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On Šnirelman's constant under the Riemann hypothesis

by

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1. Introduction and statement of the Theorem. The well known Goldbach conjecture states that every integer $n > 5$ is a sum of three primes. The conjecture itself remains unsolved today, but a significant progress has been made by applying either analytical, elementary or sieve theory methods. One of the most important results belongs to I. M. Vinogradov, who in 1937 proved using the Hardy–Littlewood circle method that there exists a natural number n_0 such that every odd $n \geq n_0$ is a sum of three primes. J. R. Chen and T. Z. Wang [1] proved recently that one can take $n_0 = e^{e^{11.503}}$.

Another line of attack has been proposed by Šnirelman, who proved by elementary means that there exists a positive constant S_0, now called *Šnirelman's constant*, such that every integer > 1 is a sum of at most S_0 primes. The numerical value of S_0 in Šnirelman's original proof was very large; it was then reduced among others by M. Deshouillers [3] ($S_0 \leq 26$), H. Riesel and R. C. Vaughan [10] ($S_0 \leq 19$) and recently by O. Ramaré [9] ($S_0 \leq 7$).

Connections between the Goldbach conjecture and the Riemann Hypothesis (R.H.) that all the non-trivial zeros of the Riemann zeta function lie on the critical line are not clear. In particular, it is not known if the Goldbach conjecture is a corollary to R.H. At least a partial explanation of this phenomenon is that the distribution of zeros of the Riemann zeta function *alone* does not enter seriously into the circle method when applied to this particular problem. From this point of view assumption of the R.H. in connection with the Goldbach conjecture seems rather modest.

Let us denote by G_2 the least upper bound for the number G with the property that all even natural numbers $4 \leq n \leq G$ are sums of two primes. The Goldbach conjecture asserts obviously that $G_2 = \infty$. With this notation we have the following result.

THEOREM. *Suppose the Riemann Hypothesis is true. Then every odd natural number can be written as a sum of at most five primes. If $G_2 > 1.405 \cdot 10^{12}$ then every even natural number can be written as a sum of at most four primes.*

In a recent paper M. K. Sinisalo [14] proved that $G_2 \ge 4 \cdot 10^{11}$ improving earlier results by M. K. Shen [13] ($G_2 \ge 3.3 \cdot 10^7$), M. L. Stein and P. R. Stein [15] ($G_2 \ge 10^8$) and A. Granville, J. van de Lune and H. J. J. te Riele [5] ($G_2 \ge 2 \cdot 10^{10}$). In a letter Prof. M. K. Sinisalo informed the present author that his algorithm for checking the Goldbach conjecture is sufficiently efficient to cover the missing range between $4 \cdot 10^{11}$ and $1.405 \cdot 10^{12}$. This would however need a lot of computer's running time.

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2. Lemmas. In this section we formulate some lemmas needed in the proof of the Theorem.

LEMMA 1. *For $s \ne 1$, $s \ne \varrho$, $s \ne -2q$, where ϱ denotes non-trivial zeros of the Riemann zeta function $\zeta(s)$, the following identity holds*:

$$\frac{\zeta'}{\zeta}(s) = \frac{-1}{s-1} + \sum_{\varrho}\left(\frac{1}{s-\varrho} + \frac{1}{\varrho}\right) + \sum_{q=1}^{\infty}\left(\frac{1}{s+2q} - \frac{1}{2q}\right) + \log(2\pi) - 1. \tag{1}$$

P r o o f. See [8], pp. 218 and 394. ∎

Let us write as usual

$$\Lambda(n) = \begin{cases} \log p & \text{for } n = p^k,\ k \text{ natural, } p \text{ prime,} \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, for $x, a > 1$ we write

$$\Lambda_{x,a}(n) = \begin{cases} \Lambda(n) & \text{for } 1 \le n \le x, \\ \dfrac{\Lambda(n)}{a-1} \cdot \dfrac{\log(x^a/n)}{\log x} & \text{for } x \le n \le x^a. \end{cases}$$

LEMMA 2. *For $s \ne 1$, $s \ne \varrho$, $s \ne -2q$ we have*

$$\frac{\zeta'}{\zeta}(s) = -\sum_{n<x^a}\frac{\Lambda_{x,a}(n)}{n^s} + \frac{1}{(a-1)\log x}\biggl(\frac{x^{a(1-s)} - x^{1-s}}{(1-s)^2} + \sum_{q=1}^{\infty}\frac{x^{-(2q+s)} - x^{-a(2q+s)}}{(2q+s)^2} + \sum_{\varrho}\frac{x^{\varrho-s} - x^{a(\varrho-s)}}{(\varrho-s)^2}\biggr). \tag{2}$$

P r o o f. This is a slightly modified Selberg formula (see [12], Lemma 2). ∎

LEMMA 3. *Suppose R.H. is true and let $s = \sigma + it$, $\sigma = 1/2 + c/\log x \le 3/4$, $t \ge 14$, $c > 0$, $x > 1$, $a > 1$ and $(a-1)c > e^{-c} + e^{-ac}$. Then*

$$(3)\qquad \left|\frac{\zeta'}{\zeta}(s)\right| \le \frac{(a-1)c}{(a-1)c - e^{-c} - e^{-ac}}\Bigg[\Bigg|\sum_{n<x^a}\frac{\Lambda_{x,a}(n)}{n^s}\Bigg| + \frac{e^{-c}+e^{-ac}}{2(a-1)c}\log t + \frac{1}{(a-1)t^2\log x} \times\left(e^{-ac}x^{a/2} + e^{-c}x^{1/2} + \frac{1}{x^{\sigma}(x^2-1)} + \frac{1}{x^{a\sigma}(x^{2a}-1)}\right)\Bigg].$$

Proof. (See also [6] and [12], Lemma 3.) Since

$$\left|\frac{x^{a(1-s)} - x^{1-s}}{(1-s)^2}\right| < \frac{e^{-ac}x^{a/2} + e^{-c}x^{1/2}}{t^2},$$

$$\left|\sum_{q=1}^{\infty}\frac{x^{-(2q+s)} - x^{-a(2q+s)}}{(2q+s)^2}\right| < \frac{1}{t^2}\left(\frac{1}{x^{\sigma}(x^2-1)} + \frac{1}{x^{a\sigma}(x^{2a}-1)}\right)$$

and

$$\left|\sum_{\varrho}\frac{x^{\varrho-s} - x^{a(\varrho-s)}}{(\varrho-s)^2}\right| \le (e^{-c}+e^{-ac})\sum_{\gamma}\frac{1}{(\sigma-1/2)^2+(t-\gamma)^2}, \quad \text{where } \varrho = 1/2 + i\gamma,$$

we obtain from (2)

$$(4)\qquad \frac{\zeta'}{\zeta}(s) = -\sum_{n<x^a}\frac{\Lambda_{x,a}(n)}{n^s} + \frac{1}{(a-1)\log x}\Bigg[v_1\frac{e^{-ac}x^{a/2}+e^{-c}x^{1/2}}{t^2} + \frac{v_2}{t^2}\left(\frac{1}{x^{\sigma}(x^2-1)} + \frac{1}{x^{a\sigma}(x^{2a}-1)}\right) + v_3(e^{-c}+e^{-ac})\sum_{\gamma}\frac{1}{(\sigma-1/2)^2+(t-\gamma)^2}\Bigg],$$

where $v_j = v_j(\sigma,t)$ and $|v_j| \le 1$ for $j = 1,2,3$.

According to (1) we have

$$(5)\quad \Re\frac{\zeta'}{\zeta}(s) = \frac{1-\sigma}{(1-\sigma)^2+t^2} + \sum_{\gamma}\left(\frac{\sigma-1/2}{(\sigma-1/2)^2+(t-\gamma)^2} + \frac{1/2}{1/4+\gamma^2}\right) + \log(2\pi) - 1 - \sum_{n=1}^{\infty}\frac{\sigma^2+2n\sigma+t^2}{2n[(\sigma+2n)^2+t^2]}.$$

Since

$$0 < \frac{1-\sigma}{(1-\sigma)^2+t^2} < \frac{1}{392.125},$$

$$0 < \sum_{n=1}^{\infty} \frac{\sigma^2+2n\sigma+t^2}{2n[(\sigma+2n)^2+t^2]} < 0.5427592 + \frac{1}{2}\log\frac{\sqrt{4+t^2}}{2},$$

and

$$k_1 := \sum_{\gamma} \frac{1/2}{1/4+\gamma^2} = 0.023\ldots \qquad (\text{cf. [2], p. 82}) \tag{6}$$

we can write (5) in the following way:

$$\begin{aligned} v_6\frac{\zeta'}{\zeta}(s) = {} & \frac{v_4}{392.125} + \sum_{\gamma} \frac{\sigma-1/2}{(\sigma-1/2)^2+(t-\gamma)^2} + k_1 + \log(2\pi) - 1 \\ & - v_5\biggl(0.5427592 + \frac{1}{2}\log\frac{\sqrt{4+t^2}}{2}\biggr), \end{aligned} \tag{7}$$

where $v_j = v_j(\sigma,t)$ for $j = 4,5,6$, $v_4, v_5 \in (0,1)$ and $|v_6| \le 1$.

Comparing (4) and (7) we obtain

$$\begin{aligned} & \frac{\zeta'}{\zeta}(s)\biggl(1 - v_3v_6\frac{e^{-c}+e^{-ac}}{(a-1)\log x(\sigma-1/2)}\biggr) \\ & \quad = -\sum_{n<x^a} \frac{\Lambda_{x,a}(n)}{n^s} + v_3\frac{e^{-c}+e^{-ac}}{(a-1)\log x(\sigma-1/2)} \\ & \qquad \times\biggl[-\frac{v_4}{392.125} - k_1 - \log(2\pi) + 1 + v_5\biggl(0.5427592 + \frac{1}{2}\log\frac{\sqrt{4+t^2}}{2}\biggr)\biggr] \\ & \qquad + \frac{1}{(a-1)t^2\log x}\biggl[v_1(e^{-ac}x^{a/2}+e^{-c}x^{1/2}) + \frac{v_2}{x^{\sigma}(x^2-1)} + \frac{v_2}{x^{a\sigma}(x^{2a}-1)}\biggr]. \end{aligned}$$

Since $\sigma - 1/2 = c/\log x$ the lemma follows. ∎

LEMMA 4. *Let* $T > 7.02, c > 0, \alpha > 1, a > 1$, $e^{\delta} = 1 + 1/T$ *and* $\sigma = 1/2 + ac/(\log T) \le 3/4$. *Then under the Riemann Hypothesis we have*

$$\int_0^{\infty} \biggl|\frac{e^{\delta(\sigma+it)}-1}{\sigma+it}\biggr|^2 \biggl|\frac{\zeta'}{\zeta}(\sigma+it)\biggr|^2\,dt \le f(a,c,T,\alpha)\frac{\log^2 T}{T}, \tag{8}$$

where

$$f(a,c,T,\alpha) = f_1 + \Bigl(\sum_{j=2}^{4} f_j^{1/2}\Bigr)^2 f_5,$$

and

$$f_1 = f_1(T) = \frac{504}{T\log T}\left(\frac{1}{1-7.02/T}\right)^2,$$

$$f_2 = f_2(a,c,T) = \frac{\pi}{1-\sigma/T}\left(\frac{1}{4a^2c^2} - \frac{0.144}{\log^2 T}\right),$$

$$f_3 = f_3(a,c,T,\alpha) = \left[\frac{\alpha^2\pi}{1-\sigma/T} + \left(2+\frac{1}{T}\right)^2\left(\frac{\alpha^2}{T^{\alpha-1}} + \frac{2\alpha}{T^{\alpha-1}\log T} + \frac{2}{T^{\alpha-1}\log^2 T}\right)\right] \times \left(\frac{e^{-c}+e^{-ac}}{2(a-1)c}\right)^2,$$

$$f_4 = f_4(a,c,T) = \frac{a^2\beta^2(1+1/T)^{2\sigma}}{8232(a-1)^2\log^4 T},$$

where $\beta = \beta(a,c,T)$ *is defined by* (14) *below and for sufficiently large* T *is equivalent to* e^{-ac}, *and finally*

$$f_5 = f_5(a,c) = \left(\frac{(a-1)c}{(a-1)c - e^{-c} - e^{-ac}}\right)^2.$$

Proof. We make use of the following integral formula:

$$\int_0^\infty \left|\frac{e^{\delta(\sigma+it)}-1}{\sigma+it}\right|^2 \cos(At)\,dt = \begin{cases} 0 & \text{for } A \ge \delta, \\ \frac{\pi}{2\sigma}(e^{\sigma(2\delta-A)} - e^{\sigma A}) & \text{for } 0 \le A < \delta, \end{cases} \tag{9}$$

which easily follows from the following well known identity:

$$\int_0^\infty \frac{\cos x}{x^2+h^2}\,dx = \frac{\pi}{2he^h},$$

valid for every positive h.

We also need the following elementary inequality:

$$\left|\frac{e^s-1}{s}\right| \le \sum_{k=1}^\infty \frac{|s|^{k-1}}{k!} \le \frac{1}{1-|s|/2}, \tag{10}$$

satisfied for every complex $|s| < 2$.

We split the range of integration on the left-hand side of (8) into two intervals $(0,14), [14,\infty)$. The first part contributes at most

$$\frac{504}{T^2}\left(\frac{1}{1-7.02/T}\right)^2$$

since according to (10) we have

$$\left|\frac{e^{\delta(\sigma+it)}-1}{\sigma+it}\right| < \frac{\delta}{1-\delta|\sigma+it|/2} < \frac{1}{T(1-7.02/T)}$$

and

$$\left|\frac{\zeta'}{\zeta}(\sigma+it)\right| \le 6 \quad \text{for } 0 < t \le 14.$$

The last inequality follows easily from (1) taking into account the numerical values of the first three zeros lying on the critical line (given for example in [4], p. 96), and the known value of k_1 (see (6)).

In order to estimate the second integral we use Lemma 3. Writing

$$\left|\frac{\zeta'}{\zeta}(s)\right| < \sqrt{f_5}(A+B+C),$$

say, after applying Minkowski's inequality we obtain

$$\int_{14}^{\infty}\left|\frac{e^{\delta(\sigma+it)}-1}{\sigma+it}\right|^2\left|\frac{\zeta'}{\zeta}(\sigma+it)\right|^2 dt < f_5\Big(\sum_{j=1}^{3} J_j^{1/2}\Big)^2,$$

where

$$J_1 = \int_{14}^{\infty}\left|\frac{e^{\delta(\sigma+it)}-1}{\sigma+it}\right|^2\left|\sum_{n<x^a}\frac{\Lambda_{x,a}(n)}{n^{\sigma+it}}\right|^2 dt,$$

$$J_2 = \left(\frac{e^{-c}+e^{-ac}}{2(a-1)c}\right)^2\int_{14}^{\infty}\left|\frac{e^{\delta(\sigma+it)}-1}{\sigma+it}\right|^2\log^2 t\,dt$$

and

$$J_3 = \int_{14}^{\infty}\left|\frac{e^{\delta(\sigma+it)}-1}{\sigma+it}\right|^2\frac{1}{(a-1)^2t^4\log^2 x} \times\left(e^{-ac}x^{a/2}+e^{-c}x^{1/2}+\frac{1}{x^{\sigma}(x^2-1)}+\frac{1}{x^{a\sigma}(x^{2a}-1)}\right)^2 dt.$$

Taking now $T = x^a$ in the preceding lemma we obtain

$$\begin{aligned} J_1 &< \int_0^{\infty}\left|\frac{e^{\delta(\sigma+it)}-1}{\sigma+it}\right|^2\sum_{n<x^a}\frac{\Lambda_{x,a}^2(n)}{n^{2\sigma}}\,dt \\ &\quad + \int_0^{\infty}\left|\frac{e^{\delta(\sigma+it)}-1}{\sigma+it}\right|^2\sum\sum_{m,n<x^a,\, m\ne n}\frac{\Lambda_{x,a}(m)\Lambda_{x,a}(n)}{(mn)^{\sigma}(n/m)^{it}}\,dt \\ &= \sum_{n<x^a}\frac{\Lambda_{x,a}^2(n)}{n^{2\sigma}}\cdot\frac{\pi(e^{2\delta\sigma}-1)}{2\sigma} \end{aligned}$$

$$+\sum\sum_{n<m<x^a}\frac{\Lambda_{x,a}(m)\Lambda_{x,a}(n)}{(mn)^\sigma}\cdot 2\int\limits_0^\infty\left|\frac{e^{\delta(\sigma+it)}-1}{\sigma+it}\right|^2\cos\left(t\log\frac{m}{n}\right)dt$$
$$=\delta\pi\frac{e^{2\delta\sigma}-1}{2\delta\sigma}\sum_{n<x^a}\frac{\Lambda_{x,a}^2(n)}{n^{2\sigma}}.$$

Since $\log\frac{m}{n}\ge\log\frac{m}{m-1}>\log(1+1/T)$, all the integrals vanish according to (9).

Next using (1) we have

$$\sum_{n<x^a}\frac{\Lambda_{x,a}^2(n)}{n^{2\sigma}}\le\sum_{n<x^a}\frac{\Lambda^2(n)}{n^{2\sigma}}<\sum_{n=1}^\infty\frac{\Lambda(n)}{n^{2\sigma}}\log n=\frac{d}{ds}\left\{\frac{\zeta'}{\zeta}(s)\right\}_{s=2\sigma}$$
$$=\left\{\frac{1}{(1-s)^2}-\sum_\varrho\frac{1}{(s-\varrho)^2}-\sum_{n=1}^\infty\frac{1}{(s+2n)^2}\right\}_{s=2\sigma}$$
$$=\frac{1}{(2\sigma-1)^2}-\sum_\gamma\frac{1}{(2\sigma-1/2-i\gamma)^2}-\sum_{n=1}^\infty\frac{1}{4(\sigma+n)^2}$$
$$<\frac{\log^2T}{4a^2c^2}+2\sum_{\gamma>0}\frac{1}{1/4+\gamma^2}-2\sum_{\gamma>0}\frac{1/4}{(1+\gamma^2)^2}-\sum_{n=1}^\infty\frac{1}{4(3/4+n)^2}$$
$$<\frac{\log^2T}{4a^2c^2}-0.144.$$

Hence by (10),

$$J_1<\frac{\pi}{T(1-\sigma/T)}\left(\frac{\log^2T}{4a^2c^2}-0.144\right). \tag{11}$$

Using (9), (10) and the easy inequality

$$|e^{\delta(\sigma+it)}-1|<2+1/T \tag{12}$$

we have

$$\int\limits_{14}^\infty\left|\frac{e^{\delta(\sigma+it)}-1}{\sigma+it}\right|^2\log^2t\,dt$$
$$=\int\limits_{14}^{T^\alpha}+\int\limits_{T^\alpha}^\infty<\int\limits_0^\infty\left|\frac{e^{\delta(\sigma+it)}-1}{\sigma+it}\right|^2\log^2T^\alpha\,dt+\int\limits_{T^\alpha}^\infty\frac{|e^{\delta(\sigma+it)}-1|^2}{t^2}\log^2t\,dt$$
$$<\alpha^2\log^2T\cdot\frac{\pi}{2\sigma}(e^{2\delta\sigma}-1)+\left(2+\frac{1}{T}\right)^2\int\limits_{T^\alpha}^\infty\frac{\log^2t}{t^2}\,dt$$
$$<\alpha^2\log^2T\,\frac{\pi}{T(1-\sigma/T)}+\left(2+\frac{1}{T}\right)^2\left(\frac{\alpha^2\log^2T}{T^\alpha}+\frac{2\alpha\log T}{T^\alpha}+\frac{2}{T^\alpha}\right),$$

and hence finally,

$$(13)\quad J_2 < \left[\alpha^2\log^2 T\,\frac{\pi}{T(1-\sigma/T)} + \left(2+\frac{1}{T}\right)^2\left(\frac{\alpha^2\log^2 T}{T^\alpha}+\frac{2\alpha\log T}{T^\alpha}+\frac{2}{T^\alpha}\right)\right] \times \left(\frac{e^{-c}+e^{-ac}}{2(a-1)c}\right)^2.$$

Now let us define β so that

$$(14)\qquad e^{-ac}x^{a/2}+e^{-c}x^{1/2}+\frac{1}{x^\sigma(x^2-1)}+\frac{1}{x^{a\sigma}(x^{2a}-1)} = \beta T^{1/2}.$$

We easily get

$$(15)\qquad \frac{1}{\delta}\left|\frac{e^{\delta(\sigma+it)}-1}{\sigma+it}\right| \le \left(1+\frac{1}{T}\right)^\sigma$$

and hence

$$(16)\qquad J_3 = \int_{14}^{\infty}\left|\frac{e^{\delta(\sigma+it)}-1}{\sigma+it}\right|^2 \frac{a^2\beta^2 T}{(a-1)^2 t^4\log^2 T}\,dt \le \delta^2\left(1+\frac{1}{T}\right)^{2\sigma}\frac{a^2\beta^2 T}{(a-1)^2\log^2 T}\int_{14}^{\infty}\frac{dt}{t^4} < \left(1+\frac{1}{T}\right)^{2\sigma}\frac{a^2\beta^2}{8232(a-1)^2 T\log^2 T}.$$

Now our lemma follows easily from (11), (13) and (16). ∎

Let as usual $(x>1)$

$$\theta(x) = \sum_{p\le x,\, p\ \mathrm{prime}} \log p,\qquad \theta_0(x) = \frac{1}{2}(\theta(x-0)+\theta(x+0)).$$

LEMMA 5. *Suppose R.H. is true. Then*

$$(17)\qquad I := \int_1^\infty \left|\frac{\theta(y+y/T)-\theta(y)-y/T}{y^{1/2+\sigma}}\right|^2 dy \le \frac{1}{\pi}(f^{1/2}+f_6^{1/2}+f_7^{1/2})^2\frac{\log^2 T}{T},$$

where

$$f_6 = f_6(a,c,T) = \frac{\pi}{(1-\sigma/T)\log^2 T}\left(0.886+\frac{\log^2 T}{8Ta^2c^2}+\frac{\log T}{4T^2ac}\right),$$

$$f_7 = f_7(a,c,T) = \frac{4.51\pi}{(1-\sigma/T)\log^2 T}$$

and f has the same meaning as in Lemma 4.

Proof. The following identity holds (see [12]):

$$\frac{\theta_0(e^{\delta+\tau})-\theta_0(e^\tau)-e^\tau/T}{e^{\sigma\tau}} = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{\delta(\sigma+it)}-1}{\sigma+it}e^{it\tau}\left\{\frac{\zeta'}{\zeta}(\sigma+it)+g(\sigma+it)\right\}dt, \tag{18}$$

where $\tau>0$ and

$$g(s)=\sum_{r=2}^{\infty}\sum_{p\ \mathrm{prime}}\frac{\log p}{p^{rs}}\qquad \text{for } \Re s>1/2.$$

Thus by the Parseval theorem

$$\int_0^{\infty}\left|\frac{\theta_0(e^{\delta+\tau})-\theta_0(e^\tau)-e^\tau/T}{e^{\sigma\tau}}\right|^2 d\tau \le \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{e^{\delta(\sigma+it)}-1}{\sigma+it}\right|^2\left|\frac{\zeta'}{\zeta}(\sigma+it)+g(\sigma+it)\right|^2 dt.$$

Putting on the left-hand side $e^\tau=y$ and writing θ instead of θ_0 we obtain

$$I\le\frac{1}{\pi}\int_0^{\infty}\left|\frac{e^{\delta(\sigma+it)}-1}{\sigma+it}\right|^2\left|\frac{\zeta'}{\zeta}(\sigma+it)+g(\sigma+it)\right|^2 dt. \tag{19}$$

For $\Re s>1/2$ we have

$$g(s)=\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{2s}}+\sum_{p\ \mathrm{prime}}\frac{\log p}{p^s(p^{2s}-1)}\,. \tag{20}$$

Inserting this identity to (19) and applying Minkowski's inequality again we obtain

$$I\le\frac{1}{\pi}\Big(\sum_{j=1}^{3}I_j^{1/2}\Big)^2,$$

where

$$I_1=\int_0^{\infty}\left|\frac{e^{\delta(\sigma+it)}-1}{\sigma+it}\right|^2\left|\frac{\zeta'}{\zeta}(\sigma+it)\right|^2 dt,$$

$$I_2=\int_0^{\infty}\left|\frac{e^{\delta(\sigma+it)}-1}{\sigma+it}\right|^2\left|\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{2(\sigma+it)}}\right|^2 dt,$$

$$I_3=\int_0^{\infty}\left|\frac{e^{\delta(\sigma+it)}-1}{\sigma+it}\right|^2\left|\sum_{p\ \mathrm{prime}}\frac{\log p}{p^{\sigma+it}(p^{2(\sigma+it)}-1)}\right|^2 dt.$$

The first integral is estimated in Lemma 4. To estimate I_2 we apply (9), (10) and (1):

$$I_2 = \sum_{n=1}^{\infty} \frac{\Lambda^2(n)}{n^{4\sigma}} \int_0^{\infty} \left| \frac{e^{\delta(\sigma+it)}-1}{\sigma+it} \right|^2 dt$$

$$+ 2 \sum\sum_{m>n} \frac{\Lambda(m)\Lambda(n)}{(mn)^{2\sigma}} \int_0^{\infty} \left| \frac{e^{\delta(\sigma+it)}-1}{\sigma+it} \right|^2 \cos\left(t \log \frac{m^2}{n^2}\right) dt$$

$$= \delta\pi \sum_{n=1}^{\infty} \frac{\Lambda^2(n)}{n^{4\sigma}} \cdot \frac{e^{2\delta\sigma}-1}{2\delta\sigma}$$

$$+ 2 \sum\sum_{m>n} \frac{\Lambda(m)\Lambda(n)}{(mn)^{2\sigma}} \cdot \begin{cases} 0 & \text{for } \log \frac{m^2}{n^2} \ge \delta \\ \frac{\pi}{2\sigma}(e^{2\delta\sigma}-1)\left(\frac{m}{n}\right)^{2\sigma} & \text{for } \log \frac{m^2}{n^2} < \delta \end{cases}$$

$$< \frac{\pi}{T(1-\sigma/T)} \left(\frac{d}{dt} \left\{ \frac{\zeta'}{\zeta}(s) \right\}_{s=4\sigma} + 2 \sum\sum_{n^2<m^2<n^2(1+1/T)} \frac{\Lambda(m)\Lambda(n)}{n^{4\sigma}} \right)$$

$$< \frac{\pi}{T(1-\sigma/T)} \left[0.886 + \frac{2}{T} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{4\sigma-1}} \left(\log n + \frac{1}{2T} \right) \right]$$

$$= \frac{\pi}{T(1-\sigma/T)} \left(0.886 + \frac{2}{T} \cdot \frac{d}{ds} \left\{ \frac{\zeta'}{\zeta}(s) \right\}_{s=4\sigma-1} - \frac{1}{T^2} \cdot \frac{\zeta'}{\zeta}(4\sigma-1) \right)$$

$$< \frac{\pi}{T(1-\sigma/T)} \left(0.886 + \frac{\log^2 T}{8Ta^2c^2} + \frac{\log T}{4T^2ac} \right)$$

and hence

$$I_2 < \frac{\pi}{T(1-\sigma/T)} \left(0.886 + \frac{\log^2 T}{8Ta^2c^2} + \frac{\log T}{4T^2ac} \right). \tag{21}$$

Finally, applying (9) and (10) once more we have

$$I_3 \le \left(\sum_{p \text{ prime}} \frac{\log p}{p^{\sigma}(p^{2\sigma}-1)} \right)^2 \int_0^{\infty} \left| \frac{e^{\delta(\sigma+it)}-1}{\sigma+it} \right|^2 dt \tag{22}$$

$$< \frac{\pi}{T(1-\sigma/T)} \left(\sum_{p \text{ prime}} \frac{\log p}{\sqrt{p}(p-1)} \right)^2$$

$$< 4.51 \frac{\pi}{T(1-\sigma/T)}.$$

Gathering (19), (21) and (22) the result follows. ∎

Let (p,p') denote a generic pair of two consecutive primes. Let us write

$$\lambda(x)=\begin{cases}2 & \text{for } 0<x\le 7,\\ \max_{p\le x}(p'-p) & \text{for } x>7.\end{cases}$$

LEMMA 6. *Under the Riemann Hypothesis we have*

$$\lambda(x)<\frac{1}{4\pi}\sqrt{\left(1+\frac{1}{16597}\right)x}\ \log^2 x \quad \text{for } x\ge 2.3\cdot 10^9.$$

Proof. This follows easily from Theorems 10 and 12 of [11]. ∎

LEMMA 7. *Every interval of the form* $(x,x+\lambda(\lambda(x)))$, $x>4$, *contains a sum of two odd primes.*

Proof. Let p_1 be the greatest odd prime smaller than x and let p_2 be the smallest odd prime greater than $x-p_1$. Then $p_1+p_2>x$ and

$$p_1+p_2-x=p_2-(x-p_1)<\lambda(x-p_1)<\lambda(\lambda(x)).$$

The lemma hence follows. ∎

3. Proof of the Theorem. Obviously it is enough to prove our Theorem for integers $\le n_0:=e^{e^{11.503}}$ since for larger numbers the situation is clear (cf. [1]). We indicate first a real constant $h\ge 7$ with the property that every interval of the form $[x,x+h]$, $0\le x\le n_0$, contains a sum of two odd primes. Suppose on the contrary that a certain interval of this sort contains no such sum, and put

$$\begin{aligned}A&=\{0\le y\le x:(y,y+h/2)\cap\mathcal{P}_0=\emptyset\},\\ B&=\{0\le y\le x:(y,y+h/2)\cap\mathcal{P}_0\ne\emptyset\},\\ C&=\{0\le y\le x:(x-y,x-y+h/2)\cap\mathcal{P}_0\ne\emptyset\},\end{aligned}$$

$\mathcal{P}_0$ denoting the set of all odd primes. Then clearly $B\cup C\subset[0,x]$, and both sets have the same Lebesgue measure: $\mu(B)=\mu(C)$. Moreover, $B\cap C=\emptyset$. Indeed, otherwise there would exist two odd primes p_1, p_2 and a real number $y_0\in[0,x]$ such that $p_1\in(y_0,y_0+h/2)$ and $p_2\in(x-y_0,x-y_0+h/2)$. Then p_1+p_2 would belong to $(x,x+h)$, which is impossible. Thus we have $\mu(B)\le x/2$, and consequently $\mu(A)=x-\mu(B)\ge x/2$.

We consider the following ranges for x separately: $4<x\le 7.263\cdot 10^{13}$, $7.263\cdot 10^{13}<x\le e^{78}$, $e^{78}<x\le e^{84000}$ and $e^{84000}<x\le n_0$.

In the first interval the situation is clear. According to [16] the maximal gap between prime numbers up to $7.263\cdot 10^{13}$ is at most 778. Hence by Lemma 7 we can take $h=\lambda(778)=18$ in this case.

For $7.263\cdot 10^{13}<x\le e^{78}$ according to Lemma 6 we have $\lambda(x)<4.1926\cdot 10^{19}$ and thus we can take $h=\lambda(4.1926\cdot 10^{19})<1.052\cdot 10^{12}$.

For larger x we assume additionally that $h \le 2e^{-12}x$. Consider the integral

$$J := \int_2^\infty \left| \frac{\theta(y+y/T) - \theta(y) - y/T}{y^{1/2+\sigma}} \right|^2 dy$$

with

$$T = \frac{2x}{h}, \quad \sigma = \frac{1}{2} + \frac{ac}{\log T} \quad \text{and} \quad 1.6 \le a \le 2, \quad 1 \le c \le 1.5.$$

Then since $A \subset (2, x]$ we have

$$J \ge \int_A \left| \frac{\theta(y+y/T) - \theta(y) - y/T}{y^{1/2+\sigma}} \right|^2 dy = \frac{1}{T^2} \int_A y^{1-2\sigma}\, dy \tag{23}$$
$$\ge \frac{1}{T^2} x^{1-2\sigma} \mu(A) \ge \frac{1}{2T^2} x^{2(1-\sigma)}.$$

In case $e^{78} < x \le e^{84000}$, using Lemmas 7 and 6 again we have $T \ge e^{51}$. Hence taking $a = 1.96745$, $c = 1.32149$ and $\alpha = 1.07$ in Lemmas 4 and 5 we obtain

$$f_1(T) < 10^{-22}, \quad f_2(a,c,T) < 0.11619, \quad f_3(a,c,T,\alpha) < 0.06636,$$
$$f_4(a,c,T) < 5 \cdot 10^{-13}, \quad f_5(a,c) < 1.85985, \quad f(a,c,T,\alpha) < 0.66614,$$
$$f_6(a,c,T) < 0.0010702, \quad f_7(a,c,T) < 0.00544736$$

and consequently

$$J < 0.271 \frac{\log^2 T}{T}\,. \tag{24}$$

Comparing (23) and (24) we obtain

$$h < 4 \cdot 0.271 e^{2ac \frac{\log x}{\log((2x)/h)}} \log^2 \frac{2x}{h}, \tag{25}$$

which after some elementary computations yields $h < 1.4 \cdot 10^{12}$.

For $x > e^{84000}$, from Lemma 6 we have

$$\lambda(x) < x^{0.50024}, \quad \lambda(\lambda(x)) < x^{0.25035}.$$

Then from Lemma 7 and inequality (25) we obtain

$$h < 1116 \log^2 x. \tag{26}$$

In this case we have $T \ge e^{83970}$. With the same parameters a and c in Lemmas 4 and 5 and with $\alpha = 1.00013$ we have

$$f_1(T) < 10^{-36474}, \quad f_2(a,c,T) < 0.11619, \quad f_3(a,c,T,\alpha) < 0.055895,$$
$$f_4(a,c,T) < 10^{-25}, \quad f_5(a,c) < 1.85985, \quad f(a,c,T,\alpha) < 0.619816,$$
$$f_6(a,c,T) < 4 \cdot 10^{-10}, \quad f_7(a,c,T) < 2.01 \cdot 10^{-9}.$$

Then

$$J < 0.1973261 \frac{\log^2 T}{T},$$

and by (26),

$$h < 143.3263 \log^2 \frac{2x}{h}, \tag{27}$$

which for $x \le n_0$ yields $h < 1.4045 \cdot 10^{12}$.

Hence every interval of the form $[x, x + 1.4045 \cdot 10^{12}]$, $x \le n_0$, contains at least one sum of two odd primes. Applying this observation with $x = n - 2$ we infer that for every $9 < n \le n_0$ there exist two odd prime numbers, p_1 and p_2 say, such that

$$3 < m := n - (p_1 + p_2) < 1.405 \cdot 10^{12}.$$

If n is odd then m is odd as well, and according to [14] and [16] it is a sum of at most three primes. Hence n itself is a sum of at most five primes. In case of even n we argue similarly. If $G_2 > 1.405 \cdot 10^{12}$, m is a sum of at most two primes, and n a sum of at most four primes. Our Theorem therefore follows. ∎

Remark. Inequality (27) gives us the following result, connected with Theorem 2 of [7]:

If R.H. is true then for $x \ge e^{84000}$ every interval $[x, x + 144 \log^2 x]$ contains a sum of two primes.

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