamma-1-5\varepsilon}|D(\alpha)|^2 + N^{3\gamma/2-1/2-5\varepsilon}|f(\alpha)D(\alpha)|\\ &\quad + N^{3\gamma/2-1/2-5\varepsilon}|f(\alpha)S(\alpha)|. \end{aligned}$$

Thus

$$\Sigma_1 = \int_0^1 g(\alpha)D^2(\alpha)e(-\alpha N)\, d\alpha = \int_0^1 f(\alpha)S^2(\alpha)e(-\alpha N)\, d\alpha + \Psi,$$

where

$$(24)\quad \Psi \ll N^{2\gamma-1-5\varepsilon}\int_0^1 |D(\alpha)|^2\,d\alpha + N^{3\gamma/2-1/2-5\varepsilon}\Big(\int_0^1 |f(\alpha)|^2\,d\alpha\Big)^{1/2}$$

$$\times\Big(\int_0^1 |D(\alpha)|^2\,d\alpha\Big)^{1/2} + N^{3\gamma/2-1/2-5\varepsilon}\Big(\int_0^1 |f(\alpha)|^2\,d\alpha\Big)^{1/2}$$

$$\times\Big(\int_0^1 |S(\alpha)|^2\,d\alpha\Big)^{1/2}$$

$$\ll N^{3\gamma-1-4\varepsilon},$$

where we note that $N(p)=0$ or 1 and that $p\in P_\gamma$ is equivalent to $N(p)=1$; we also use the estimation $\sum_{p\le N} N(p)\le \sum_{n\le N} N(n)\ll N^\gamma$.

In the following we investigate

$$\Sigma_3=\int_0^1 f(\alpha)S^2(\alpha)e(-\alpha N)\,d\alpha.$$

Let $Q=N\log^{-80}N$. We divide the interval $[-1/Q,1-1/Q)$ into two parts:

$$E_1=\{\alpha:\alpha\in[-1/Q,1-1/Q),\alpha=a/q+\beta,q\le\log^{80}N,$$
$$0\le a\le q-1,(a,q)=1,\ |\beta|\le 1/(qQ)\},$$
$$E_2=[-1/Q,1-1/Q)-E_1.$$

Then

$$\Sigma_3=\Big(\int_{E_1}+\int_{E_2}\Big)f(\alpha)S^2(\alpha)e(-\alpha N)\,d\alpha.$$

For any $\alpha\in E_2$, there is one q ($\log^{80}N<q\le Q$) such that $|\alpha-a/q|<1/(qQ)$. Lemma 5 yields $S(\alpha)\ll N^\gamma\log^{-35}N$. Hence,

$$\int_{E_2} f(\alpha)S^2(\alpha)e(-\alpha N)\,d\alpha$$

$$\ll \frac{N^\gamma}{\log^{35}N}\Big(\int_0^1 |f(\alpha)|^2\,d\alpha\Big)^{1/2}\Big(\int_0^1 |S(\alpha)|^2\,d\alpha\Big)^{1/2}\ll \frac{N^{3\gamma-1}}{\log^{20}N}.$$

If $\alpha=a/q+\beta\in E_1$, then

$$f(\alpha)=\sum_{\substack{N/10<rl\le N\\ r\sim R}} j(r)(rl)^{\gamma-1}e(\alpha rl),$$

where $R=MK$ and

$$j(r)=\gamma\sum_{\substack{mk=r\\ m\sim M,\,k\sim K}} a(m)b(k).$$

If $q \nmid b$, then

$$\sum_{l \le x} e(bl/q) = O(q).$$

From this and partial summation, we know that

$$\begin{aligned}
&\sum_{\substack{N/10<rl\le N\\ r\sim R,\, q\nmid r}} j(r)(rl)^{\gamma-1}e(\alpha rl)\\
&\qquad = \sum_{\substack{r\sim R\\ q\nmid r}} j(r)r^{\gamma-1}\sum_{N/(10r)<l\le N/r} l^{\gamma-1}e\Big(\frac{arl}{q}+\beta rl\Big)\\
&\qquad \ll N^{\gamma-1}\sum_{r\sim R} d(r)\log^{80}N \ll N^{\gamma-\varepsilon}.
\end{aligned}$$

On the other hand,

$$\sum_{\substack{N/10<rl\le N\\ r\sim R,\, q\mid r}} j(r)(rl)^{\gamma-1}e(\alpha rl) = \sum_{\substack{r\sim R\\ q\mid r}} j(r)r^{\gamma-1}\sum_{N/(10r)<l\le N/r} l^{\gamma-1}e(\beta rl).$$

Now,

$$\begin{aligned}
&\sum_{N/(10r)<l\le N/r} l^{\gamma-1}e(\beta rl)\\
&\qquad = \int_{N/(10r)}^{N/r} t^{\gamma-1}e(\beta rt)\,d[t]\\
&\qquad = \int_{N/(10r)}^{N/r} t^{\gamma-1}e(\beta rt)\,dt - \int_{N/(10r)}^{N/r} t^{\gamma-1}e(\beta rt)\,d(\{t\})\\
&\qquad = \frac{1}{r^{\gamma}}\int_{N/10}^{N} u^{\gamma-1}e(\beta u)\,du + O\bigg(\Big(\frac{N}{r}\Big)^{\gamma-1}\log^{80}N\bigg)\\
&\qquad = \frac{1}{r^{\gamma}}\sum_{N/10<s\le N} s^{\gamma-1}e(\beta s) + O\bigg(\Big(\frac{N}{r}\Big)^{\gamma-1}\log^{80}N\bigg).
\end{aligned}$$

Thus,

$$f(\alpha) = \sum_{\substack{r\sim R\\ q\mid r}} \frac{j(r)}{r}\sum_{N/10<s\le N} s^{\gamma-1}e(\beta s) + O(N^{\gamma-\varepsilon}). \tag{25}$$

The prime number theorem for arithmetic progressions (refer to Section 22 of [2]) yields that, for $q \le \log^{80} N$, $(l,q)=1$ and $N/10 < t \le N$,

$$\pi(t;l,q)=\sum_{\substack{N/10<p\le t\\ p\equiv l\,(\mathrm{mod}\,q)}}1=\frac{1}{\varphi(q)}\int_{N/10}^{t}\frac{du}{\log u}+O(N\exp(-c\sqrt{\log N})).\tag{26}$$

Now,

$$\sum_{N/10<p\le N}p^{\gamma-1}e(\alpha p)=\sum_{\substack{l=1\\ (l,q)=1}}^{q}e\left(\frac{al}{q}\right)\sum_{\substack{N/10<p\le N\\ p\equiv l\,(\mathrm{mod}\,q)}}p^{\gamma-1}e(\beta p),$$

and so

$$\begin{aligned}\sum_{\substack{N/10<p\le N\\ p\equiv l\,(\mathrm{mod}\,q)}}p^{\gamma-1}e(\beta p)&=\int_{N/10}^{N}t^{\gamma-1}e(\beta t)\,d(\pi(t;l,q))\\&=\frac{1}{\varphi(q)}\int_{N/10}^{N}\frac{t^{\gamma-1}e(\beta t)}{\log t}\,dt\\&\quad+O(N^{\gamma}\exp(-c_1\sqrt{\log N}))\\&=\frac{1}{\varphi(q)}\sum_{N/10<s\le N}\frac{s^{\gamma-1}e(\beta s)}{\log s}\\&\quad+O(N^{\gamma}\exp(-c_1\sqrt{\log N})).\end{aligned}$$

Combining the above discussion with Lemma 7, we have

$$\begin{aligned}\sum_{N/10<p\le N}p^{\gamma-1}e(\alpha p)&=\frac{\mu(q)}{\varphi(q)}\sum_{N/10<s\le N}\frac{s^{\gamma-1}e(\beta s)}{\log s}\\&\quad+O(N^{\gamma}\exp(-c_2\sqrt{\log N})).\end{aligned}\tag{27}$$

Altogether, we get

$$\begin{aligned}\Sigma_4&=\int_{E_1}f(\alpha)S^2(\alpha)e(-\alpha N)\,d\alpha\\&=\sum_{q\le\log^{80}N}\sum_{\substack{a=0\\ (a,q)=1}}^{q-1}e\left(-\frac{aN}{q}\right)\int_{-1/(qQ)}^{1/(qQ)}f\left(\frac{a}{q}+\beta\right)S^2\left(\frac{a}{q}+\beta\right)e(-\beta N)\,d\beta\\&=\gamma^2\sum_{q\le\log^{80}N}\frac{\mu^2(q)C(q,-N)}{\varphi^2(q)}\sum_{\substack{r\sim R\\ q\mid r}}\frac{j(r)}{r}\int_{-1/(qQ)}^{1/(qQ)}\Big(\sum_{N/10<s\le N}s^{\gamma-1}e(\beta s)\Big)\\&\quad\times\Big(\sum_{N/10<s\le N}\frac{s^{\gamma-1}e(\beta s)}{\log s}\Big)^2e(-\beta N)\,d\beta+O\left(\frac{N^{3\gamma-1}}{\log^{20}N}\right).\end{aligned}$$

Since

$$\int_{1/(qQ)}^{1/2} \Big(\sum_{N/10<s\le N} s^{\gamma-1}e(\beta s)\Big)\Big(\sum_{N/10<s\le N} \frac{s^{\gamma-1}e(\beta s)}{\log s}\Big)^2 e(-\beta N)\,d\beta$$

$$\ll \int_{1/(qQ)}^{1/2} N^{3(\gamma-1)}\frac{d\beta}{\beta^3} \ll \frac{q^2N^{3\gamma-1}}{\log^{160} N},$$

we obtain

$$\Sigma_4 = \frac{1}{\gamma}I(N)\sum_{q\le\log^{80}N}\frac{\mu^2(q)C(q,-N)}{\varphi^2(q)}\sum_{\substack{r\sim R\\ q\mid r}}\frac{j(r)}{r} + O\bigg(\frac{N^{3\gamma-1}}{\log^{20}N}\bigg),$$

where

$$I(N) = \sum_{\substack{N=n_1+n_2+n_3\\ N/10<n_1,n_2,n_3\le N}}\frac{\gamma^3(n_1n_2n_3)^{\gamma-1}}{\log n_2\log n_3}.$$

Let

$$\begin{aligned}\Omega &= \sum_{q\le\log^{80}N}\frac{\mu^2(q)C(q,-N)}{\varphi^2(q)}\sum_{\substack{r\sim R\\ q\mid r}}\frac{j(r)}{r}\\ &= \sum_{r\sim R}\frac{j(r)}{r}\sum_{\substack{q\le\log^{80}N\\ q\mid r}}\frac{\mu^2(q)C(q,-N)}{\varphi^2(q)}.\end{aligned}$$

Now,

$$\sum_{r\sim R}\frac{j(r)}{r}\sum_{\substack{q>\log^{80}N\\ q\mid r}}\frac{\mu^2(q)C(q,-N)}{\varphi^2(q)} \ll \frac{1}{\log^{60}N}\sum_{r\sim R}\frac{d^2(r)}{r} \ll \frac{1}{\log^{50}N},$$

so that

$$\begin{aligned}\Omega &= \sum_{r\sim R}\frac{j(r)}{r}\sum_{q\mid r}\frac{\mu^2(q)C(q,-N)}{\varphi^2(q)} + O\bigg(\frac{1}{\log^{50}N}\bigg)\\ &= \sum_{r\sim R}\frac{\omega(r)j(r)}{r} + O\bigg(\frac{1}{\log^{50}N}\bigg)\\ &= \gamma\sum_{m\sim M}\sum_{k\sim K}a(m)b(k)\frac{\omega(mk)}{mk} + O\bigg(\frac{1}{\log^{50}N}\bigg).\end{aligned}$$

Hence

$$\Sigma_4 = I(N)\sum_{m\sim M}\sum_{k\sim K}a(m)b(k)\frac{\omega(mk)}{mk} + O\bigg(\frac{N^{3\gamma-1}}{\log^{20}N}\bigg).$$

Finally,

$$\Sigma_1 = I(N) \sum_{m\sim M} \sum_{k\sim K} a(m)b(k)\frac{\omega(mk)}{mk} + O\biggl(\frac{N^{3\gamma-1}}{\log^{20} N}\biggr).$$

The proof of Lemma 10 is complete.

LEMMA 11. *Assume that* $M, K \ll N^{5/16}$ *and that* $a(m)$, $b(k) = O(1)$. *Let* $\omega(r)$ *be defined in* (21). *Let*

$$J_1(N) = \sum_{N^{5/16}<p_1\le N^{1/2}} \frac{1}{p_1} \sum_{\substack{N=n_1+n_2+n_3\\ N/10<n_1,n_2,n_3\le N}} \frac{\gamma^3(n_1n_2n_3)^{\gamma-1}}{\log\frac{n_2}{p_1}\log n_3}, \tag{28}$$

$$J_2(N) = \sum_{N^{5/16}<p_1\le N^{1/3}} \sum_{p_1<p_2<\sqrt{N/p_1}} \frac{1}{p_1p_2} \times \sum_{\substack{N=n_1+n_2+n_3\\ N/10<n_1,n_2,n_3\le N}} \frac{\gamma^3(n_1n_2n_3)^{\gamma-1}}{\log\frac{n_2}{p_1p_2}\log n_3}. \tag{29}$$

Then

$$\sum_{m\sim M} \sum_{k\sim K} a(m)b(k)\biggl(\sum_{\substack{N=mkl+p_1p_2+p_3\\ N/10<mkl\le N\\ N/10<p_1p_2,p_3\le N\\ N^{5/16}<p_1\le N^{1/2}\\ p_1<p_2}} N(mkl)N(p_1p_2)N(p_3) - \frac{\omega(mk)}{mk}J_1(N)\biggr) = O\biggl(\frac{N^{3\gamma-1}}{\log^{20} N}\biggr)$$

and

$$\sum_{m\sim M} \sum_{k\sim K} a(m)b(k)\biggl(\sum_{\substack{N=mkl+p_1p_2p_3+p_4\\ N/10<mkl\le N\\ N/10<p_1p_2p_3,p_4\le N\\ N^{5/16}<p_1\le N^{1/3}\\ p_1<p_2<p_3}} N(mkl)N(p_1p_2p_3)N(p_4) - \frac{\omega(mk)}{mk}J_2(N)\biggr) = O\biggl(\frac{N^{3\gamma-1}}{\log^{20} N}\biggr).$$

This can be proved in almost the same way as Lemma 10.

4. Sieve method. Assume that

$$\mathcal{A}=\{a:a=N-p_1-p_2, N(a)=N(p_1)=N(p_2)=1,$$
$$N/10<p_1,p_2\le N, p_1+p_2<9N/10\},$$
$$\mathcal{B}=\{b:b=N-d-p_4, N(b)=N(d)=N(p_4)=1,$$
$$N/10<d,p_4\le N, d+p_4<9N/10, d=p_1p_2\ (N^{5/16}<p_1\le N^{1/2},$$
$$p_1<p_2)\text{ or } d=p_1p_2p_3\ (N^{5/16}<p_1\le N^{1/3}, p_1<p_2<p_3)\}$$

and that

$$P(z)=\prod_{p<z}p,\quad \mathcal{S}(\mathcal{A},z)=\sum_{\substack{a\in\mathcal{A}\\(a,P(z))=1}}1,\quad \mathcal{S}(\mathcal{B},w)=\sum_{\substack{b\in\mathcal{B}\\(b,P(w))=1}}1.$$

Note once again that $p\in P_\gamma$ is equivalent to $N(p)=1$. Applying Buchstab's identity, we get

$$\begin{aligned}(30)\qquad T_1(N)&\ge\mathcal{S}(\mathcal{A},N^{1/2})\\&=\mathcal{S}(\mathcal{A},N^{3/16})-\sum_{N^{3/16}<p\le N^{5/16}}\mathcal{S}(\mathcal{A}_p,p)\\&\quad-\sum_{N^{5/16}<p\le N^{1/2}}\mathcal{S}(\mathcal{A}_p,p)\\&=\mathcal{S}_1-\mathcal{S}_2-\mathcal{S}_3.\end{aligned}$$

Using Buchstab's identity again, we obtain

$$\begin{aligned}(31)\qquad \mathcal{S}_1&=\mathcal{S}(\mathcal{A},N^{2.5/16})-\sum_{N^{2.5/16}<p\le N^{3/16}}\mathcal{S}(\mathcal{A}_p,p)\\&=\mathcal{S}(\mathcal{A},N^{2.5/16})-\sum_{N^{2.5/16}<p\le N^{3/16}}\mathcal{S}\Big(\mathcal{A}_p,\Big(\frac{N^{10/16}}{p}\Big)^{1/5}\Big)\\&\quad+\sum_{N^{2.5/16}<p\le N^{3/16}}\ \sum_{(N^{10/16}/p)^{1/5}<q<N^{5/16}/p}\mathcal{S}(\mathcal{A}_{pq},q)\\&\quad+\sum_{N^{2.5/16}<p\le N^{3/16}}\ \sum_{N^{5/16}/p<q<(N^{10/16}/p)^{1/3}}\mathcal{S}(\mathcal{A}_{pq},q)\\&\quad+\sum_{N^{2.5/16}<p\le N^{3/16}}\ \sum_{(N^{10/16}/p)^{1/3}<q<p}\mathcal{S}(\mathcal{A}_{pq},q)\\&=\Phi_1-\Phi_2+\Phi_3+\Phi_4+\Phi_5.\end{aligned}$$

Next,

$$(32)\qquad \mathcal{S}_3=\sum_{N^{5/16}<p\le N^{1/2}}\mathcal{S}(\mathcal{A}_p,p)$$

$$
\begin{aligned}
&= \#\{d : d = N - p_4 - p_5, N(d) = N(p_4) = N(p_5) = 1,\\
&\qquad N/10 < p_4, p_5 \le N, p_4 + p_5 < 9N/10,\\
&\qquad d = p_1p_2 \ (N^{5/16} < p_1 \le N^{1/2}, p_1 < p_2)\\
&\qquad \text{or } d = p_1p_2p_3 \ (N^{5/16} < p_1 \le N^{1/3}, p_1 < p_2 < p_3)\}\\
&= \#\{p_4 : p_4 = N - d - p_5, N(p_4) = N(d) = N(p_5) = 1,\\
&\qquad N/10 < d, p_5 \le N, d + p_5 < 9N/10,\\
&\qquad d = p_1p_2 \ (N^{5/16} < p_1 \le N^{1/2}, p_1 < p_2)\\
&\qquad \text{or } d = p_1p_2p_3 \ (N^{5/16} < p_1 \le N^{1/3}, p_1 < p_2 < p_3)\}\\
&= \mathcal{S}(\mathcal{B}, N^{1/2}).
\end{aligned}
$$

Using Buchstab's identity, we have

$$
\begin{aligned}
\mathcal{S}(\mathcal{B}, N^{1/2}) &= \mathcal{S}(\mathcal{B}, N^{2.5/16}) - \sum_{N^{2.5/16} < p \le N^{3/16}} \mathcal{S}(\mathcal{B}_p, p)\\
&\quad - \sum_{N^{3/16} < p \le N^{5/16}} \mathcal{S}(\mathcal{B}_p, p) - \sum_{N^{5/16} < p \le N^{1/2}} \mathcal{S}(\mathcal{B}_p, p)\\
&\le \mathcal{S}(\mathcal{B}, N^{2.5/16}) - \sum_{N^{2.5/16} < p \le N^{3/16}} \mathcal{S}\Big(\mathcal{B}_p, \Big(\frac{N^{10/16}}{p}\Big)^{1/5}\Big)\\
&\quad + \sum_{N^{2.5/16} < p \le N^{3/16}} \ \sum_{(N^{10/16}/p)^{1/5} < q < N^{5/16}/p} \mathcal{S}(\mathcal{B}_{pq}, q)\\
&\quad + \sum_{N^{2.5/16} < p \le N^{3/16}} \ \sum_{N^{5/16}/p < q < p} \mathcal{S}(\mathcal{B}_{pq}, q)\\
&\quad - \sum_{N^{3/16} < p \le N^{5/16}} \mathcal{S}(\mathcal{B}_p, p)\\
&= \mathit{\Gamma}_1 - \mathit{\Gamma}_2 + \mathit{\Gamma}_3 + \mathit{\Gamma}_4 - \mathit{\Gamma}_5.
\end{aligned}
\tag{33}
$$

LEMMA 12.

$$\mathit{\Phi}_1 = \mathcal{S}(\mathcal{A}, N^{2.5/16}) \ge 3.515559 Z(\gamma) C(N) \frac{N^{3\gamma-1}}{\log^3 N},$$

where $C(N)$ *is defined in* (2) *and*

$$Z(\gamma) = \gamma^3 \int_{1/10}^{8/10} u_1^{\gamma-1}\, du_1 \int_{1/10}^{9/10-u_1} u_2^{\gamma-1}(1 - u_1 - u_2)^{\gamma-1}\, du_2. \tag{34}$$

Proof. Take

$$X = I(N) = \sum_{\substack{N = n_1 + n_2 + n_3\\ N/10 < n_1, n_2, n_3 \le N}} \frac{\gamma^3 (n_1n_2n_3)^{\gamma-1}}{\log n_2 \log n_3}$$

and

$$r(d) = \#\mathcal{A}_d - \frac{\omega(d)}{d} X,$$

where $\omega(d)$ is defined in (21).

By Theorem 7.11 of [10], we have

$$W(z) = \prod_{p<z} \left(1 - \frac{\omega(p)}{p}\right) = C(N) \frac{e^{-\gamma}}{\log z} \left(1 + O\left(\frac{1}{\log z}\right)\right),$$

where γ is Euler's constant.

Let $z = N^{2.5/16}$, $D = N^{10/16}$. By Iwaniec's bilinear sieve method (see Theorem 1 in [5]), we obtain

$$\Phi_1 \geq \frac{C(N)X}{\log z} \left(f\left(\frac{\log D}{\log z}\right) - O(\varepsilon)\right) - \sum_{m<N^{5/16}} \sum_{k<N^{5/16}} a(m)b(k)r(mk),$$

where $f(u)$ is a standard function. In particular,

$$f(u) = \begin{cases} \dfrac{2}{u} \log(u-1), & 2 \leq u \leq 4; \\ \dfrac{2}{u}\left(\log(u-1) + \displaystyle\int_3^{u-1} \frac{dt}{t} \int_2^{t-1} \frac{\log(s-1)}{s}\, ds \right), & 4 \leq u \leq 6. \end{cases}$$

By Lemma 10, we have

$$\sum_{m<N^{5/16}} \sum_{k<N^{5/16}} a(m)b(k)r(mk) = O\left(\frac{N^{3\gamma-1}}{\log^{10} N}\right).$$

On the other hand,

$$\begin{aligned} X &= \frac{(1+O(\varepsilon))\gamma^3}{\log^2 N} \sum_{\substack{N=n_1+n_2+n_3 \\ N/10<n_1,n_2,n_3\leq N}} (n_1 n_2 n_3)^{\gamma-1} \\ &= \frac{(1+O(\varepsilon))\gamma^3}{\log^2 N} \int_{N/10}^{8N/10} t_1^{\gamma-1}\, dt_1 \int_{N/10}^{9N/10-t_1} t_2^{\gamma-1} (N-t_1-t_2)^{\gamma-1}\, dt_2 \\ &= \frac{(1+O(\varepsilon))\gamma^3}{\log^2 N} N^{3\gamma-1} \int_{1/10}^{8/10} u_1^{\gamma-1}\, du_1 \int_{1/10}^{9/10-u_1} u_2^{\gamma-1} (1-u_1-u_2)^{\gamma-1}\, du_2. \end{aligned}$$

Hence,

$$\Phi_1 \geq 3.515559 Z(\gamma) C(N) \frac{N^{3\gamma-1}}{\log^3 N}.$$

LEMMA 13.

$$\Phi_2 = \sum_{N^{2.5/16} < p \le N^{3/16}} \mathcal{S}\left(\mathcal{A}_p, \left(\frac{N^{10/16}}{p}\right)^{1/5}\right) \le 1.130791 Z(\gamma) C(N) \frac{N^{3\gamma-1}}{\log^3 N}.$$

Proof. Take

$$z(p) = \left(\frac{N^{10/16}}{p}\right)^{1/5}, \quad D(p) = \frac{N^{10/16}}{p}.$$

By Iwaniec's bilinear sieve method, we obtain

$$\Phi_2 \le (1 + O(\varepsilon)) C(N) X \sum_{N^{2.5/16} < p \le N^{3/16}} \frac{1}{p \log z(p)} F\left(\frac{\log D(p)}{\log z(p)}\right) + R^+,$$

where

$$R^+ = \sum_{N^{2.5/16} < p \le N^{3/16}} \sum_{h < N^{5/16}/p} \sum_{k < N^{5/16}} c(h) b(k) r(phk),$$

and $F(u)$ is a standard function. In particular,

$$F(u) = \begin{cases} \dfrac{2}{u}, & 2 \le u \le 3; \\ \dfrac{2}{u}\left(1 + \displaystyle\int_2^{u-1} \frac{\log(t-1)}{t}\, dt\right), & 3 \le u \le 5. \end{cases}$$

In R^+, let $ph = m$. By Lemma 10, we have

$$R^+ = O\left(\frac{N^{3\gamma-1}}{\log^{10} N}\right).$$

From the above discussion and the prime number theorem, we have

$$\begin{aligned} \Phi_2 &\le Z(\gamma) C(N) \frac{N^{3\gamma-1}}{\log^2 N} \sum_{N^{2.5/16} < p \le N^{3/16}} \frac{5F(5)}{p \log \frac{N^{10/16}}{p}} + O\left(\frac{\varepsilon N^{3\gamma-1}}{\log^3 N}\right) \\ &= Z(\gamma) C(N) \frac{N^{3\gamma-1}}{\log^3 N} \int_{2.5/16}^{3/16} \frac{2dt}{t(\frac{10}{16} - t)} \left(1 + \int_2^4 \frac{\log(u-1)}{u}\, du\right) \\ &\quad + O\left(\frac{\varepsilon N^{3\gamma-1}}{\log^3 N}\right) \\ &\le 1.130791 Z(\gamma) C(N) \frac{N^{3\gamma-1}}{\log^3 N}. \end{aligned}$$

LEMMA 14.

$$\varPhi_4 = \sum_{N^{2.5/16}<p\le N^{3/16}} \sum_{N^{5/16}/p<q<(N^{10/16}/p)^{1/3}} \mathcal{S}(\mathcal{A}_{pq}, q) \ge 0.011651 Z(\gamma)C(N)\frac{N^{3\gamma-1}}{\log^3 N}.$$

Proof. Take

$$D(p,q) = \frac{N^{10/16}}{pq}.$$

By Iwaniec's bilinear sieve method, we have

$$\varPhi_4 \ge (1+O(\varepsilon))C(N)X \sum_{N^{2.5/16}<p\le N^{3/16}} \sum_{N^{5/16}/p<q<(N^{10/16}/p)^{1/3}} \frac{1}{pq\log q} \times f\left(\frac{\log D(p,q)}{\log q}\right) - R^-,$$

where

$$R^- = \sum_{N^{2.5/16}<p\le N^{3/16}} \sum_{N^{5/16}/p<q<(N^{10/16}/p)^{1/3}} \sum_{h<N^{5/16}/p} \sum_{g<N^{5/16}/q} c(h)v(g)r(pqhg).$$

In R^-, let $ph = m, qg = k$. By Lemma 10, we have

$$R^- = O\left(\frac{N^{3\gamma-1}}{\log^{10} N}\right).$$

Hence,

$$\varPhi_4 \ge Z(\gamma)C(N)\frac{N^{3\gamma-1}}{\log^3 N}\int_{2.5/16}^{3/16}\frac{dt}{t}\int_{5/16-t}^{(10/16-t)/3}\frac{2}{w(\frac{10}{16}-t-w)} \times \log\left(\frac{(\frac{10}{16}-t)}{w}-2\right)dw + O\left(\frac{\varepsilon N^{3\gamma-1}}{\log^3 N}\right) \ge 0.011651 Z(\gamma)C(N)\frac{N^{3\gamma-1}}{\log^3 N}.$$

LEMMA 15.

$$\varGamma_1 = \mathcal{S}(\mathcal{B}, N^{2.5/16}) \le 2.926882 Z(\gamma)C(N)\frac{N^{3\gamma-1}}{\log^3 N}.$$

Proof. We take $Y = J_1(N) + J_2(N)$, where $J_1(N), J_2(N)$ are defined in (28) and (29) respectively, and

$$r(d) = \#\mathcal{B}_d - \frac{\omega(d)}{d}Y.$$

By Iwaniec's bilinear sieve method, we get

$$\varGamma_1 \le \frac{C(N)Y}{\log N} \cdot \frac{16}{2.5}(F(4) + O(\varepsilon)) + \sum_{m<N^{5/16}} \sum_{k<N^{5/16}} a(m)b(k)r(mk).$$

Applying Lemma 11, we get

$$\sum_{m<N^{5/16}} \sum_{k<N^{5/16}} a(m)b(k)r(mk) = O\bigg(\frac{N^{3\gamma-1}}{\log^{10} N}\bigg).$$

Hence,

$$\varGamma_1 \le 3.671111 \cdot \frac{C(N)Y}{\log N}.$$

Now,

$$\begin{aligned} Y &= \frac{(1+O(\varepsilon))\gamma^3}{\log N} \sum_{\substack{N=n_1+n_2+n_3\\ N/10<n_1,n_2,n_3\le N}} (n_1n_2n_3)^{\gamma-1}\bigg(\sum_{N^{5/16}<p_1\le N^{1/2}} \frac{1}{p_1 \log \frac{N}{p_1}} \\ &\quad + \sum_{N^{5/16}<p_1\le N^{1/3}} \sum_{p_1<p_2<\sqrt{N/p_1}} \frac{1}{p_1p_2 \log \frac{N}{p_1p_2}}\bigg) \\ &= (1+O(\varepsilon))Z(\gamma)\frac{N^{3\gamma-1}}{\log^2 N}\bigg(\int_{5/16}^{1/2} \frac{dt}{t(1-t)} + \int_{5/16}^{1/3} \frac{dt}{t} \int_{t}^{(1-t)/2} \frac{dw}{w(1-t-w)}\bigg) \\ &\le 0.797274 Z(\gamma)\frac{N^{3\gamma-1}}{\log^2 N}. \end{aligned}$$

Therefore,

$$\varGamma_1 \le 2.926882 Z(\gamma)C(N)\frac{N^{3\gamma-1}}{\log^3 N}.$$

LEMMA 16.

$$\begin{aligned} \varGamma_2 &= \sum_{N^{2.5/16}<p\le N^{3/16}} \mathcal{S}\bigg(\mathcal{B}_p, \Big(\frac{N^{10/16}}{p}\Big)^{1/5}\bigg) \\ &\ge 0.898396 Z(\gamma)C(N)\frac{N^{3\gamma-1}}{\log^3 N}. \end{aligned}$$

P r o o f. Using Lemma 11, in almost the same way as in Lemma 13, we obtain

$$\begin{aligned} \varGamma_2 &\ge (1+O(\varepsilon))C(N)Y \sum_{N^{2.5/16}<p\le N^{3/16}} \frac{5f(5)}{p \log \frac{N^{10/16}}{p}} \\ &= (1+O(\varepsilon))Z(\gamma)C(N)\frac{N^{3\gamma-1}}{\log^3 N} \int_{2.5/16}^{3/16} \frac{2\,dt}{t(\frac{10}{16}-t)} \end{aligned}$$

$$\times\left(\log 4+\int\limits_3^4\frac{dt}{t}\int\limits_2^{t-1}\frac{\log(s-1)}{s}\,ds\right)$$

$$\times\left(\int\limits_{5/16}^{1/2}\frac{dt}{t(1-t)}+\int\limits_{5/16}^{1/3}\frac{dt}{t}\int\limits_{t}^{(1-t)/2}\frac{dw}{w(1-t-w)}\right)$$

$$\geq 0.898396Z(\gamma)C(N)\frac{N^{3\gamma-1}}{\log^3 N}.$$

LEMMA 17.

$$\Gamma_4=\sum_{N^{2.5/16}<p\leq N^{3/16}}\ \sum_{N^{5/16}/p<q<p}\mathcal{S}(\mathcal{B}_{pq},q)$$
$$\leq 0.194188Z(\gamma)C(N)\frac{N^{3\gamma-1}}{\log^3 N}.$$

Proof. Applying Lemma 11, in almost the same way as in Lemma 14, we get

$$\Gamma_4\leq\sum_{N^{2.5/16}<p\leq N^{3/16}}\ \sum_{N^{5/16}/p<q<p}\mathcal{S}\left(\mathcal{B}_{pq},\left(\frac{N^{10/16}}{pq}\right)^{1/3}\right)$$
$$\leq(1+O(\varepsilon))C(N)Y\sum_{N^{2.5/16}<p\leq N^{3/16}}\ \sum_{N^{5/16}/p<q<p}\frac{3F(3)}{pq\log\frac{N^{10/16}}{pq}}$$
$$=(1+O(\varepsilon))Z(\gamma)C(N)\frac{N^{3\gamma-1}}{\log^3 N}\int\limits_{2.5/16}^{3/16}\frac{dt}{t}\int\limits_{5/16-t}^{t}\frac{2\,dw}{w(\frac{10}{16}-t-w)}$$
$$\times\left(\int\limits_{5/16}^{1/2}\frac{dt}{t(1-t)}+\int\limits_{5/16}^{1/3}\frac{dt}{t}\int\limits_{t}^{(1-t)/2}\frac{dw}{w(1-t-w)}\right)$$
$$\leq 0.194188Z(\gamma)C(N)\frac{N^{3\gamma-1}}{\log^3 N}.$$

5. Asymptotic formulas

LEMMA 18. *Assume that* $N^{11/16}\ll M\ll N^{13/16}$, $0\leq a(m)=O(1)$ *and that* $a(m)=0$ *if* m *has a prime factor* $<N^{\varepsilon}$. *Then*

$$\Sigma=\sum_{\substack{N=mp_1+p_2+p_3\\ N/10<mp_1,p_2,p_3\leq N\\ m\sim M}}a(m)N(mp_1)N(p_2)N(p_3)$$
$$=(1+O(\varepsilon))Z(\gamma)C(N)\frac{N^{3\gamma-1}}{N\log^2 N}\sum_{m\sim M}a(m)\sum_{N/m<p\leq 2N/m}1$$
$$+O\left(\frac{N^{3\gamma-1}}{\log^{10} N}\right).$$

P r o o f. We have

$$\Sigma = \int_0^1 \sum_{\substack{N/10 < mp_1 \le N \\ m \sim M}} a(m) N(mp_1) e(\alpha m p_1) \times \Big(\sum_{N/10 < p \le N} N(p) e(\alpha p) \Big)^2 e(-\alpha N)\, d\alpha.$$

Using the same reasoning as in Lemma 10, we get

$$\Sigma = \int_{E_1} g(\alpha) S^2(\alpha) e(-\alpha N)\, d\alpha + O\bigg(\frac{N^{3\gamma-1}}{\log^{10} N} \bigg),$$

where E_1 is defined in Lemma 10,

$$g(\alpha) = \gamma \sum_{\substack{N/10 < mp_1 \le N \\ m \sim M}} a(m) (mp_1)^{\gamma-1} e(\alpha m p_1)$$

and

$$\mathcal{S}(\alpha) = \gamma \sum_{N/10 < p \le N} p^{\gamma-1} e(\alpha p).$$

Note that if $a(m) \ne 0$ and $q \le \log^{80} N$, then $(m,q) = 1$. Thus,

$$g(\alpha) = \gamma \sum_{\substack{l=1 \\ (l,q)=1}}^{q} e\bigg(\frac{al}{q}\bigg) \sum_{m \sim M} a(m) m^{\gamma-1} \sum_{\substack{N/(10m) < p_1 \le N/m \\ mp_1 \equiv l \,(\mathrm{mod}\, q)}} p_1^{\gamma-1} e(\beta m p_1).$$

Let $\overline{m}$ be a number such that $\overline{m} m \equiv 1 \pmod q$. Using the discussion in Lemma 10, we have

$$\sum_{\substack{N/(10m) < p_1 \le N/m \\ mp_1 \equiv l \,(\mathrm{mod}\, q)}} p_1^{\gamma-1} e(\beta m p_1)$$

$$= \sum_{\substack{N/(10m) < p_1 \le N/m \\ p_1 \equiv \overline{m} l \,(\mathrm{mod}\, q)}} p_1^{\gamma-1} e(\beta m p_1) = \int_{N/(10m)}^{N/m} t^{\gamma-1} e(\beta m t)\, d(\pi(t; \overline{m} l, q))$$

$$= \frac{1}{\varphi(q)} \int_{N/(10m)}^{N/m} \frac{t^{\gamma-1} e(\beta m t)}{\log t}\, dt + O\bigg(\Big(\frac{N}{m}\Big)^{\gamma} \exp(-c\sqrt{\log N}) \bigg)$$

$$= \frac{m^{-\gamma}}{\varphi(q)} \int_{N/10}^{N} \frac{u^{\gamma-1} e(\beta u)}{\log \frac{u}{m}}\, du + O\bigg(\Big(\frac{N}{m}\Big)^{\gamma} \exp(-c\sqrt{\log N}) \bigg)$$

$$= \frac{m^{-\gamma}}{\varphi(q)} \sum_{N/10<s\le N} \frac{s^{\gamma-1}e(\beta s)}{\log \frac{s}{m}} + O\bigg(\Big(\frac{N}{m}\Big)^{\gamma} \exp(-c\sqrt{\log N})\bigg).$$

Therefore,

$$g(\alpha) = \gamma \frac{\mu(q)}{\varphi(q)} \sum_{m\sim M} \frac{a(m)}{m} \sum_{N/10<s\le N} \frac{s^{\gamma-1}e(\beta s)}{\log \frac{s}{m}} + O(N^{\gamma}\exp(-c_1\sqrt{\log N})),$$

and

$$\mathcal{S}(\alpha) = \gamma \frac{\mu(q)}{\varphi(q)} \sum_{N/10<s\le N} \frac{s^{\gamma-1}e(\beta s)}{\log s} + O(N^{\gamma}\exp(-c_2\sqrt{\log N})).$$

Hence,

$$\begin{aligned}
\varSigma &= \sum_{q\le \log^{80} N} \sum_{\substack{a=0\\(a,q)=1}}^{q-1} e\Big(-\frac{aN}{q}\Big) \int_{-1/(qQ)}^{1/(qQ)} g\Big(\frac{a}{q}+\beta\Big) S^2\Big(\frac{a}{q}+\beta\Big) e(-\beta N)\, d\beta \\
&\quad + O\Big(\frac{N^{3\gamma-1}}{\log^{10} N}\Big) \\
&= \gamma^3 \sum_{q\le \log^{80} N} \frac{\mu(q)C(q,-N)}{\varphi^3(q)} \int_{-1/(qQ)}^{1/(qQ)} \sum_{m\sim M} \frac{a(m)}{m} \sum_{N/10<s\le N} \frac{s^{\gamma-1}e(\beta s)}{\log \frac{s}{m}} \\
&\quad \times \Big(\sum_{N/10<s\le N} \frac{s^{\gamma-1}e(\beta s)}{\log s}\Big)^2 e(-\beta N)\, d\beta + O\Big(\frac{N^{3\gamma-1}}{\log^{10} N}\Big) \\
&= \sum_{q\le \log^{80} N} \frac{\mu(q)C(q,-N)}{\varphi^3(q)} K(N) + O\Big(\frac{N^{3\gamma-1}}{\log^{10} N}\Big),
\end{aligned}$$

where

$$\begin{aligned}
K(N) &= \gamma^3 \sum_{m\sim M} \frac{a(m)}{m} \sum_{\substack{N=n_1+n_2+n_3\\ N/10<n_1,n_2,n_3\le N}} \frac{(n_1n_2n_3)^{\gamma-1}}{\log \frac{n_1}{m} \log n_2 \log n_3} \\
&= (1+O(\varepsilon))Z(\gamma) \frac{N^{3\gamma-1}}{\log^2 N} \sum_{m\sim M} \frac{a(m)}{m\log \frac{N}{m}} \\
&= (1+O(\varepsilon))Z(\gamma) \frac{N^{3\gamma-1}}{N\log^2 N} \sum_{m\sim M} a(m) \sum_{N/m<p\le 2N/m} 1.
\end{aligned}$$

Moreover,

$$\sum_{q\le\log^{80}N}\frac{\mu(q)C(q,-N)}{\varphi^3(q)}=\sum_{q=1}^{\infty}\frac{\mu(q)C(q,-N)}{\varphi^3(q)}+O\biggl(\frac{1}{\log^{40}N}\biggr)$$
$$=C(N)+O\biggl(\frac{1}{\log^{40}N}\biggr).$$

The above discussion yields

$$\Sigma=(1+O(\varepsilon))Z(\gamma)C(N)\frac{N^{3\gamma-1}}{N\log^2N}\sum_{m\sim M}a(m)\sum_{N/m<p\le 2N/m}1$$
$$+O\biggl(\frac{N^{3\gamma-1}}{\log^{10}N}\biggr).$$

The proof of Lemma 18 is complete.

Lemma 19.

$$\mathcal{S}_2=\sum_{N^{3/16}<p\le N^{5/16}}\mathcal{S}(\mathcal{A}_p,p)\le 1.198136Z(\gamma)C(N)\frac{N^{3\gamma-1}}{\log^3N}.$$

P r o o f. From Lemmas 18, 9 and 8, it follows that

$$\mathcal{S}_2=\sum_{\substack{N=rp+p_2+p_3\\ N/10<rp,p_2,p_3\le N\\ N^{3/16}<p\le N^{5/16}\\ (r,P(p))=1}}N(rp)N(p_2)N(p_3)$$
$$=(1+O(\varepsilon))Z(\gamma)C(N)\frac{N^{3\gamma-1}}{N\log^2N}\sum_{N^{3/16}<p\le N^{5/16}}\sum_{\substack{N/p<r\le 2N/p\\ (r,P(p))=1}}1$$
$$+O\biggl(\frac{N^{3\gamma-1}}{\log^6N}\biggr)$$
$$=Z(\gamma)C(N)\frac{N^{3\gamma-1}}{\log^2N}\sum_{N^{3/16}<p\le N^{5/16}}\frac{1}{p\log p}w\biggl(\frac{\log\frac{N}{p}}{\log p}\biggr)+O\biggl(\frac{\varepsilon N^{3\gamma-1}}{\log^3N}\biggr)$$
$$=Z(\gamma)C(N)\frac{N^{3\gamma-1}}{\log^3N}\int_{3/16}^{5/16}\frac{1}{u^2}w\biggl(\frac{1-u}{u}\biggr)\,du+O\biggl(\frac{\varepsilon N^{3\gamma-1}}{\log^3N}\biggr)$$
$$=Z(\gamma)C(N)\frac{N^{3\gamma-1}}{\log^3N}\int_{11/5}^{13/3}w(t)\,dt+O\biggl(\frac{\varepsilon N^{3\gamma-1}}{\log^3N}\biggr)$$

$$\leq Z(\gamma)C(N)\frac{N^{3\gamma-1}}{\log^3 N}\bigg(\int\limits_{11/5}^{3}\frac{(1+\log(u-1))}{u}\,du$$
$$+\int\limits_{3}^{4}\bigg\{\frac{1+\log(u-1)}{u}+\frac{1}{u}\int\limits_{2}^{u-1}\frac{\log(t-1)}{t}dt\bigg\}\,du+0.5644\int\limits_{4}^{13/3}du\bigg)$$
$$\leq 1.198136Z(\gamma)C(N)\frac{N^{3\gamma-1}}{\log^3 N}.$$

Lemma 20.
$$\Phi_3=\sum_{N^{2.5/16}<p\leq N^{3/16}}\ \sum_{(N^{10/16}/p)^{1/5}<q<N^{5/16}/p}\mathcal{S}(\mathcal{A}_{pq},q)$$
$$\geq 0.399722Z(\gamma)C(N)\frac{N^{3\gamma-1}}{\log^3 N}.$$

P r o o f. We have
$$\Phi_3=\sum_{\substack{N=rpq+p_2+p_3\\ N/10<rpq,p_2,p_3\leq N\\ N^{2.5/16}<p\leq N^{3/16}\\ (N^{10/16}/p)^{1/5}<q<N^{5/16}/p\\ (r,P(q))=1}}N(rpq)N(p_2)N(p_3).$$

Note that $N^{3/16}\ll pq\ll N^{5/16}$ and that $N^{11/16}\ll r\ll N^{13/16}$. By Lemma 18 with a small modification, and Lemmas 9 and 8, we have

$$\Phi_3=(1+O(\varepsilon))Z(\gamma)C(N)\frac{N^{3\gamma-1}}{N\log^2 N}$$
$$\times\sum_{N^{2.5/16}<p\leq N^{3/16}}\ \sum_{(N^{10/16}/p)^{1/5}<q<N^{5/16}/p}\ \sum_{\substack{N/(pq)<r\leq 2N/(pq)\\ (r,P(q))=1}}1$$
$$+O\bigg(\frac{N^{3\gamma-1}}{\log^6 N}\bigg)$$
$$=Z(\gamma)C(N)\frac{N^{3\gamma-1}}{\log^2 N}\sum_{N^{2.5/16}<p\leq N^{3/16}}\ \sum_{(N^{10/16}/p)^{1/5}<q<N^{5/16}/p}\frac{1}{pq\log q}$$
$$\times w\bigg(\frac{\log\frac{N}{pq}}{\log q}\bigg)+O\bigg(\frac{\varepsilon N^{3\gamma-1}}{\log^3 N}\bigg)$$
$$\geq 0.5606Z(\gamma)C(N)\frac{N^{3\gamma-1}}{\log^3 N}\int\limits_{2.5/16}^{3/16}\frac{dt}{t}\int\limits_{(10/16-t)/5}^{5/16-t}\frac{dw}{w^2}$$
$$\geq 0.399722Z(\gamma)C(N)\frac{N^{3\gamma-1}}{\log^3 N}.$$

LEMMA 21. *Assume that* $N^{11/16} \ll M \ll N^{13/16}$, $0 \le a(m) = O(1)$ *and that* $a(m) = 0$ *if* m *has a prime factor which is* $< N^{\varepsilon}$. *Let*

$$\Sigma = \sum_{\substack{N=mp+d+p_4\\ N/10<mp,d,p_4\le N\\ m\sim M}} a(m)N(mp)N(d)N(p_4),$$

where $d = p_1p_2$ $(N^{5/16} < p_1 \le N^{1/2}, p_1 < p_2)$ *or* $d = p_1p_2p_3$ $(N^{5/16} < p_1 \le N^{1/3}, p_1 < p_2 < p_3)$. *Then*

$$\Sigma = (1+O(\varepsilon))Z(\gamma)C(N)\frac{N^{3\gamma-1}}{N\log^2 N}\sum_{m\sim M} a(m)\sum_{N/m<p\le 2N/m} 1$$
$$\times\Bigg(\int_{5/16}^{1/2}\frac{dt}{t(1-t)} + \int_{5/16}^{1/3}\frac{dt}{t}\int_{t}^{(1-t)/2}\frac{dw}{w(1-t-w)}\Bigg) + O\bigg(\frac{N^{3\gamma-1}}{\log^6 N}\bigg).$$

P r o o f. In almost the same way as in Lemma 18, referring to Lemma 11, we can get

$$\Sigma = (1+O(\varepsilon))C(N)\sum_{m\sim M}\frac{a(m)}{m}$$
$$\times\Bigg(\sum_{N^{5/16}<p_1\le N^{1/2}}\frac{1}{p_1}\sum_{\substack{N=n_1+n_2+n_3\\ N/10<n_1,n_2,n_3\le N}}\frac{\gamma^3(n_1n_2n_3)^{\gamma-1}}{\log\frac{n_1}{m}\log\frac{n_2}{p_1}\log n_3}$$
$$+\sum_{N^{5/16}<p_1\le N^{1/3}}\ \sum_{p_1<p_2<\sqrt{N/p_1}}\frac{1}{p_1p_2}$$
$$\times\sum_{\substack{N=n_1+n_2+n_3\\ N/10<n_1,n_2,n_3\le N}}\frac{\gamma^3(n_1n_2n_3)^{\gamma-1}}{\log\frac{n_1}{m}\log\frac{n_2}{p_1p_2}\log n_3}\Bigg) + O\bigg(\frac{N^{3\gamma-1}}{\log^6 N}\bigg)$$
$$= (1+O(\varepsilon))\frac{C(N)\gamma^3}{\log N}\sum_{m\sim M}\frac{a(m)}{m\log\frac{N}{m}}\Bigg(\sum_{N^{5/16}<p_1\le N^{1/2}}\frac{1}{p_1\log\frac{N}{p_1}}$$
$$+\sum_{N^{5/16}<p_1\le N^{1/3}}\ \sum_{p_1<p_2<\sqrt{N/p_1}}\frac{1}{p_1p_2\log\frac{N}{p_1p_2}}\Bigg)$$
$$\times\sum_{\substack{N=n_1+n_2+n_3\\ N/10<n_1,n_2,n_3\le N}}(n_1n_2n_3)^{\gamma-1} + O\bigg(\frac{N^{3\gamma-1}}{\log^6 N}\bigg)$$
$$= (1+O(\varepsilon))Z(\gamma)C(N)\frac{N^{3\gamma-1}}{N\log^2 N}\sum_{m\sim M} a(m)\sum_{N/m<p\le 2N/m} 1$$

$$\times\Big(\int_{5/16}^{1/2}\frac{dt}{t(1-t)}+\int_{5/16}^{1/3}\frac{dt}{t}\int_{t}^{(1-t)/2}\frac{dw}{w(1-t-w)}\Big)+O\Big(\frac{N^{3\gamma-1}}{\log^6 N}\Big).$$

LEMMA 22.

$$\Gamma_5=\sum_{N^{3/16}<p\le N^{5/16}}\mathcal{S}(\mathcal{B}_p,p)\ge 0.954253Z(\gamma)C(N)\frac{N^{3\gamma-1}}{\log^3 N}.$$

Proof. We have

$$\Gamma_5=\sum_{\substack{N=rp+d+p_4\\ N/10<rp,d,p_4\le N\\ N^{3/16}<p\le N^{5/16}\\ (r,P(p))=1}}N(rp)N(d)N(p_4),$$

where $d=p_1p_2$ ($N^{5/16}<p_1\le N^{1/2}, p_1<p_2$) or $d=p_1p_2p_3$ ($N^{5/16}<p_1\le N^{1/3}, p_1<p_2<p_3$). By Lemmas 21 and 9, in almost the same way as in Lemma 19, we have

$$\Gamma_5=Z(\gamma)C(N)\frac{N^{3\gamma-1}}{\log^3 N}\int_{11/5}^{13/3}w(t)\,dt$$
$$\times\Big(\int_{5/16}^{1/2}\frac{dt}{t(1-t)}+\int_{5/16}^{1/3}\frac{dt}{t}\int_{t}^{(1-t)/2}\frac{dw}{w(1-t-w)}\Big)+O\Big(\frac{\varepsilon N^{3\gamma-1}}{\log^3 N}\Big).$$

Using Lemma 8, we get

$$\int_{11/5}^{13/3}w(t)\,dt\ge\int_{11/5}^{3}\frac{(1+\log(u-1))}{u}\,du+\int_{3}^{4}\Big\{\frac{1+\log(u-1)}{u}$$
$$+\frac{1}{u}\int_{2}^{u-1}\frac{\log(t-1)}{t}\,dt\Big\}\,du+0.5607\int_{4}^{13/3}du$$
$$\ge 1.196900.$$

Hence

$$\Gamma_5\ge 0.954253Z(\gamma)C(N)\frac{N^{3\gamma-1}}{\log^3 N}.$$

LEMMA 23.

$$\Gamma_3=\sum_{N^{2.5/16}<p\le N^{3/16}}\sum_{(N^{10/16}/p)^{1/5}<q<N^{5/16}/p}\mathcal{S}(\mathcal{B}_{pq},q)$$
$$\le 0.320849Z(\gamma)C(N)\frac{N^{3\gamma-1}}{\log^3 N}.$$

Proof. We have

$$\Gamma_3 = \sum_{\substack{N=rpq+d+p_4\\ N/10<rpq,d,p_4\le N\\ N^{2.5/16}<p\le N^{3/16}\\ (N^{10/16}/p)^{1/5}<q<N^{5/16}/p\\ (r,P(q))=1}} N(rpq)N(d)N(p_4),$$

where $d = p_1p_2$ ($N^{5/16} < p_1 \le N^{1/2}$, $p_1 < p_2$) or $d = p_1p_2p_3$ ($N^{5/16} < p_1 \le N^{1/3}$, $p_1 < p_2 < p_3$). By Lemma 21 and the deduction in Lemma 20, we get

$$\Gamma_3 \le 0.5644Z(\gamma)C(N)\frac{N^{3\gamma-1}}{\log^3 N}\int\limits_{2.5/16}^{3/16}\frac{dt}{t}\int\limits_{(10/16-t)/5}^{5/16-t}\frac{dw}{w^2}$$
$$\times\Bigg(\int\limits_{5/16}^{1/2}\frac{dt}{t(1-t)}+\int\limits_{5/16}^{1/3}\frac{dt}{t}\int\limits_{t}^{(1-t)/2}\frac{dw}{w(1-t-w)}\Bigg)+O\bigg(\frac{\varepsilon N^{3\gamma-1}}{\log^3 N}\bigg)$$
$$\le 0.320849Z(\gamma)C(N)\frac{N^{3\gamma-1}}{\log^3 N}.$$

6. The proof of the Theorem. Applying Lemmas 12, 13, 20 and 14 to the expression in (31), we get

$$\mathcal{S}_1 \ge 2.796141Z(\gamma)C(N)\frac{N^{3\gamma-1}}{\log^3 N}.$$

Applying Lemmas 15, 16, 23, 17 and 22 to the expression in (33), we obtain

$$\mathcal{S}_3 \le 1.589270Z(\gamma)C(N)\frac{N^{3\gamma-1}}{\log^3 N}.$$

In (30), the above two inequalities and Lemma 19 yield

$$T_1(N) \ge 0.008734Z(\gamma)C(N)\frac{N^{3\gamma-1}}{\log^3 N} \ge \frac{\varrho_0 C(N)N^{3\gamma-1}}{\log^3 N}.$$

Hence, the Theorem follows.

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