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Ideal class groups of cyclotomic number fields I

by

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1. Notation. Let $K \subset L$ be number fields; we will use the following notation:

- $\mathfrak{O}_K$ is the ring of integers of K;
- E_K is its group of units;
- W_K is the group of roots of unity contained in K;
- w_K is the order of W_K;
- $\mathrm{Cl}(K)$ is the ideal class group of K;
- $[\mathfrak{a}]$ is the ideal class generated by the ideal $\mathfrak{a}$;
- K^1 denotes the Hilbert class field of K, that is the maximal abelian extension of K which is unramified at all places;
- $j_{K\to L}$ denotes the transfer of ideal classes for number fields $K \subset L$, i.e. the homomorphism $\mathrm{Cl}(K) \to \mathrm{Cl}(L)$ induced by mapping an ideal $\mathfrak{a}$ to $\mathfrak{a}\mathfrak{O}_L$;
- $\kappa_{L/K}$ denotes the capitulation kernel $\ker j_{K\to L}$;

Now let K be a CM-field, i.e. a totally complex quadratic extension of a totally real number field; the following definitions are standard:

- σ is complex conjugation;
- K^+ denotes the maximal real subfield of K; this is the subfield fixed by σ;
- $\mathrm{Cl}^-(K)$ is the kernel of the map $N_{K/K^+} : \mathrm{Cl}(K) \to \mathrm{Cl}(K^+)$ and is called the minus class group;
- $h^-(K)$ is the order of $\mathrm{Cl}^-(K)$, the minus class number;
- $Q(K) = (E_K : W_K E_{K^+}) \in \{1, 2\}$ is Hasse's unit index.

We will need a well known result from class field theory. Assume that $K \subset L$ are CM-fields; then $\ker(N_{L/K} : \mathrm{Cl}(L) \to \mathrm{Cl}(K))$ has order $(L \cap K^1 : K)$. Since K/K^+ is ramified at the infinite places, the norm $N_{K/K^+} : \mathrm{Cl}(K) \to \mathrm{Cl}(K^+)$ is onto.

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2. Hasse's unit index. Hasse's book [H] contains numerous theorems (Sätze 14–29) concerning the unit index $Q(L) = (E_L : W_L E_K)$, where $K = L^+$ is the maximal real subfield of a cyclotomic number field L. Hasse considered only abelian number fields $L/\mathbb{Q}$, hence he was able to describe these fields in terms of their character groups $X(L)$; as we are interested in results on general CM-fields, we have to proceed in a different manner. But first we will collect some of the most elementary properties of $Q(L)$ (see also [H] and [W]; a reference "Satz $*$" always refers to Hasse's book [H]) in

PROPOSITION 1. *Let $K \subset L$ be CM-fields; then*

(a) (Satz 14) $Q(L) = (E_L : W_L E_{L^+}) = (E_L^{\sigma-1} : W_L^2) = (E_L^{\sigma+1} : E_{L^+}^2)$; *in particular*, $Q(L) \in \{1, 2\}$;

(b) (Satz 16, 17) *If $Q(L) = 2$ then $\kappa_{L/L^+} = 1$;*

(c) (Satz 25) *If L^+ contains units with any given signature, then $Q(L) = 1$;*

(d) (Satz 29) $Q(K) \,|\, Q(L) \cdot (W_L : W_K)$;

(e) (compare Satz 26) *Suppose that $N_{L/K} : W_L/W_L^2 \to W_K/W_K^2$ is onto. Then $Q(L) \mid Q(K)$;*

(f) ([HY, Lemma 2]) *If $(L : K)$ is odd, then $Q(L) = Q(K)$;*

(g) (Satz 27) *If $L = \mathbb{Q}(\zeta_m)$, where $m \not\equiv 2 \bmod 4$ is composite, then $Q(L) = 2$;*

(h) (see Example 4 below) *Let $K_1 \subseteq \mathbb{Q}(\zeta_m)$ and $K_2 \subseteq \mathbb{Q}(\zeta_n)$ be abelian CM-fields, where $m = p^\mu$ and $n = q^\nu$ are prime powers such that $p \neq q$, and let $K = K_1 K_2$; then $Q(K) = 2$.*

The proofs are straightforward:

(a) The map $\varepsilon \to \varepsilon^{\sigma-1}$ induces an epimorphism $E_L \to E_L^{\sigma-1}/W_L^2$. If $\varepsilon^{\sigma-1} = \zeta^2$ for some $\zeta \in W_L$, then $(\zeta\varepsilon)^{\sigma-1} = 1$, and $\zeta\varepsilon \in E_{L^+}$. This shows that $\sigma - 1$ gives rise to an isomorphism $E_L/W_L E_{L^+} \to E_L^{\sigma-1}/W_L^2$, hence we have $(E_L : W_L E_{L^+}) = (E_L^{\sigma-1} : W_L^2)$. The other claim is proved similarly.

(b) Since W_L/W_L^2 is cyclic of order 2, the first claim follows immediately from (a). Now let $\mathfrak{a}$ be an ideal in $\mathfrak{O}_K$ such that $\mathfrak{a}\mathfrak{O}_L = \alpha\mathfrak{O}_L$. Then $\alpha^{\sigma-1} = \zeta$ for some root of unity $\zeta \in L$, and $Q(L) = 2$ shows that $\zeta = \varepsilon^{\sigma-1}$ for some $\varepsilon \in E_L$. Now $\alpha\varepsilon^{-1}$ generates $\mathfrak{a}$ and is fixed by σ, hence lies in K. This shows that $\mathfrak{a}$ is principal in K, i.e. that $\kappa_{L/L^+} = 1$.

(c) Units in L^+ which are norms from L are totally positive; our assumption implies that totally positive units are squares, hence we get $E_L^{\sigma+1} = E_{L^+}^2$, and our claim follows from (a).

(d) First note that $(W_L : W_K) = (W_L^2 : W_K^2)$; then

$$Q(L)\cdot(W_L : W_K) = (E_L^{\sigma-1} : W_L^2)(W_L^2 : W_K^2) = (E_L^{\sigma-1} : E_K^{\sigma-1})(E_K^{\sigma-1} : W_K^2)$$
$$= (E_L^{\sigma-1} : E_K^{\sigma-1})\cdot Q(K)$$

proves the claim.

(e) Since $Q(L) = 2$, there is a unit $\varepsilon \in E_L$ such that $\varepsilon^{\sigma-1} = \zeta$ generates W_L/W_L^2. Taking the norm to K shows that $(N_{L/K}\varepsilon)^{\sigma-1} = N_{L/K}(\zeta)$ generates W_K/W_K^2, i.e. we have $Q(K) = 2$.

(f) If $(L : K)$ is odd, then $(W_L : W_K)$ is odd, too, and we get $Q(K) \mid Q(L)$ from (d) and $Q(L) \mid Q(K)$ from (e).

(g) In this case, $1 - \zeta_m$ is a unit, and we find $(1-\zeta_m)^{1-\sigma} = -\zeta_m$. Since $-\zeta_m \in W_L \setminus W_L^2$, we must have $Q(L) = 2$.

(h) First assume that m and n are odd. A subfield $F \subseteq L = \mathbb{Q}(\zeta_m)$, where $m = p^\mu$ is an odd prime power, is a CM-field if and only if it contains the maximal 2-extension contained in L, i.e. if and only if $(L : F)$ is odd. Since $(\mathbb{Q}(\zeta_m) : K_1)$ and $(\mathbb{Q}(\zeta_n) : K_2)$ are both odd, so is $(\mathbb{Q}(\zeta_{mn}) : K_1K_2)$; moreover, $\mathbb{Q}(\zeta_{mn})$ has unit index $Q = 2$, hence the assertion follows from (f) and (g).

Now assume that $p = 2$. If $\sqrt{-1} \in K_1$, then we must have $K_1 = \mathbb{Q}(\zeta_m)$ for $m = 2^\alpha$ and some $\alpha \geq 2$ (complex subfields of the field of 2^μth roots of unity containing $\sqrt{-1}$ necessarily have this form). Now n is odd and $K_2 \subseteq \mathbb{Q}(\zeta_n)$ is complex, hence $(\mathbb{Q}(\zeta_n) : K_2)$ is odd. By (f) it suffices to show that $K_1(\zeta_n) = \mathbb{Q}(\zeta_{mn})$ has unit index 2, and this follows from (g).

If $\sqrt{-1} \notin K_1$, let $\widetilde{K}_1 = K_1(i)$; then $\widetilde{K}_1 = \mathbb{Q}(\zeta_m)$ for $m = 2^\alpha$ and some $\alpha \geq 2$, and in the last paragraph we have seen that $Q(\widetilde{K}_1K_2) = 2$. Hence we only need to show that the norm map

$$N : W_{\widetilde{K}_1}/W_{\widetilde{K}_1}^2 \to W_{K_1}/W_{K_1}^2$$

is onto: since $(W_{\widetilde{K}_1K_2} : W_{\widetilde{K}_1})$ is odd, this implies $2 = Q(\widetilde{K}_1K_2) \mid Q(K_1K_2)$ by (e). But the observation that the non-trivial automorphism of $\mathbb{Q}(\zeta_m)/K_1$ maps ζ_m to $-\zeta_m^{-1}$ implies at once that $N(\zeta_m) = -1$, and -1 generates $W_{K_1}/W_{K_1}^2$. ∎

Now let L be a CM-field with maximal real subfield K; we will call L/K *essentially ramified* if $L = K(\sqrt{\alpha})$ and there is a prime ideal $\mathfrak{p}$ in $\mathfrak{O}_K$ such that the exact power of $\mathfrak{p}$ dividing $\alpha\mathfrak{O}_K$ is odd; it is easily seen that this does not depend on which α we choose. Moreover, every ramified prime ideal $\mathfrak{p}$ above an odd prime p is necessarily essentially ramified. We leave it as an exercise to the reader to verify that our definition of essential ramification coincides with Hasse's [H, Sect. 22]; the key observation is the ideal equation $(4\alpha) = \mathfrak{a}^2\mathfrak{d}$, where $\mathfrak{d} = \mathrm{disc}(K(\sqrt{\alpha})/K)$ and $\mathfrak{a}$ is an integral ideal in $\mathfrak{O}_K$.

We will also need certain totally real elements of norm 2 in the field of

2^mth roots of unity: to this end we define

$$\begin{aligned}\pi_2 &= 2 = 2+\zeta_4+\zeta_4^{-1},\\ \pi_3 &= 2+\sqrt{2} = 2+\zeta_8+\zeta_8^{-1},\\ &\vdots\\ \pi_n &= 2+\sqrt{\pi_{n-1}} = 2+\zeta_{2^n}+\zeta_{2^n}^{-1}.\end{aligned}$$

Let $m \ge 2$, $L = \mathbb{Q}(\zeta_{2^{m+1}})$ and $K = \mathbb{Q}(\pi_m)$; then L/K is an extension of type $(2,2)$ with subfields $K_1 = \mathbb{Q}(\zeta_{2^m})$, $K_2 = \mathbb{Q}(\sqrt{\pi_m})$ and $K_3 = \mathbb{Q}(\sqrt{-\pi_m})$. Moreover, K_2/K and K_3/K are essentially ramified, whereas K_1/K is not.

THEOREM 1. *Let L be a CM-field with maximal real subfield K.*

(i) *If $w_L \equiv 2 \bmod 4$, then*:
  1. *If L/K is essentially ramified, then $Q(L) = 1$, and $\kappa_{L/K} = 1$.*
  2. *If L/K is not essentially ramified, then $L = K(\sqrt{\alpha})$ for some $\alpha \in \mathfrak{O}_K$ such that $\alpha\mathfrak{O}_K = \mathfrak{a}^2$, where $\mathfrak{a}$ is an integral ideal in $\mathfrak{O}_K$, and*
     (a) *$Q(L) = 2$ if $\mathfrak{a}$ is principal, and*
     (b) *$Q(L) = 1$ and $\kappa_{L/K} = \langle[\mathfrak{a}]\rangle$ if $\mathfrak{a}$ is not principal.*

(ii) *If $w_L \equiv 2^m \bmod 2^{m+1}$, where $m \ge 2$, then L/K is not essentially ramified, and*:
  1. *If $\pi_m\mathfrak{O}_K$ is not an ideal square, then $Q(L) = 1$ and $\kappa_{L/K} = 1$.*
  2. *If $\pi_m\mathfrak{O}_K = \mathfrak{b}^2$ for some integral ideal $\mathfrak{b}$, then*
     (a) *$Q(L) = 2$ if $\mathfrak{b}$ is principal, and*
     (b) *$Q(L) = 1$ and $\kappa_{L/K} = \langle[\mathfrak{b}]\rangle$ if $\mathfrak{b}$ is not principal.*

For the proof of Theorem 1 we will need the following

LEMMA 1. *Let $L = K(\sqrt{\pi})$, and let σ denote the non-trivial automorphism of L/K. Moreover, let $\mathfrak{b}$ be an ideal in $\mathfrak{O}_K$ such that $\mathfrak{b}\mathfrak{O}_L = (\beta)$ and $\beta^{\sigma-1} = -1$ for some $\beta \in L$. Then $\pi\mathfrak{O}_K$ is an ideal square in $\mathfrak{O}_K$. If, on the other hand, $\beta^{\sigma-1} = \zeta$, where ζ is a primitive 2^mth root of unity, then $\pi_m\mathfrak{O}_K$ is an ideal square in $\mathfrak{O}_K$.*

Proof. We have $(\beta\sqrt{\pi})^{\sigma-1} = 1$, hence $\beta\sqrt{\pi} \in K$. Therefore $\mathfrak{b}$ and $\mathfrak{c} = (\beta\sqrt{\pi})$ are ideals in $\mathfrak{O}_K$, and $(\mathfrak{c}\mathfrak{b}^{-1})^2 = \pi\mathfrak{O}_K$ proves our claim.

Now assume that $\beta^{\sigma-1} = \zeta$; then σ fixes $(1-\zeta)\beta^{-1}$, hence $((1-\zeta)\beta)$ and $\mathfrak{c} = (1-\zeta) = \mathfrak{c}^\sigma$ are ideals in $\mathfrak{O}_K$, and $\mathfrak{c}^2 = N_{L/K}(1-\zeta) = (2+\zeta+\zeta^{-1})\mathfrak{O}_K$ is indeed an ideal square in $\mathfrak{O}_K$ as claimed. ∎

Proof of Theorem 1. (i) Assume that $w_L \equiv 2 \bmod 4$.

Case 1: *L/K is essentially ramified.* Assume we had $Q(L) = 2$; then $E_L^{\sigma-1} = W_L$, hence there is a unit $\varepsilon \in E_L$ such that $\varepsilon^{\sigma-1} = -1$. Write $L = K(\sqrt{\pi}\,)$, and apply Lemma 1 to $\mathfrak{b} = (1), \beta = \varepsilon$: this will yield the contradiction that L/K is not essentially ramified.

Case 2: *L/K is not essentially ramified.* Then $L = K(\sqrt{\alpha}\,)$ for some $\alpha \in \mathfrak{O}_K$ such that $\alpha\mathfrak{O}_K = \mathfrak{a}^2$, where $\mathfrak{a}$ is an integral ideal in $\mathfrak{O}_K$.

(a) If $\mathfrak{a}$ is principal, say $\mathfrak{a} = \beta\mathfrak{O}_K$, then there is a unit $\varepsilon \in E_K$ such that $\alpha = \beta^2\varepsilon$, and we see that $L = K(\sqrt{\varepsilon}\,)$. Now $\sqrt{\varepsilon}^{\,\sigma-1} = -1$ is no square since $w_L \equiv 2 \bmod 4$, and Proposition 1(a) gives $Q(L) = 2$.

(b) If $\mathfrak{a}$ is not principal, then the ideal class $[\mathfrak{a}]$ capitulates in L/K because $\mathfrak{a}\mathfrak{O}_L = \sqrt{\alpha}\mathfrak{O}_L$. Proposition 1(b) shows that $Q(L) = 1$.

(ii) Assume that $w_L \equiv 2^m \bmod 2^{m+1}$ for some $m \geq 2$.

Case 1: *Assume that $Q(L) = 2$ or $\kappa_{L/K} \neq 1$.* Then Lemma 1 says that $\pi_m\mathfrak{O}_K = \mathfrak{b}^2$ is an ideal square in $\mathfrak{O}_K$ contrary to our assumption.

Case 2: *$\pi_m = \mathfrak{b}^2$ is an ideal square in $\mathfrak{O}_K$.* If $\mathfrak{b}$ is not principal, then $\mathfrak{b}\mathfrak{O}_L = (1-\zeta)$ shows that $\kappa_{L/K} = \langle[\mathfrak{b}]\rangle$, and Proposition 1(b) gives $Q(L) = 1$. If, on the other hand, $\mathfrak{b} = \beta\mathfrak{O}_K$, then $\eta\beta^2 = \pi_m$ for some unit $\eta \in E_K$. If η were a square in $\mathfrak{O}_K$, then π_m would also be a square, and $L = K(\sqrt{-1}\,)$ would contain the 2^{m+1}th roots of unity. Now $\eta\beta^2 = \pi_m = \zeta^{-1}(1+\zeta)^2$, hence $\eta\zeta$ is a square in L, and we have $Q(L) = 2$ as claimed. ∎

Remark. For $L/\mathbb{Q}$ abelian, Theorem 1 is equivalent to Hasse's Satz 22; we will again only sketch the proof: suppose that $w_L \equiv 2^m \bmod 2^{m+1}$ for some $m \geq 2$, and define $L' = L(\zeta_{2^{m+1}})$, $K' = L' \cap \mathbb{R}$. Then K'/K is essentially ramified if and only if π_m is not an ideal square in $\mathfrak{O}_K$ (because $K' = K(\pi_{m+1}) = K(\sqrt{\pi_m})$). The asserted equivalence should now be clear. Except for the results on capitulation, Theorem 1 is also contained in [O] (for general CM-fields).

Examples. 1. Complex subfields L of $\mathbb{Q}(\zeta_{p^m})$, where p is prime, have unit index $Q(L) = 1$ (Hasse's Satz 23) and $\kappa_{L/L^+} = 1$: since p ramifies completely in $\mathbb{Q}(\zeta_{p^m})/\mathbb{Q}$, L/L^+ is essentially ramified if $p \neq 2$, and the claim follows from Theorem 1. If $p = 2$ and L/L^+ is not essentially ramified, then we must have $L = \mathbb{Q}(\zeta_{2^\mu})$ for some $\mu \in \mathbb{N}$, and we find $Q(L) = 1$ by Theorem 1(ii.1).

2. $L = \mathbb{Q}(\zeta_m)$ has unit index $Q(L) = 1$ if and only if $m \not\equiv 2 \bmod 4$ is a prime power (Satz 27). This follows from Example 1 and Proposition 1(e).

3. If K is a CM-field, which is essentially ramified at a prime ideal $\mathfrak{p}$ above $p \in \mathbb{N}$, and if F is a totally real field such that $p \nmid \operatorname{disc} F$, then $Q(L) = 1$ and $\kappa_{L/L^+} = 1$ for $L = KF$: this is again due to the fact that either L/L^+ is essentially ramified at the prime ideals above $\mathfrak{p}$, or $p = 2$ and

$K = K^+(\sqrt{-1})$. In the first case, we have $Q(L) = 1$ by Theorem 1(i.1), and in the second case by Theorem 1(ii.1).

4. Suppose that the abelian CM-field K is the compositum $K = K_1 \dots K_t$ of fields with pairwise different prime power conductors; then $Q(K) = 1$ if and only if exactly one of the K_i is imaginary (Uchida [U, Prop. 3]). The proof is easy: if there is exactly one complex field among the K_j, then $Q(K) = 1$ by Example 3. Now suppose that K_1 and K_2 are imaginary; we know $Q(K_1K_2) = 2$ (Proposition 1(h)), and from the fact that the K_j have pairwise different conductors we deduce that $(W_K : W_{K_1K_2}) \equiv 1 \bmod 2$. Now the claim follows from Hasse's Satz 29 (Proposition 1(c)). Observe that $\kappa_{K/K^+} = 1$ in all cases.

5. Cyclic extensions $L/\mathbb{Q}$ have unit index $Q(L) = 1$ (Hasse's Satz 24): Let F be the maximal subfield of L such that $(F : \mathbb{Q})$ is odd. Then F is totally real, and $2 \nmid \operatorname{disc} F$ (this follows from the theorem of Kronecker and Weber). Similarly, let K be the maximal subfield of L such that $(K : \mathbb{Q})$ is a 2-power: then K is a CM-field, and $L = FK$. If K/K^+ is essentially ramified at a prime ideal $\mathfrak{p}$ above an odd prime p, then so is L/L^+, because $L/\mathbb{Q}$ is abelian, and all prime ideals in F have odd ramification index. Hence the claim in this case follows by Example 3 above.

If, however, K/K^+ is not essentially ramified at a prime ideal $\mathfrak{p}$ above an odd prime p, then $\operatorname{disc} K$ is a 2-power (recall that $K/\mathbb{Q}$ is cyclic of 2-power degree). Applying the theorem of Kronecker and Weber, we find that $K \subseteq \mathbb{Q}(\zeta)$, where ζ is some primitive 2^mth root of unity. If K/K^+ is essentially ramified at a prime ideal above 2, then so is L/L^+, and Theorem 1 gives us $Q(L) = 1$. If K/K^+ is not essentially ramified at a prime ideal above 2, then we must have $K = \mathbb{Q}(\zeta)$, where ζ is a primitive 2^mth root of unity; but now $\pi_m \mathfrak{O}_{L^+}$ is not the square of an integral ideal, and we have $Q(L) = 1$ by Theorem 1. Alternatively, we may apply Proposition 1(e) and observe that $Q(K) = 1$ by Example 1.

6. Let $p \equiv 1 \bmod 8$ be a prime such that the fundamental unit ε_{2p} of $\mathbb{Q}(\sqrt{2p})$ has norm $+1$ (by [S], there are infinitely many such primes; note also that $N\varepsilon_{2p} = +1 \Leftrightarrow (2, \sqrt{2p})$ is principal). Put $K = \mathbb{Q}(i, \sqrt{2p})$ and $L = \mathbb{Q}(i, \sqrt{2}, \sqrt{p})$. Then $Q(K) = 2$ by Theorem 1(ii.2)(a), whereas the fact that L is the compositum of $\mathbb{Q}(\zeta_8)$ and $\mathbb{Q}(\sqrt{p})$ shows that $Q(L) = 1$ (Example 4). This generalization of Lenstra's example given by Martinet in [H] is contained in Theorem 4 of [HY], where several other results of this kind can be found.

3. Masley's theorem $h_m^- \mid h_{mn}^-$. Now we can prove a theorem which will contain Masley's result $h^-(K) \mid h^-(L)$ for cyclotomic fields $K = \mathbb{Q}(\zeta_m)$ and $L = \mathbb{Q}(\zeta_{mn})$ as a special case:

THEOREM 2. *Let $K \subset L$ be CM-fields; then*

$$h^-(K) \,|\, h^-(L) \cdot |\kappa_{L/L^+}| \cdot \frac{(L \cap K^1 : K)}{(L^+ \cap (K^+)^1 : K^+)},$$

and the last quotient is a power of 2.

Proof. Let ν_K and ν_L denote the norms N_{K/K^+} and N_{L/L^+}, respectively; then the following diagram is exact and commutative:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{Cl}^-(L) & \longrightarrow & \mathrm{Cl}(L) & \xrightarrow{\nu_L} & \mathrm{Cl}(L^+) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle N^-} & & \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle N^+} & & \\ 1 & \longrightarrow & \mathrm{Cl}^-(K) & \longrightarrow & \mathrm{Cl}(K) & \xrightarrow{\nu_K} & \mathrm{Cl}(K^+) & \longrightarrow & 1 \end{array}$$

The snake lemma gives us an exact sequence

$$1 \to \ker N^- \to \ker N \to \ker N^+ \to \operatorname{cok} N^- \to \operatorname{cok} N \to \operatorname{cok} N^+ \to 1.$$

Let $h(L/K)$ denote the order of $\ker N$, and let $h^-(L/K)$ and $h(L^+/K^+)$ be defined accordingly. The remark at the end of Section 1 shows

$$|\operatorname{cok} N| = (L \cap K^1 : K), \qquad |\operatorname{cok} N^+| = (L^+ \cap (K^+)^1 : K^+).$$

The alternating product of the orders of the groups in exact sequences equals 1, so the above sequence implies

$$h^-(L/K) \cdot h(L^+/K^+) \cdot |\operatorname{cok} N| = h(L/K) \cdot |\operatorname{cok} N^-| \cdot |\operatorname{cok} N^+|.$$

The exact sequence

$$1 \to \ker N^- \to \mathrm{Cl}^-(L) \to \mathrm{Cl}^-(K) \to \operatorname{cok} N^- \to 1$$

gives us

$$h^-(L/K) \cdot h^-(K) = h^-(L) \cdot |\operatorname{cok} N^-|.$$

Collecting everything we find that

$$(*) \qquad h^-(K) \cdot \frac{h(L/K)}{h(L^+/K^+)} \cdot \frac{(L^+ \cap (K^+)^1 : K^+)}{(L \cap K^1 : K)} = h^-(L).$$

Now the claimed divisibility property follows if we can prove that $h(L^+/K^+)$ divides $h(L/K) \cdot |\kappa_{L/L^+}|$. But this is easy: exactly $h(L^+/K^+)/|\kappa_{L/L^+}|$ ideal classes of $\ker N^+ \subset \mathrm{Cl}(L^+)$ survive the transfer to $\mathrm{Cl}(L)$, and if the norm of L^+/K^+ kills an ideal class $c \in \mathrm{Cl}(L^+)$, the same thing happens to the transferred class c^j when the norm of L/K is applied. We remark in passing that $|\kappa_{L/L^+}| \le 2$ (see Hasse [H, Satz 18]).

It remains to show that $(L \cap K^1 : K)/(L^+ \cap (K^+)^1 : K^+)$ is a power of 2. Using induction on $(L : K)$, we see that it suffices to prove that if L/K is an unramified abelian extension of CM-fields of odd prime degree $(L : K) = q$, then so is L^+/K^+. Suppose otherwise; then there exists a

finite prime $\mathfrak{p}$ which ramifies, and since L^+/K^+ is cyclic, $\mathfrak{p}$ has ramification index q. Now L/K^+ is cyclic of order $2q$, hence K must be the inertia field of $\mathfrak{p}$, contradicting the assumption that L/K is unramified. We conclude that L^+/K^+ is also unramified, and so odd factors of $(L \cap K^1 : K)$ cancel against the corresponding factors of $(L^+ \cap (K^+)^1 : K^+)$. ∎

Corollary 1 ([LOO]). *Let $K \subset L$ be CM-fields such that $(L : K)$ is odd; then $h^-K \mid h^-(L)$.*

Proof. From $(*)$ and the fact that $(L^+ \cap (K^+)^1 : K^+) = 1$ (this index is a power of 2 and divides $(L : K)$, which is odd), we see that it is sufficient to show that $h(L^+/K^+) \mid h(L/K)$. This in turn follows if we can prove that no ideal class from $\ker N^+ \subseteq \mathrm{Cl}(L^+)$ capitulates when transferred to $\mathrm{Cl}(L)$. Assume therefore that $\kappa_{L/L^+} = \langle [\mathfrak{a}] \rangle$. If $w_L \equiv 2 \bmod 4$, then by Theorem 1(i.2) we may assume that $L = L^+(\sqrt{\alpha})$, where $\alpha\mathfrak{O}_{L^+} = \mathfrak{a}^2$. Since $(L : K)$ is odd, we can choose $\alpha \in O_{K^+}$, hence $N^+(\mathfrak{a}) = \mathfrak{a}^{(L:K)}$ shows that the ideal class $[\mathfrak{a}]$ is not contained in $\ker N^+$. The proof in the case $w_L \equiv 0 \bmod 4$ is completely analogous. ∎

Remark. For any prime p, let $\mathrm{Cl}_p^-(K)$ denote the p-Sylow subgroup of $\mathrm{Cl}^-(K)$; then $\mathrm{Cl}_p^-(K) \subseteq \mathrm{Cl}_p^-(L)$ for every $p \nmid (L : K)$. This is trivial, because ideal classes with order prime to $(L : K)$ cannot capitulate in L/K.

Corollary 2 ([MM]). *If $K = \mathbb{Q}(\zeta_m)$ and $L = \mathbb{Q}(\zeta_{mn})$ for some $m, n \in \mathbb{N}$, then $h^-(K) \mid h^-(L)$.* ∎

Proof. We have shown in Section 2 that $j_{K^+ \to K}$ and $j_{L^+ \to L}$ are injective in this case. Moreover, L/K does not contain a non-trivial subfield of K^1 (note that p is completely ramified in L/K if $n = p$, and use induction). ∎

The special case $m = p^a$, $n = p$ of Corollary 2 can already be found in [We]. Examples of CM-fields L/K such that $h^-(K) \nmid h^-(L)$ have been given by Hasse [H]; here are some more:

1. Let $d_1 \in \{-4, -8, -q \ (q \equiv 3 \bmod 4)\}$ be a prime discriminant, and suppose that $d_2 > 0$ is the discriminant of a real quadratic number field such that $(d_1, d_2) = 1$. Put $K = \mathbb{Q}(\sqrt{d_1 d_2})$ and $L = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$; then $Q(L) = 1$ and $\kappa_{L/L^+} = 1$ by Example 4, and $(L \cap K^1 : K) = 2 \cdot (L^+ \cap (K^+)^1 : K^+)$ since L/K is unramified but L^+/K^+ is not. The class number formula (1) below shows that in fact $h^-(K) \nmid h^-(L)$.

2. Let $d_1 = -4, d_2 = 8m$ for some odd $m \in \mathbb{N}$, and suppose that $\mathfrak{2} = (2, \sqrt{2m})$ is not principal in $\mathfrak{O}_k$, where $k = \mathbb{Q}(\sqrt{2m})$. Then $h^-(K) \nmid h^-(L)$ for $K = \mathbb{Q}(\sqrt{-2m})$, $L = \mathbb{Q}(\sqrt{-1}, \sqrt{2m})$. Here $(L \cap K^1 : K) = (L^+ \cap (K^+)^1 : K^+)$, but $\kappa_{L/L^+} = \langle [\mathfrak{2}] \rangle$, since $\mathfrak{2}\mathfrak{O}_L = (1+i)$. This example shows that we cannot drop the factor κ_{L/L^+} in Theorem 2.

Other examples can be found by replacing d_1 in Example 2 by $d_1 = -8$ or d_2 by $d_2 = 4m$, $m \in \mathbb{N}$ odd. The proof that in fact $h^-(K) \nmid h^-(L)$ for these fields uses Theorem 1, as well as Propositions 2 and 3 below.

4. Metsänkylä's factorization. An extension L/K is called a V_4-*extension of CM-fields* if

1. L/K is normal and $\mathrm{Gal}(L/K) \simeq V_4 = (2,2)$;
2. Exactly two of the three quadratic subfields are CM-fields; call them K_1 and K_2, respectively.

This implies, in particular, that K is totally real, and that L is a CM-field with maximal real subfield $L^+ = K_3$. We will write $Q_1 = Q(K_1), W_1 = W_{K_1}$, etc.

Louboutin [Lou, Prop. 13] has given an analytic proof of the following class number formula for V_4-extension of CM-fields, which contains Lemma 8 of Ferrero [F] as a special case:

Proposition 2. *Let L/K be a V_4-extension of CM-fields*; *then*

$$h^-(L) = \frac{Q(L)}{Q_1 Q_2} \cdot \frac{w_L}{w_1 w_2} h^-(K_1) h^-(K_2).$$

Proof. Kuroda's class number formula (for an algebraic proof see [L]) yields

$$h(L) = 2^{d-\kappa-2-\upsilon} q(L) h(K_1) h(K_2) h(L^+)/h(K)^2, \tag{1}$$

where

- $d = (K : \mathbb{Q})$ is the number of infinite primes of K ramified in L/K;
- $\kappa = d - 1$ is the $\mathbb{Z}$-rank of the unit group of K;
- $\upsilon = 1$ if and only if all three quadratic subfields of L/K can be written as $K(\sqrt{\varepsilon}\,)$ for units $\varepsilon \in E_K$, and $\upsilon = 0$ otherwise;
- $q(L) = (E_L : E_1 E_2 E_3)$ is the unit index for extensions of type $(2,2)$; here E_j is the unit group of K_j (similarly, let W_j denote the group of roots unity in L_j).

Now we need to find a relation between the unit indices involved; we assert

Proposition 3. *If L/K is a V_4-extension of CM-fields*, *then*

$$\frac{Q(L)}{Q_1 Q_2} \cdot \frac{w_L}{w_1 w_2} = 2^{-1-\upsilon} q(L).$$

Proof of Proposition 3. We start with the observation

$$Q(L) = (E_L : W_L E_3) = (E_L : E_1 E_2 E_3) \frac{(E_1 E_2 E_3 : W_1 W_2 E_3)}{(W_L E_3 : W_1 W_2 E_3)}.$$

In [L] we have defined groups $E_j^* = \{\varepsilon \in E_j : N_j\varepsilon \text{ is a square in } E_K\}$, where N_j denotes the norm of K_j/K; we have also shown that

$$(E_1E_2E_3 : E_1^*E_2^*E_3^*) = 2^{-v}\prod(E_j : E_j^*)$$

and $E_j/E_j^* \simeq E_K/N_jE_j$. Now Proposition 1(a) gives $(E_K : N_jE_j) = Q_j$ for $j = 1, 2$, and we claim

1. $(W_LE_3 : W_1W_2E_3) = (W_L : W_1W_2) = 2 \cdot \dfrac{w_L}{w_1w_2}$;
2. $E_1^*E_2^*E_3^* = W_1W_2E_3^*$;
3. $(W_1W_2E_3 : W_1W_2E_3^*) = (E_3 : E_3^*)$.

This will give us

$$Q(L) = 2^{-1-v}q(L)Q_1Q_2\frac{w_1w_2}{w_L}, \tag{2}$$

completing the proof of Proposition 3; inserting (2) into equation (1) and recalling the definition of the minus class number yields Louboutin's formula.

We still have to prove the three claims above:

1. $W_LE_3/W_1W_2E_3 \simeq W_L/(W_L \cap W_1W_2E_3) \simeq W_L/W_1W_2$, and the claim follows from $W_1 \cap W_2 = \{-1, +1\}$;

2. We only need to show that $E_1^*E_2^*E_3^* \subset W_1W_2E_3^*$; but Proposition 1(a) shows that $\varepsilon \in E_1^* \Leftrightarrow \varepsilon^{\sigma+1} \in E_K^2 \Leftrightarrow \varepsilon \in W_1E_K$, and this implies the claim;

3. $W_1W_2E_3/W_1W_2E_3^* \simeq E_3/E_3 \cap W_1W_2E_3^* \simeq E_3/E_3^*$. ∎

Combining the result of Section 3 with Proposition 2, we get the following

THEOREM 3. *Let L_1 and L_2 be CM-fields, and let $L = L_1L_2$ and $K = L_1^+L_2^+$; then L/K is a V_4-extension of CM-fields with subfields $K_1 = L_1L_2^+$, $K_2 = L_1^+L_2$, $K_3 = L^+$, and*

$$h^-(L) = \frac{Q(L)}{Q_1Q_2} \cdot \frac{w_L}{w_1w_2}h^-(L_1)h^-(L_2)T_1T_2,$$

where $T_1 = h^-(L_1L_2^+)/h^-(L_1)$ and $T_2 = h^-(L_2L_1^+)/h^-(L_2)$.

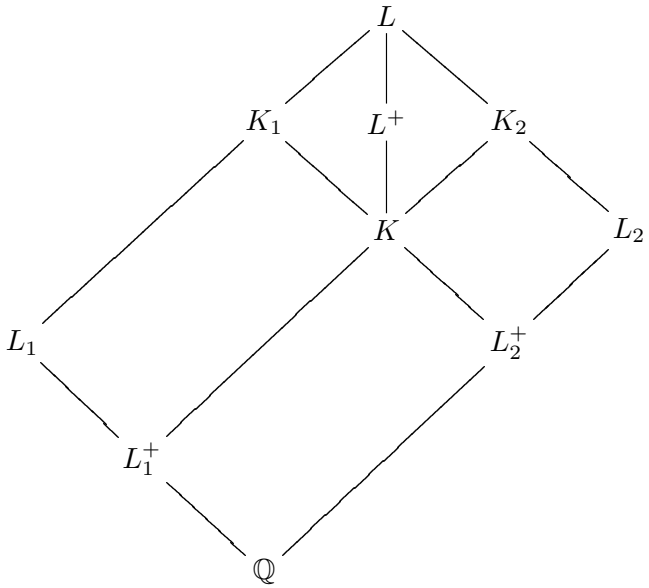
If we assume that $\kappa_1 = \kappa_2 = 1$ (κ_1 is the group of ideal classes capitulating in $L_1L_2^+/K$ and κ_2 is defined similarly) and that

$$(L_1L_2^+ \cap L_1^1 : L_1) = (L_1^+L_2^+ \cap (L_1^+)^1 : L_1^+),$$
$$(L_2L_1^+ \cap L_2^1 : L_2) = (L_2^+L_1^+ \cap (L_2^+)^1 : L_2^+),$$

then T_1 and T_2 are integers.

Proof. Theorem 3 follows directly from Theorem 2 and Proposition 2. ∎

The following Hasse diagram explains the situation:



Now let $m = p^\mu$ and $n = q^\nu$ be prime powers, and suppose that $p \neq q$. Moreover, let $L_1 \subseteq \mathbb{Q}(\zeta_m)$ and $L_2 \subseteq \mathbb{Q}(\zeta_n)$ be CM-fields. Then

(1) $Q(L) = 2$, $Q_1 = Q(L_1L_2^+) = Q_2 = Q(L_2L_1^+) = 1$: this has been proved in Proposition 1(h) and Example 4 in Section 2;

(2) $w_1w_2 = 2w_L$ (obviously);

(3) $\kappa_1 = \kappa_2 = 1$: see Example 4 in Section 2;

(4) $(L_1L_2^+ \cap L_1^1 : L_1) = (L_1^+L_2^+ \cap (L_1^+)^1 : L_1^+)$: this, as well as the corresponding property for K_2, is obvious, because the prime ideals above p and q ramify completely in L/L_2 and L/L_1, respectively.

In particular, we have the following

COROLLARY ([M]). *Let $L_1 \subseteq \mathbb{Q}(\zeta_m)$ and $L_2 \subseteq \mathbb{Q}(\zeta_n)$ be CM-fields, where $m = p^\mu$ and $n = q^\nu$ are prime powers, and let $L = L_1L_2$; then*

$$h^-(L) = h^-(L_1)h^-(L_2)T_1T_2,$$

where $T_1 = h^-(L_1L_2^+)/h^-(L_1)$ and $T_2 = h^-(L_2L_1^+)/h^-(L_2)$ are integers.

It still remains to identify the character sums T_{01} and T_{10} in [M] with the class number factors T_1 and T_2 given above. But this is easy: the character group $X(L_1)$ corresponding to the field L_1 is generated by a character χ_1, and it is easily seen that

$$X(L_1) = \langle\chi_1\rangle, \qquad X(L_1L_2^+) = \langle\chi_1, \chi_2^2\rangle,$$
$$X(L_2) = \langle\chi_2\rangle, \qquad X(L_2L_1^+) = \langle\chi_2, \chi_1^2\rangle,$$
$$X(L) = \langle\chi_1, \chi_2\rangle, \qquad X(L^+) = \langle\chi_1\chi_2, \chi_1^2\rangle.$$

The analytical class number formula for an abelian CM-field K reads

$$h^-(K) = Q(K)w_K \prod_{\chi\in X^-(K)} \frac{1}{2\mathfrak{f}(\chi)} \sum_{a \bmod^+ \mathfrak{f}(\chi)} (-\chi(a)a), \tag{3}$$

where $a \bmod^+ \mathfrak{f}(\chi)$ indicates that the sum is extended over all $1 \le a \le \mathfrak{f}(\chi)$ such that $(a, \mathfrak{f}(\chi)) = 1$, and $X^-(L) = X(L) \setminus X(L^+)$ is the set of $\chi \in X(L)$ such that $\chi(-1) = -1$. Applying formula (3) to the CM-fields listed above and noting that $Q(L) = 2, Q(L_1) = Q(L_2) = Q(L_1L_2^+) = Q(L_2L_1^+) = 1$ and $2w_L = w_1w_2$, we find

$$h^-(L) = h^-(L_1) \cdot h^-(L_2) \prod_{\chi\in X^*(L)} \frac{1}{2\mathfrak{f}(\chi)} \sum_{a \bmod^+ \mathfrak{f}(\chi)} (-\chi(a)a),$$

where $X^*(L)$ is the subset of all $\chi \in X^-(L)$ not lying in $X^-(L_1)$ or $X^-(L_2)$. Now define $X_1(L) = \{\chi = \chi_1^x\chi_2^y \in X^*(L) : x \equiv 1 \bmod 2, y \equiv 0 \bmod 2\}$, and let $X_2(L)$ be defined accordingly. Then $X^*(L) = X_1(L) \cup X_2(L)$, and

$$h^-(L_1) \cdot \prod_{\chi\in X_1(L)} \frac{1}{2\mathfrak{f}(\chi)} \sum_{a \bmod^+ \mathfrak{f}(\chi)} (-\chi(a)a) = h^-(L_1L_2^+),$$

and we have shown that

$$T_1 = \prod_{\chi\in X_1(L)} \frac{1}{2\mathfrak{f}(\chi)} \sum_{a \bmod^+ \mathfrak{f}(\chi)} (-\chi(a)a).$$

Comparing with the definition of Metsänkylä's factor T_{10}, this shows that indeed $T_1 = T_{10}$.

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