Covering the integers by arithmetic sequences

by

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1. Introduction. Let $\mathbb{R}$ be the field of real numbers and $\mathbb{R}^+$ the set of positive reals. For $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}^+$ we call

$$\alpha + \beta\mathbb{Z} = \{\ldots, \alpha - 2\beta, \alpha - \beta, \alpha, \alpha + \beta, \alpha + 2\beta, \ldots\}$$

an *arithmetic sequence with common difference* β. In the case $\alpha \in \mathbb{Z}$ and $\beta \in \mathbb{Z}^+$, $\alpha + \beta\mathbb{Z}$ is just the residue class $\alpha \bmod \beta$ with modulus β.

Let m be a positive integer. A finite system

$$(1) \qquad \mathcal{A} = \{\alpha_s + \beta_s\mathbb{Z}\}_{s=1}^k \qquad (\alpha_1, \ldots, \alpha_k \in \mathbb{R} \text{ and } \beta_1, \ldots, \beta_k \in \mathbb{R}^+)$$

of arithmetic sequences is said to be an (exact) m-*cover* of $\mathbb{Z}$ if it covers each integer at least (resp., exactly) m times. Instead of "1-cover" and "exact 1-cover" we use the terms "cover" and "exact cover" respectively.

Since they were introduced by P. Erdős ([5]) in the early 1930's, covers of $\mathbb{Z}$ by (finitely many) residue classes have been studied seriously and many nice applications have been found. (Cf. sections A19, B21, E23, F13 and F14 of R. K. Guy [9].) For problems and results in this area we refer the reader to surveys of Erdős [7, 8], Š. Porubský [13] and Š. Znám [21]. Recently further progress was made by various authors.

If a finite system

$$(2) \qquad A = \{a_s + n_s\mathbb{Z}\}_{s=1}^k \qquad (a_1, \ldots, a_k \in \mathbb{Z} \text{ and } n_1, \ldots, n_k \in \mathbb{Z}^+)$$

of residue classes forms an m-cover of $\mathbb{Z}$, then $\sum_{s=1}^k 1/n_s \geq m$, and the equality holds if and only if (2) is an exact m-cover of $\mathbb{Z}$. This becomes apparent if we calculate

$$\sum_{s=1}^k |\{0 \leq x < N : x \equiv a_s \pmod{n_s}\}| = \sum_{x=0}^{N-1} |\{1 \leq s \leq k : x \equiv a_s \pmod{n_s}\}|$$

where N is the least common multiple of $n_1, \ldots, n_k$.

In this paper we investigate properties of m-covers of $\mathbb{Z}$ in the form (1). In the next section we shall give three equivalent conditions for (1) to be an m-cover of $\mathbb{Z}$. One is that (1) covers W consecutive integers at least m times where

$$W = \left| \left\{ \left\{ \sum_{s \in I} \frac{1}{\beta_s} \right\} : I \subseteq \{1, \ldots, k\} \right\} \right|$$

($[x]$ and $\{x\}$ stand for the integral and fractional parts of a real x respectively throughout the paper), the other two are finite systems of equalities (not inequalities) involving roots of unity. Our tools used to deduce them include Vandermonde determinants, Stirling numbers, a little analysis and linear algebra.

In Sections 3 and 4 we will derive a number of results including the following ones:

(I) Let (1) be an m-cover of $\mathbb{Z}$ and $J \subseteq \{1, \ldots, k\}$. Then

$$\left\{ \sum_{s \in I} \frac{1}{\beta_s} \right\} = \left\{ \sum_{s \in J} \frac{1}{\beta_s} \right\} \quad \text{for some } I \subseteq \{1, \ldots, k\} \text{ with } I \neq J,$$

provided $\sum_{s=1}^{k} 1/\beta_s = m$ (e.g. (1) is an exact m-cover of $\mathbb{Z}$ with $\alpha_s \in \mathbb{Z}$ and $\beta_s \in \mathbb{Z}^+$ for $s = 1, \ldots, k$) we have $\sum_{s \in I} 1/\beta_s = \sum_{s \in J} 1/\beta_s$ for some $I \subseteq \{1, \ldots, k\}$ with $I \neq J$ if $\emptyset \neq J \subset \{1, \ldots, k\}$, when $\sum_{s \in I} 1/\beta_s = \sum_{s \in J} 1/\beta_s$ for no $I \subseteq \{1, \ldots, k\}$ with $I \neq J$ there are at least m nonzero integers of the form $\sum_{s \in I} 1/\beta_s - \sum_{s \in J} 1/\beta_s$ where $I \subseteq \{1, \ldots, k\}$.

(II) Let $k \geq l \geq 0$ be integers. Then $2^{k-l}(l+1)$ is the smallest $n \in \mathbb{Z}^+$ such that any system of k arithmetic sequences with at least l equal common differences covers an arithmetic sequence at least m times if it covers n consecutive terms in the sequence at least m times.

The last section contains some unsolved problems related to possible extensions.

2. Characterizations of m-covers. Let us provide several technical lemmas the first of which serves as the starting point of our new approach.

LEMMA 1. *Let $m \in \mathbb{Z}^+$ and $x \in \mathbb{R}$. Then (1) covers x at least m times if and only if*

$$(3) \qquad \prod_{s=1}^{k} (1 - r^{1/\beta_s} e^{2\pi i (\alpha_s - x)/\beta_s}) = o((1 - r)^{m-1}) \quad (r \to 1).$$

P r o o f. Set $I = \{1 \leq s \leq k : x \in \alpha_s + \beta_s \mathbb{Z}\}$ and $I' = \{1, \ldots, k\} \setminus I$.

Clearly,

$$\lim_{r \to 1} \frac{\prod_{s=1}^{k}(1 - r^{1/\beta_s} e^{2\pi i(\alpha_s - x)/\beta_s})}{(1 - r)^{|I|}}$$

$$= \lim_{r \to 1} \prod_{s \in I'} (1 - r^{1/\beta_s} e^{2\pi i(\alpha_s - x)/\beta_s}) \cdot \lim_{r \to 1} \prod_{s \in I} \frac{1 - r^{1/\beta_s}}{1 - r}$$

$$= \prod_{s \in I'} (1 - e^{2\pi i(\alpha_s - x)/\beta_s}) \cdot \prod_{s \in I} \frac{d}{dr}(r^{1/\beta_s})\Big|_{r=1}$$

$$= \prod_{s \in I'} (1 - e^{2\pi i(\alpha_s - x)/\beta_s}) \cdot \prod_{s \in I} \beta_s^{-1} \neq 0,$$

and hence

$$\lim_{r \to 1} \frac{\prod_{s=1}^{k}(1 - r^{1/\beta_s} e^{2\pi i(\alpha_s - x)/\beta_s})}{(1 - r)^{m-1}}$$

$$= \lim_{r \to 1} \frac{\prod_{s=1}^{k}(1 - r^{1/\beta_s} e^{2\pi i(\alpha_s - x)/\beta_s})}{(1 - r)^{|I|}} (1 - r)^{|I|-m+1}$$

$$= \begin{cases} 0 & \text{if } |I| > m - 1, \\ \infty & \text{if } |I| < m - 1. \end{cases}$$

Now it is apparent that $|I| \geq m$ if and only if (3) holds. We are done.

LEMMA 2. *Let $\theta_1, \ldots, \theta_n$ be real numbers with distinct fractional parts. For any $\varepsilon > 0$ there exists a $\delta > 0$ such that if*

$$\left| \sum_{t=1}^{n} e^{2\pi i s \theta_t} x_t \right| < \delta$$

for every $s = 1, \ldots, n$ then $|x_t| < \varepsilon$ for all $t = 1, \ldots, n$.

Proof. Let A be the matrix $(e^{2\pi i s \theta_t})_{\substack{1 \leq s \leq n \\ 1 \leq t \leq n}}$. Then

$$\frac{|A|}{e^{2\pi i \theta_1} e^{2\pi i \theta_2} \dots e^{2\pi i \theta_n}} = \begin{vmatrix} 1 & 1 & \dots & 1 \\ e^{2\pi i \theta_1} & e^{2\pi i \theta_2} & \dots & e^{2\pi i \theta_n} \\ (e^{2\pi i \theta_1})^2 & (e^{2\pi i \theta_2})^2 & \dots & (e^{2\pi i \theta_n})^2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ (e^{2\pi i \theta_1})^{n-1} & (e^{2\pi i \theta_2})^{n-1} & \dots & (e^{2\pi i \theta_n})^{n-1} \end{vmatrix}$$

is a determinant of Vandermonde's type. As $|A| \neq 0$ the inverse matrix of A exists; we denote it by $B = (b_{st})_{\substack{1 \leq s \leq n \\ 1 \leq t \leq n}}$.

Let $b = \max\{|b_{st}| : s, t = 1, \ldots, n\} > 0$ and $\delta = \varepsilon/(bn)$. Let $x_1, \ldots, x_n$

be any complex numbers, and set

$$y_s = \sum_{t=1}^{n} e^{2\pi i s \theta_t} x_t \quad \text{for } s = 1, \dots, n.$$

Let

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \text{and} \quad \vec{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}.$$

Then $\vec{x} = BA\vec{x} = B\vec{y}$. If $|y_s| < \delta$ for every $s = 1, \dots, n$, then

$$|x_s| = \left| \sum_{t=1}^{n} b_{st} y_t \right| \le \sum_{t=1}^{n} b|y_t| < bn\delta = \varepsilon \quad \text{for all } s = 1, \dots, n.$$

This concludes the proof.

LEMMA 3. *Let* $m \in \mathbb{Z}^+$. *Then*

$$(4) \qquad \sum_{n=0}^{m-1} a_n t^{n-m+1} = o(1) \quad (t \to 0)$$

if and only if $a_0 = \dots = a_{m-1} = 0$.

Proof. The "if" direction is trivial. When $a_0, \dots, a_{m-1}$ are not all zero, for the least k such that $a_k \ne 0$ we have

$$\sum_{n=0}^{m-1} a_n (x^{-1})^{n-m+1} = \sum_{n=k}^{m-1} a_n x^{m-1-n} \sim a_k x^{m-1-k} \quad (x \to \infty),$$

which contradicts (4). This ends the proof.

LEMMA 4. *Let* $n \ge m > 0$ *be integers and* $a_1, \dots, a_n$ *distinct numbers. Then the system*

$$(5) \qquad \begin{cases} x_1 + \dots + x_n = 0, \\ a_1 x_1 + \dots + a_n x_n = 0, \\ a_1^2 x_1 + \dots + a_n^2 x_n = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_1^{m-1} x_1 + \dots + a_n^{m-1} x_n = 0, \end{cases}$$

is equivalent to

$$(6) \qquad \begin{cases} a_{11} x_1 + \dots + a_{1n} x_n = 0, \\ a_{21} x_1 + \dots + a_{2n} x_n = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1} x_1 + \dots + a_{mn} x_n = 0, \end{cases}$$

where

$$a_{st} = \prod_{\substack{i=1 \\ i \neq s}}^{m} \frac{a_i - a_t}{a_i - a_s} \quad for \ s = 1, \ldots, m \ and \ t = 1, \ldots, n.$$

P r o o f. Rewrite (5) in the form

$$
\begin{cases}
x_1 + \ldots + x_m = -\sum_{m < t \leq n} x_t, \\
a_1 x_1 + \ldots + a_m x_m = -\sum_{m < t \leq n} a_t x_t, \\
a_1^2 x_1 + \ldots + a_m^2 x_m = -\sum_{m < t \leq n} a_t^2 x_t, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
a_1^{m-1} x_1 + \ldots + a_m^{m-1} x_m = -\sum_{m < t \leq n} a_t^{m-1} x_t.
\end{cases}
$$

By Cramer's rule, this says that

$$
x_s = \begin{vmatrix}
1 & \ldots & 1 & -\sum_{m<t\leq n} x_t & 1 & \ldots & 1 \\
a_1 & \ldots & a_{s-1} & -\sum_{m<t\leq n} a_t x_t & a_{s+1} & \ldots & a_m \\
a_1^2 & \ldots & a_{s-1}^2 & -\sum_{m<t\leq n} a_t^2 x_t & a_{s+1}^2 & \ldots & a_m^2 \\
\cdots & & & \cdots & & & \cdots \\
a_1^{m-1} & \ldots & a_{s-1}^{m-1} & -\sum_{m<t\leq n} a_t^{m-1} x_t & a_{s+1}^{m-1} & \ldots & a_m^{m-1}
\end{vmatrix}
$$

$$
\times \begin{vmatrix}
1 & \ldots & 1 \\
a_1 & \ldots & a_m \\
a_1^2 & \ldots & a_m^2 \\
\cdots & & \cdots \\
a_1^{m-1} & \ldots & a_m^{m-1}
\end{vmatrix}^{-1}
$$

$$
= -\sum_{m<t\leq n} x_t \begin{vmatrix}
1 & \ldots & 1 & 1 & 1 & \ldots & 1 \\
a_1 & \ldots & a_{s-1} & a_t & a_{s+1} & \ldots & a_m \\
a_1^2 & \ldots & a_{s-1}^2 & a_t^2 & a_{s+1}^2 & \ldots & a_m^2 \\
\cdots & & & & & & \cdots \\
a_1^{m-1} & \ldots & a_{s-1}^{m-1} & a_t^{m-1} & a_{s+1}^{m-1} & \ldots & a_m^{m-1}
\end{vmatrix}
$$

$$
\times \begin{vmatrix}
1 & \ldots & 1 \\
a_1 & \ldots & a_m \\
a_1^2 & \ldots & a_m^2 \\
\cdots & & \cdots \\
a_1^{m-1} & \ldots & a_m^{m-1}
\end{vmatrix}^{-1}
$$

$$\prod_{1 \le i < s} (a_t - a_i) \cdot \prod_{s < i \le m} (a_i - a_t) \cdot \prod_{\substack{1 \le i < j \le m \\ i,j \ne s}} (a_j - a_i)$$

$$= - \sum_{m < t \le n} x_t \frac{}{\prod_{1 \le i < s} (a_s - a_i) \cdot \prod_{s < i \le m} (a_i - a_s) \cdot \prod_{\substack{1 \le i < j \le m \\ i,j \ne s}} (a_j - a_i)}$$

$$= - \sum_{m < t \le n} a_{st} x_t \qquad\qquad \text{(Vandermonde)}$$

for every $s = 1, \ldots, m$, i.e.

$$\sum_{t=1}^{m} \delta_{st} x_t + \sum_{m < t \le n} a_{st} x_t = 0 \quad \text{ for } s = 1, \ldots, m$$

where δ_{st} is the Kronecker delta. Since $a_{st} = \delta_{st}$ for $s, t = 1, \ldots, m$, we have finished the proof.

Now we are ready to present

THEOREM 1. *Let* $\mathcal{A} = \{\alpha_s + \beta_s \mathbb{Z}\}_{s=1}^{k}$, *where* $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$ *and* $\beta_1, \ldots, \beta_k \in \mathbb{R}^+$. *Let* $m \in \mathbb{Z}^+$ *and*

$$S = \left\{ 0 \le \theta < 1 : \left\{ \sum_{s \in I} \frac{1}{\beta_s} \right\} = \theta \text{ for some } I \subseteq \{1, \ldots, k\} \right\}.$$

Let

$$V(\theta) = \left\{ \sum_{s \in I} \frac{1}{\beta_s} : I \subseteq \{1, \ldots, k\} \text{ and } \sum_{s \in I} \frac{1}{\beta_s} - \theta \in \mathbb{Z} \right\}$$

and $U(\theta)$ *be a set of* m *distinct numbers comparable with* $V(\theta)$ *(i.e.* $|U(\theta)| = m$, *and either* $U(\theta) \subseteq V(\theta)$ *or* $U(\theta) \supseteq V(\theta)$). *Then the following statements are equivalent:*

(a) $\mathcal{A}$ *is an* m-*cover of* $\mathbb{Z}$.
(b) $\mathcal{A}$ *covers* $|S|$ *consecutive integers at least* m *times.*
(c) *For each* $\theta \in S$,

$$(7) \qquad \sum_{\substack{I \subseteq \{1,\ldots,k\} \\ \{\Sigma_{s \in I} 1/\beta_s\} = \theta}} (-1)^{|I|} \binom{[\sum_{s \in I} 1/\beta_s]}{n} e^{2\pi i \Sigma_{s \in I} \alpha_s / \beta_s} = 0$$

holds for every $n = 0, 1, \ldots, m - 1$. $\left(\text{As usual } \binom{x}{n} \text{ denotes } \frac{x(x-1)\ldots(x-n+1)}{1 \cdot 2 \cdot \ldots \cdot (n-1)n}.\right)$
(d) *For any* $\theta \in S$,

$$(8) \qquad \sum_{v \in V(\theta)} a_{uv} f(v) = 0 \quad \text{ for all } u \in U(\theta),$$

where

$$a_{uv} = \prod_{\substack{x \in U(\theta) \\ x \neq u}} \frac{x-v}{x-u} \quad and \quad f(v) = \sum_{\substack{I \subseteq \{1,\dots,k\} \\ \Sigma_{s \in I} 1/\beta_s = v}} (-1)^{|I|} e^{2\pi i \Sigma_{s \in I} \alpha_s / \beta_s}.$$

P r o o f. (a)$\Rightarrow$(b). This is obvious.

(b)$\Rightarrow$(c). Suppose that each of $x+1, \dots, x+|S|$ is covered by $\mathcal{A}$ at least m times, where x is an integer. By Lemma 1 for every $n = 1, \dots, |S|$ we have

$$
\begin{aligned}
0 &= \lim_{r \to 1} \frac{\prod_{s=1}^{k}(1 - r^{1/\beta_s} e^{2\pi i(\alpha_s - x - n)/\beta_s})}{(1-r)^{m-1}} \\
&= \lim_{r \to 1} \Big((1-r)^{1-m} \\
&\quad \times \sum_{I \subseteq \{1,\dots,k\}} (-1)^{|I|} r^{\Sigma_{s \in I} 1/\beta_s} e^{2\pi i \Sigma_{s \in I}(\alpha_s - x)/\beta_s} e^{-2\pi i n \Sigma_{s \in I} 1/\beta_s} \Big) \\
&= \lim_{r \to 1} \sum_{\theta \in S} F(r,\theta) e^{-2\pi i n \theta},
\end{aligned}
$$

where

$$F(r,\theta) = \sum_{\substack{I \subseteq \{1,\dots,k\} \\ \{\Sigma_{s \in I} 1/\beta_s\} = \theta}} (-1)^{|I|} r^{\Sigma_{s \in I} 1/\beta_s} e^{2\pi i \Sigma_{s \in I} \alpha_s / \beta_s} e^{-2\pi i x \theta} / (1-r)^{m-1}.$$

Let ε be an arbitrary positive number. By Lemma 2 there is an $\eta > 0$ such that if

$$\Big| \sum_{\theta \in S} e^{-2\pi i n \theta} x_\theta \Big| < \eta$$

for every $n = 1, \dots, |S|$ then $|x_\theta| < \varepsilon$ for all $\theta \in S$. Since

$$\sum_{\theta \in S} F(r,\theta) e^{-2\pi i n \theta} = o(1) \quad (r \to 1) \quad \text{for } n = 1, \dots, |S|,$$

there exists a $\delta > 0$ such that whenever $|r - 1| < \delta$,

$$\Big| \sum_{\theta \in S} F(r,\theta) e^{-2\pi i n \theta} \Big| < \eta \quad \text{for all } n = 1, \dots, |S|$$

and hence by the above $|F(r,\theta)| < \varepsilon$ for each $\theta \in S$. This shows that $\lim_{r \to 1} F(r,\theta) = 0$ for every $\theta \in S$.

For any $\theta \in S$ we have

$$
\begin{aligned}
0 &= \lim_{r \to 1} \sum_{\substack{I \subseteq \{1,\ldots,k\} \\ \{\Sigma_{s \in I} 1/\beta_s\} = \theta}} (-1)^{|I|} r^{\Sigma_{s \in I} 1/\beta_s} e^{2\pi i \Sigma_{s \in I} \alpha_s/\beta_s} / (1-r)^{m-1} \\
&= \lim_{t \to 0} \sum_{\substack{I \subseteq \{1,\ldots,k\} \\ \{\Sigma_{s \in I} 1/\beta_s\} = \theta}} (-1)^{|I|} (1-t)^{[\Sigma_{s \in I} 1/\beta_s] + \theta} t^{1-m} e^{2\pi i \Sigma_{s \in I} \alpha_s/\beta_s} \\
&= \lim_{t \to 0} \sum_{\substack{I \subseteq \{1,\ldots,k\} \\ \{\Sigma_{s \in I} 1/\beta_s\} = \theta}} (-1)^{|I|} \sum_{n=0}^{[\Sigma_{s \in I} 1/\beta_s]} \binom{[\Sigma_{s \in I} 1/\beta_s]}{n} (-t)^n t^{1-m} e^{2\pi i \Sigma_{s \in I} \alpha_s/\beta_s} \\
&= \lim_{t \to 0} \sum_{\substack{I \subseteq \{1,\ldots,k\} \\ \{\Sigma_{s \in I} 1/\beta_s\} = \theta}} \Bigg((-1)^{|I|} e^{2\pi i \Sigma_{s \in I} \alpha_s/\beta_s} \\
&\qquad \times \sum_{\substack{n=0 \\ n \le [\Sigma_{s \in I} 1/\beta_s]}}^{m-1} \binom{[\Sigma_{s \in I} 1/\beta_s]}{n} (-1)^n t^{n-m+1} \Bigg) \\
&= \lim_{t \to 0} \sum_{n=0}^{m-1} (-1)^n \Bigg(\sum_{\substack{I \subseteq \{1,\ldots,k\} \\ \{\Sigma_{s \in I} 1/\beta_s\} = \theta}} (-1)^{|I|} \binom{[\Sigma_{s \in I} 1/\beta_s]}{n} e^{2\pi i \Sigma_{s \in I} \alpha_s/\beta_s} \Bigg) t^{n-m+1}.
\end{aligned}
$$

In view of Lemma 3, (7) holds for every $n = 0, 1, \ldots, m-1$. Therefore part (c) follows.

(c)$\Rightarrow$(d). Fix $\theta \in S$. For each $n = 0, 1, \ldots, m-1$,

$$
x^n = \sum_{j=0}^{n} S(n,j) x(x-1) \ldots (x-j+1)
$$

where $S(n,j)$ $(0 \le j \le n)$ are Stirling numbers of the second kind, so by (c) we have

$$
\sum_{\substack{I \subseteq \{1,\ldots,k\} \\ \{\Sigma_{s \in I} 1/\beta_s\} = \theta}} (-1)^{|I|} \Big[\sum_{s \in I} 1/\beta_s \Big]^n e^{2\pi i \Sigma_{s \in I} \alpha_s/\beta_s}
$$

$$
= \sum_{j=0}^{n} j! S(n,j) \sum_{\substack{I \subseteq \{1,\ldots,k\} \\ \{\Sigma_{s \in I} 1/\beta_s\} = \theta}} (-1)^{|I|} \binom{[\Sigma_{s \in I} 1/\beta_s]}{j} e^{2\pi i \Sigma_{s \in I} \alpha_s/\beta_s} = 0,
$$

i.e.

$$
\sum_{v \in V(\theta)} \sum_{\substack{I \subseteq \{1,\ldots,k\} \\ \Sigma_{s \in I} 1/\beta_s = v}} (-1)^{|I|} e^{2\pi i \Sigma_{s \in I} \alpha_s/\beta_s} [v]^n = 0.
$$

C a s e 1: $|V(\theta)| \leq m$. In this case

$$\sum_{v \in V(\theta)} [v]^n f(v) = 0 \quad \text{for every } n = 0, 1, \ldots, |V(\theta)| - 1.$$

Hence (8) holds since $f(v) = 0$ for all $v \in V(\theta)$ (Vandermonde).

C a s e 2: $|V(\theta)| > m$. In this case, $U(\theta) \subset V(\theta)$ and

$$\sum_{v \in U(\theta)} [v]^n f(v) + \sum_{v \in V(\theta) \setminus U(\theta)} [v]^n f(v) = 0$$

for each $n = 0, 1, \ldots, m - 1$. According to Lemma 4,

$$\sum_{v \in V(\theta)} a_{uv} f(v) = \sum_{v \in V(\theta)} \left(\prod_{\substack{x \in U(\theta) \\ x \neq u}} \frac{[x] - [v]}{[x] - [u]} \right) f(v) = 0 \quad \text{for all } u \in U(\theta).$$

So in either case we have (8).

(d)$\Rightarrow$(a). Assume that (d) holds. Let $\theta \in S$. For $u, v \in U(\theta)$,

$$a_{uv} = \prod_{\substack{x \in U(\theta) \\ x \neq u}} \frac{x - v}{x - u} = \begin{cases} 1 & \text{if } u = v, \\ 0 & \text{if } u \neq v. \end{cases}$$

C a s e 1: $|V(\theta)| \leq m$. In this case $V(\theta) \subseteq U(\theta)$. As

$$f(u) = \sum_{v \in V(\theta)} a_{uv} f(v) = 0 \quad \text{for each } u \in V(\theta),$$

we get

$$\sum_{v \in V(\theta)} f(v) [v]^n = 0 \quad \text{for all } n = 0, 1, 2, \ldots$$

C a s e 2: $|V(\theta)| > m$. In this case $U(\theta) \subset V(\theta)$, so for any $u \in U(\theta)$ and $v \in V(\theta)$ we have $\{u\} = \{v\} = \theta$ and hence $[u] - [v] = u - v$. Since

$$\sum_{v \in V(\theta)} \left(\prod_{\substack{x \in U(\theta) \\ x \neq u}} \frac{[x] - [v]}{[x] - [u]} \right) f(v) = \sum_{v \in V(\theta)} a_{uv} f(v) = 0$$

for every $u \in U(\theta)$, it follows from Lemma 4 that

$$\sum_{v \in V(\theta)} f(v) [v]^n = \sum_{v \in U(\theta)} [v]^n f(v) + \sum_{v \in V(\theta) \setminus U(\theta)} [v]^n f(v) = 0$$

for all $n = 0, 1, \ldots, m - 1$.

In both cases,

$$\sum_{v \in V(\theta)} f(v) [v]^n = 0 \quad \text{for } n = 0, 1, \ldots, m - 1.$$

Thus for each nonnegative integer $n < m$,

$$\sum_{v \in V(\theta)} f(v) \binom{[v]}{n} = \sum_{v \in V(\theta)} f(v) \sum_{j=0}^{n} (-1)^{n-j} s(n,j)[v]^j / n!$$

$$= \frac{1}{n!} \sum_{j=0}^{n} (-1)^{n-j} s(n,j) \sum_{v \in V(\theta)} f(v)[v]^j = 0,$$

where $s(n,j)$ $(0 \le j \le n)$ are Stirling numbers of the first kind, i.e.

$$\sum_{\substack{I \subseteq \{1,\ldots,k\} \\ \{\Sigma_{s \in I} 1/\beta_s\} = \theta}} (-1)^{|I|} \binom{[\sum_{s \in I} 1/\beta_s]}{n} e^{2\pi i \Sigma_{s \in I} \alpha_s / \beta_s}$$

$$= \sum_{v \in V(\theta)} \sum_{\substack{I \subseteq \{1,\ldots,k\} \\ \Sigma_{s \in I} 1/\beta_s = v}} \binom{[v]}{n} (-1)^{|I|} e^{2\pi i \Sigma_{s \in I} \alpha_s / \beta_s} = 0.$$

Therefore by the proof of (b)$\Rightarrow$(c),

$$\lim_{r \to 1} \sum_{\substack{I \subseteq \{1,\ldots,k\} \\ \{\Sigma_{s \in I} 1/\beta_s\} = \theta}} (-1)^{|I|} r^{\Sigma_{s \in I} 1/\beta_s} e^{2\pi i \Sigma_{s \in I} \alpha_s / \beta_s} / (1-r)^{m-1}$$

$$= \lim_{t \to 0} \sum_{n=0}^{m-1} (-1)^n \left(\sum_{\substack{I \subseteq \{1,\ldots,k\} \\ \{\Sigma_{s \in I} 1/\beta_s\} = \theta}} (-1)^{|I|} \binom{[\sum_{s \in I} 1/\beta_s]}{n} e^{2\pi i \Sigma_{s \in I} \alpha_s / \beta_s} \right) t^{n-m+1}$$

$$= 0.$$

Now for every integer x,

$$\prod_{s=1}^{k} (1 - r^{1/\beta_s} e^{2\pi i (\alpha_s - x)/\beta_s})$$

$$= \sum_{I \subseteq \{1,\ldots,k\}} (-1)^{|I|} r^{\Sigma_{s \in I} 1/\beta_s} e^{2\pi i \Sigma_{s \in I} (\alpha_s - x)/\beta_s}$$

$$= \sum_{\theta \in S} e^{-2\pi i x \theta} \sum_{\substack{I \subseteq \{1,\ldots,k\} \\ \{\Sigma_{s \in I} 1/\beta_s\} = \theta}} (-1)^{|I|} r^{\Sigma_{s \in I} 1/\beta_s} e^{2\pi i \Sigma_{s \in I} \alpha_s / \beta_s}$$

$$= \sum_{\theta \in S} e^{-2\pi i x \theta} o((1-r)^{m-1}) = o((1-r)^{m-1}) \quad (r \to 1).$$

Applying Lemma 1 we then obtain part (a).

The proof of Theorem 1 is now complete.

3. Reciprocals of common differences. In 1989 M. Z. Zhang [19] showed the following surprising result analytically: Provided that (2) is a cover of $\mathbb{Z}$, $\sum_{s \in I} 1/n_s \in \mathbb{Z}^+$ for some $I \subseteq \{1, \ldots, k\}$. Here we give

THEOREM 2. *Let* (1) *be a cover of* $\mathbb{Z}$. *Then for any* $J \subseteq \{1, \ldots, k\}$ *there is an* $I \subseteq \{1, \ldots, k\}$ *with* $I \neq J$ *such that*

$$(9) \qquad \sum_{s \in I} \frac{1}{\beta_s} - \sum_{s \in J} \frac{1}{\beta_s} \in \mathbb{Z}.$$

Proof. Set $\theta = \left\{ \sum_{s \in J} 1/\beta_s \right\}$. By Theorem 1,

$$\sum_{\substack{I \subseteq \{1,\ldots,k\} \\ \{\Sigma_{s \in I} 1/\beta_s\} = \theta}} (-1)^{|I|} \binom{[\sum_{s \in I} 1/\beta_s]}{0} e^{2\pi i \Sigma_{s \in I} \alpha_s / \beta_s} = 0,$$

that is,

$$\sum_{\substack{J \neq I \subseteq \{1,\ldots,k\} \\ \{\Sigma_{s \in I} 1/\beta_s\} = \theta}} (-1)^{|I|} e^{2\pi i \Sigma_{s \in I} \alpha_s / \beta_s} = -(-1)^{|J|} e^{2\pi i \Sigma_{s \in J} \alpha_s / \beta_s}.$$

Therefore

$$\left\{ I \subseteq \{1, \ldots, k\} : I \neq J \text{ and } \left\{ \sum_{s \in I} \frac{1}{\beta_s} \right\} = \theta \right\} \neq \emptyset.$$

We are done.

In the case $J = \emptyset$, Theorem 2 yields a generalization of Zhang's result ([19]).

Provided that (1) is an m-cover of $\mathbb{Z}$ with $m \in \mathbb{Z}^+$, Theorem 2 asserts that for any $J \subseteq \{1, \ldots, k\}$,

$$(10) \qquad S(J) = \left\{ I \subseteq \{1, \ldots, k\} : I \neq J \text{ and } \sum_{s \in I} \frac{1}{\beta_s} - \sum_{s \in J} \frac{1}{\beta_s} \in \mathbb{Z} \right\}$$

is nonempty. This becomes trivial if

$$(11) \qquad \sum_{s \in I} \frac{1}{\beta_s} = \sum_{s \in J} \frac{1}{\beta_s} \quad \text{for some } I \subseteq \{1, \ldots, k\} \text{ with } I \neq J.$$

What can we say about

$$(12) \qquad Z(J) = \left\{ \sum_{s \in I} \frac{1}{\beta_s} - \sum_{s \in J} \frac{1}{\beta_s} : I \in S(J) \right\}$$

if it does not contain zero? The following theorem gives us more information.

THEOREM 3. *Assume that* (1) *is an* m-cover *of* $\mathbb{Z}$. *Let* J *be a subset of* $\{1, \ldots, k\}$ *such that* (11) *fails, i.e.* $0 \notin Z(J)$ *where* $S(J)$ *and* $Z(J)$ *are given by* (10) *and* (12). *Then*

(i) $|Z(J)| \geq m$ *and hence*

(13)
$$\sum_{s=1}^{k} \frac{1}{\beta_s} \geq md(J) + \left\{\sum_{s \in J} \frac{1}{\beta_s}\right\} \geq m,$$

where $d(J)$ is the least positive integer that can be written as the difference of two (distinct) numbers of the form

$$\sum_{s \in I} \frac{1}{\beta_s} \in \mathbb{Z} + \sum_{s \in J} \frac{1}{\beta_s} \quad \text{where } I \subseteq \{1, \ldots, k\}.$$

(ii) *When $d(J) \geq [\sum_{s=1}^{k} 1/\beta_s]/m$, $d(J)$ equals $[\sum_{s=1}^{k} 1/\beta_s]/m$ and divides $[\sum_{s \in J} 1/\beta_s]$, and for every $n = 0, 1, \ldots, m$ there exist at least*

$$\binom{m}{n} \Big/ \binom{m}{m[\sum_{s \in J} 1/\beta_s]/[\sum_{s=1}^{k} 1/\beta_s]}$$

subsets I of $\{1, \ldots, k\}$ such that

(14)
$$\sum_{s \in I} \frac{1}{\beta_s} = \frac{n}{m}\left[\sum_{s=1}^{k} \frac{1}{\beta_s}\right] + \left\{\sum_{s \in J} \frac{1}{\beta_s}\right\},$$

hence

$$|S(J)| \geq 2^m \Big/ \binom{m}{m[\sum_{s \in J} 1/\beta_s]/[\sum_{s=1}^{k} 1/\beta_s]} - 1 \quad \text{and} \quad |Z(J)| = m.$$

Proof. Let $\theta = \{\sum_{s \in J} 1/\beta_s\}$, $V(\theta)$, $U(\theta)$ and $f(x)$ be as in Theorem 1. If $|V(\theta)| \leq m$, then $V(\theta) \subseteq U(\theta)$, hence by Theorem 1 for all $u \in V(\theta) \subseteq U(\theta)$,

$$f(u) = \sum_{v \in V(\theta)} \left(\prod_{\substack{x \in U(\theta) \\ x \neq u}} \frac{x - v}{x - u}\right) f(v) = 0,$$

which is impossible since $0 \notin Z(J)$ and

$$f\left(\sum_{s \in J} \frac{1}{\beta_s}\right) = (-1)^{|J|} e^{2\pi i \Sigma_{s \in J} \alpha_s/\beta_s} \neq 0.$$

Thus $|V(\theta)| > m$.

(i) Let $v_0 < v_1 < \ldots < v_m$ be the first $m + 1$ elements of $V(\theta)$ in ascending order. Clearly

$$1 + |Z(J)| = |Z(J) \cup \{0\}| = \left|\left\{v - \sum_{s \in J} \frac{1}{\beta_s} : v \in V(\theta)\right\}\right| = |V(\theta)| \geq m + 1$$

and

$$\sum_{s=1}^{k} \frac{1}{\beta_s} \geq \max_{v \in V(\theta)} v \geq v_m = \sum_{i=0}^{m-1} (v_{i+1} - v_i) + v_0 \geq md(J) + \theta.$$

(ii) If $|V(\theta)| > m + 1$ then

$$\sum_{s=1}^{k} \frac{1}{\beta_s} \geq \max_{v \in V(\theta)} v \geq v_m + 1 \geq 1 + md(J) + \theta.$$

Now suppose that $d(J) \geq [\sum_{s=1}^{k} 1/\beta_s]/m$. Then we must have $|V(\theta)| = m + 1$, thus $V(\theta) = \{v_0, v_1, \ldots, v_m\}$ and $|Z(J)| = |V(\theta)| - 1 = m$. As

$$md(J) \geq \left[\sum_{s=1}^{k} \frac{1}{\beta_s}\right] \geq [v_m] = v_0 - \theta + \sum_{i=0}^{m-1} (v_{i+1} - v_i) \geq [v_0] + md(J),$$

$$md(J) = \left[\sum_{s=1}^{k} \frac{1}{\beta_s}\right], \quad [v_0] = 0$$

and

$$[v_n] = v_0 - \theta + \sum_{i=0}^{n-1} (v_{i+1} - v_i) = 0 + \sum_{i=0}^{n-1} d(J) = nd(J)$$

for $n = 1, \ldots, m$.

Choose $0 \leq j \leq m$ such that $v_j = \sum_{s \in J} 1/\beta_s$. Then

$$j = \frac{[v_j]}{d(J)} = m \left[\sum_{s \in J} \frac{1}{\beta_s}\right] \Big/ \left[\sum_{s=1}^{k} \frac{1}{\beta_s}\right].$$

Set

$$U'(\theta) = \{v_i : 0 \leq i \leq m, \ i \neq j\}.$$

By Theorem 1, for any $n = 0, 1, \ldots, m$ with $n \neq j$,

$$0 = \sum_{v \in V(\theta)} \left(\prod_{\substack{x \in U'(\theta) \\ x \neq v_n}} \frac{x - v}{x - v_n} \right) f(v) = \sum_{t=0}^{m} \left(\prod_{\substack{i=0 \\ i \neq j, n}}^{m} \frac{v_i - v_t}{v_i - v_n} \right) f(v_t)$$

$$= \sum_{t=0}^{m} \left(\prod_{\substack{i=0 \\ i \neq j, n}}^{m} \frac{id(J) + \theta - (td(J) + \theta)}{id(J) + \theta - (nd(J) + \theta)} \right) f(v_t)$$

$$= \sum_{t=0}^{m} \left(\prod_{\substack{i=0 \\ i \neq j, n}}^{m} \frac{i - t}{i - n} \right) f(v_t)$$

$$= f(v_n) + \left(\prod_{\substack{i=0 \\ i \neq j, n}}^{m} \frac{i - j}{i - n} \right) f(v_j).$$

Since

$$\prod_{\substack{i=0 \\ i \neq j,n}}^{m} \frac{i-j}{i-n} = \frac{\prod_{i=0,\, i \neq j}^{m}(i-j)}{n-j} \bigg/ \frac{\prod_{i=0,\, i \neq n}^{m}(i-n)}{j-n}$$

$$= -\frac{\prod_{i=0}^{j-1}(i-j) \cdot \prod_{i=j+1}^{m}(i-j)}{\prod_{i=0}^{n-1}(i-n) \cdot \prod_{i=n+1}^{m}(i-n)}$$

$$= -\frac{(-1)^j j!(m-j)!}{(-1)^n n!(m-n)!} = (-1)^{j-n+1} \binom{m}{n} \bigg/ \binom{m}{j},$$

we have

$$\sum_{\substack{I \subseteq \{1,\ldots,k\} \\ \Sigma_{s \in I} 1/\beta_s = nd(J)+\theta}} (-1)^{|I|} e^{2\pi i \Sigma_{s \in I} \alpha_s/\beta_s}$$

$$= f(v_n) = -(-1)^{j-n+1} \binom{m}{n}\binom{m}{j}^{-1} f\left(\sum_{s \in J} \frac{1}{\beta_s}\right)$$

$$= (-1)^{j-n} \binom{m}{n}\binom{m}{j}^{-1} (-1)^{|J|} e^{2\pi i \Sigma_{s \in J} \alpha_s/\beta_s}$$

and hence

$$\left|\left\{ I \subseteq \{1,\ldots,k\} : \sum_{s \in I} \frac{1}{\beta_s} = \frac{n}{m}\left[\sum_{s=1}^{k} \frac{1}{\beta_s}\right] + \left\{\sum_{s \in J} \frac{1}{\beta_s}\right\}\right\}\right|$$

$$= \sum_{\substack{I \subseteq \{1,\ldots,k\} \\ \Sigma_{s \in I} 1/\beta_s = nd(J)+\theta}} 1$$

$$\geq \left| \sum_{\substack{I \subseteq \{1,\ldots,k\} \\ \Sigma_{s \in I} 1/\beta_s = nd(J)+\theta}} (-1)^{|I|} e^{2\pi i \Sigma_{s \in I} \alpha_s/\beta_s} \right| = \binom{m}{n} \bigg/ \binom{m}{j};$$

therefore

$$1 + |S(J)| = \left|\left\{ I \subseteq \{1,\ldots,k\} : \sum_{s \in I} \frac{1}{\beta_s} \in V(\theta) \right\}\right|$$

$$= \sum_{n=0}^{m} \left|\left\{ I \subseteq \{1,\ldots,k\} : \sum_{s \in I} \frac{1}{\beta_s} = v_n = nd(J)+\theta \right\}\right|$$

$$\geq \sum_{n=0}^{m} \binom{m}{n} \bigg/ \binom{m}{j} = 2^m \bigg/ \binom{m}{j}.$$

This ends the proof.

Now let us apply Theorem 3 to those m-covers (1) with $\sum_{s=1}^{k} 1/\beta_s = m$.

THEOREM 4. *Let* (1) *be an m-cover of* $\mathbb{Z}$ *with* $\sum_{s=1}^{k} 1/\beta_s = m \in \mathbb{Z}^+$, *which happens if* (1) *is an exact m-cover of* $\mathbb{Z}$ *by residue classes. Then*

(i) *For every* $l = 1, \ldots, k - 1$ *we have*

(15)
$$\sum_{s=l+1}^{k} \frac{1}{\beta_s} \geq \frac{1}{\max\{\beta_1, \ldots, \beta_l\}}.$$

(ii) *For any* $\emptyset \neq J \subset \{1, \ldots, k\}$ *there exists an* $I \subseteq \{1, \ldots, k\}$ *with* $I \neq J$ *such that*

(16)
$$\sum_{s \in I} \frac{1}{\beta_s} = \sum_{s \in J} \frac{1}{\beta_s},$$

furthermore when $\sum_{s \in J} 1/\beta_s \in \mathbb{Z}$ *there are at least*

$$\binom{m}{\sum_{s \in J} 1/\beta_s} \geq m > 1$$

subsets I *of* $\{1, \ldots, k\}$ *satisfying* (16).

P r o o f. (i) For $l = 1, \ldots, k - 1$ (15) follows from part (ii) in the case $J = \{l + 1, \ldots, k\}$, so we proceed to the proof of part (ii).

(ii) If (11) fails then by part (i) of Theorem 3 and the equality $\sum_{s=1}^{k} 1/\beta_s = m$ we must have

$$\left\{ \sum_{s \in J} \frac{1}{\beta_s} \right\} = 0, \quad \text{i.e.} \quad \sum_{s \in J} \frac{1}{\beta_s} \in \mathbb{Z}.$$

Observe that

$$0 < \sum_{s \in J} \frac{1}{\beta_s} < \sum_{s=1}^{k} \frac{1}{\beta_s} = m.$$

If $\sum_{s \in J} 1/\beta_s \in \mathbb{Z}$, then $m > 1$ and $\sum_{s \in J} 1/\beta_s = n$ for some $n = 1, \ldots, m-1$, by part (ii) of Theorem 3 there are at least $\binom{m}{n} / \binom{m}{m} = \binom{m}{n} \geq m$ subsets I of $\{1, \ldots, k\}$ such that

$$\sum_{s \in I} \frac{1}{\beta_s} = \frac{n}{m} \left[\sum_{s=1}^{k} \frac{1}{\beta_s} \right] + \left\{ \sum_{1 \leq s \leq k} \frac{1}{\beta_s} \right\} = n = \sum_{s \in J} \frac{1}{\beta_s}.$$

We are done.

R e m a r k. In 1992 Z. W. Sun ([17]) proved that if (2) is an exact m-cover of $\mathbb{Z}$ then for each $n = 1, \ldots, m$ there exist at least $\binom{m}{n}$ subsets I of $\{1, \ldots, k\}$ such that $\sum_{s \in I} 1/n_s$ equals n. The lower bounds $\binom{m}{n}$ $(1 \leq n \leq m)$ are best possible, and the Riemann zeta function was used in the proof.

From Theorem 3 we can also deduce the following theorem which extends Zhang's result ([19]) and the theorem of Sun [17] even in the case $l = k$.

THEOREM 5. *Let* (1) *be an* m-*cover of* $\mathbb{Z}$ *and* l *a positive integer not exceeding* k *such that*

$$(17) \qquad \min\left\{1, \frac{1}{\beta_1}, \ldots, \frac{1}{\beta_l}\right\} > \sum_{l<t\le k} \frac{1}{\beta_t},$$

where $\sum_{l<t\le k} 1/\beta_t$ *is considered to be zero for* $l = k$. *Then*

(i) *There are at least* m *positive integers representable by*

$$(18) \qquad \sum_{s\in I} \frac{1}{\beta_s} - \sum_{l<t\le k} \frac{1}{\beta_t}, \qquad \text{where } I \subseteq \{1, \ldots, k\},$$

thus we have

$$(19) \qquad \sum_{s=1}^{l} \frac{1}{\beta_s} = \sum_{s=1}^{k} \frac{1}{\beta_s} - \sum_{l<t\le k} \frac{1}{\beta_t} \ge m.$$

(ii) *Provided that any positive integer less than* $[\sum_{s=1}^{k} 1/\beta_s]/m$ *cannot be expressed as the difference of two integers of the form* (18), $[\sum_{s=1}^{k} 1/\beta_s]$ *is divisible by* m *and for each* $n = 0, 1, \ldots, m$ *there are at least* $\binom{m}{n}$ *subsets* I *of* $\{1, \ldots, k\}$ *such that*

$$(20) \qquad \sum_{s\in I} \frac{1}{\beta_s} = \frac{n}{m}\left[\sum_{s=1}^{k} \frac{1}{\beta_s}\right] + \sum_{l<t\le k} \frac{1}{\beta_t},$$

hence there exist at least $2^m - 1$ *subsets* I *of* $\{1, \ldots, k\}$ *with*

$$\sum_{s\in I} \frac{1}{\beta_s} \in \mathbb{Z}^+ + \sum_{l<t\le k} \frac{1}{\beta_t}.$$

Proof. Let $J = \{1 \le t \le k : t > l\}$. By (17),

$$\left[\sum_{t\in J} \frac{1}{\beta_t}\right] = 0 \quad \text{and} \quad \left\{\sum_{t\in J} \frac{1}{\beta_t}\right\} = \sum_{l<t\le k} \frac{1}{\beta_t}.$$

For any $I \subseteq \{1, \ldots, k\}$, if $I \subset J$ then

$$0 < \sum_{t\in J} \frac{1}{\beta_t} - \sum_{s\in I} \frac{1}{\beta_s} < 1,$$

and if $I \not\subseteq J$ then

$$\sum_{s\in I} \frac{1}{\beta_s} - \sum_{t\in J} \frac{1}{\beta_t} \ge \min\left\{\frac{1}{\beta_s} : 1 \le s \le l\right\} - \sum_{l<t\le k} \frac{1}{\beta_t} > 0.$$

So (11) fails, moreover $Z(J)$ given by (12) contains only positive integers. Applying Theorem 3 we obtain the desired results.

Erdős conjectured (before 1950) that if (2) is a cover of $\mathbb{Z}$ with $1 < n_1 < n_2 < \ldots < n_k$ then $\sum_{s=1}^{k} 1/n_s > 1$. H. Davenport, L. Mirsky, D. Newman and R. Radó confirmed this conjecture (independently) by proving that if (2) is an exact cover of $\mathbb{Z}$ with $1 < n_1 \leq \ldots \leq n_{k-1} \leq n_k$ then $n_{k-1} = n_k$. For further improvements see Znám [20], M. Newman [10], Porubský [11, 12], M. A. Berger, A. Felzenbaum and A. S. Fraenkel [1]. The best record in this direction is the following result due to the author which is partially announced in [15] and completely proved in [16]: Let $\lambda_1, \ldots, \lambda_k$ be complex numbers and $n_0 \in \mathbb{Z}^+$ a period of the function

$$\sigma(x) = \sum_{\substack{s=1 \\ x \equiv a_s \, (\mathrm{mod} \ n_s)}}^{k} \lambda_s.$$

If $d \in \mathbb{Z}^+$ does not divide n_0 and

$$\sum_{\substack{s=1 \\ d \mid n_s, \, a_s \equiv a \, (\mathrm{mod} \, d)}}^{k} \frac{\lambda_s}{n_s} \neq 0 \quad \text{for some integer } a,$$

then

$$|\{a_s \bmod d : 1 \leq s \leq k, \ d \, | \, n_s\}| \geq \min_{\substack{0 \leq s \leq k \\ d \nmid n_s}} \frac{d}{\gcd(d, n_s)} \geq p(d),$$

where $p(d)$ is the least prime divisor of d. Here we have

THEOREM 6. *Let* (1) *be an m-cover of $\mathbb{Z}$ with $\beta_1 \leq \ldots \leq \beta_{k-l} < \beta_{k-l+1} = \ldots = \beta_k$ where $1 \leq l < k$. Then either*

$$(21) \qquad l \geq \beta_k / \max\{1, \beta_{k-l}\} \quad (> 1 \text{ if } \beta_k > 1),$$

or there are at least m positive integers in the form

$$(22) \qquad \sum_{s \in I} \frac{1}{\beta_s} - \frac{l}{\beta_k}, \quad \text{where } I \subseteq \{1, \ldots, k\},$$

and hence

$$(23) \qquad \sum_{s=1}^{k} \frac{1}{\beta_s} > \sum_{s=1}^{k-l} \frac{1}{\beta_s} = \sum_{s=1}^{k} \frac{1}{\beta_s} - \frac{l}{\beta_k} \geq m.$$

(*Also,* $\sum_{s=1}^{k} 1/\beta_s > \sum_{s=1}^{k} 1/\beta_k \geq k \geq m$ *if* $\beta_k \leq 1$.)

P r o o f. Clearly $l < \beta_k / \max\{1, \beta_{k-l}\}$ if and only if

$$\min \left\{ 1, \frac{1}{\beta_1}, \ldots, \frac{1}{\beta_{k-l}} \right\} > \sum_{k-l < t \leq k} \frac{1}{\beta_t} \ (= l/\beta_k).$$

Therefore Theorem 6 follows from part (i) of Theorem 5.

Note that when $\beta_{k-l} \geq 1$ and $\beta_k / \beta_{k-l} \in \mathbb{Z}$

$$\beta_k / \max\{1, \beta_{k-l}\} = \beta_k / \beta_{k-l} \geq p(\beta_k / \beta_{k-l}) \quad (\geq p(\beta_k) \text{ if } \beta_{k-l}, \beta_k \in \mathbb{Z}).$$

4. Some local-global results. In 1958 S. K. Stein [14] conjectured that whenever the residue classes in (2) are pairwise disjoint and the moduli $n_1, \ldots, n_k > 1$ are distinct there exists an integer x with $1 \leq x \leq 2^k$ such that x is not covered by (2). Erdős [6] confirmed this conjecture with $k \cdot 2^k$ instead of 2^k. Since the Davenport–Mirsky–Newman–Radó result indicates that an exact cover of $\mathbb{Z}$ by (finitely many) residue classes cannot have its moduli distinct and greater than one, Erdős proposed the stronger conjecture that any system of k residue classes not covering all the integers omits a positive integer not exceeding 2^k. Both conjectures have some local-global character. In 1969 R. B. Crittenden and C. L. Vanden Eynden [2] claimed their positive answer to the stronger conjecture. Later in [3] a long indirect and awkward proof was given for $k \geq 20$, the authors concluded the paper with the statements: "Of course it remains to show the conjecture is true for $k < 20$. This may be checked by more special arguments."

In 1970 Crittenden and Vanden Eynden [4] conjectured further that if all the moduli n_s in (2) are greater than an integer $0 \leq l < k$ then (2) is a cover of $\mathbb{Z}$ if it covers all the integers in the interval $[1, 2^{k-l}(l+1)]$. In contrast with the Crittenden–Vanden Eynden conjecture we give

THEOREM 7. *For any* $m \in \mathbb{Z}^+$, (1) *is an* m-*cover of* $\mathbb{Z}$ *if it covers* $2^{k-M}(M+1)$ *consecutive integers at least* m *times, where*

$$(24) \qquad M = \max_{1 \leq t \leq k} |\{1 \leq s \leq k : \beta_s = \beta_t\}|.$$

Proof. Let $\beta > 0$ be a number such that $J = \{1 \leq s \leq k : \beta_s = \beta\}$ has cardinality M. As

$$\left| \left\{ \left\{ \sum_{s \in I} \frac{1}{\beta_s} \right\} : I \subseteq \{1, \ldots, k\} \right\} \right|$$

$$\leq \left| \left\{ \sum_{s \in I \cap J} \frac{1}{\beta_s} + \sum_{s \in I \setminus J} \frac{1}{\beta_s} : I \subseteq \{1, \ldots, k\} \right\} \right|$$

$$\leq \left| \left\{ \sum_{s \in I} \frac{1}{\beta} : I \subseteq J \right\} \right| \cdot \left| \left\{ \sum_{s \in I} \frac{1}{\beta_s} : I \subseteq \{1, \ldots, k\} \setminus J \right\} \right|$$

$$\leq \left| \left\{ \frac{|I|}{\beta} : I \subseteq J \right\} \right| \cdot |\{I : I \subseteq \{1, \ldots, k\} \setminus J\}|$$

$$= (|J| + 1) \cdot 2^{k-|J|} = 2^{k-M}(M+1),$$

Theorem 1 implies Theorem 7.

The following example noted by Crittenden and Vanden Eynden [4] shows that the number $g(k, M) = 2^{k-M}(M+1)$ in Theorem 7 is best possible.

EXAMPLE. Let $M = n - 1 \in \mathbb{Z}^+$. Consider the system A consisting of the following $k \geq M$ residue classes:

$$1 + n\mathbb{Z}, \quad 2 + n\mathbb{Z}, \quad \ldots, \quad M + n\mathbb{Z},$$
$$n + 2n\mathbb{Z}, \quad 2n + 2^2 n\mathbb{Z}, \quad \ldots, \quad 2^{k-M-1}n + 2^{k-M}n\mathbb{Z}.$$

Observe that A together with $2^{k-M}n\mathbb{Z}$ forms an exact cover of $\mathbb{Z}$. So A covers positive integers from 1 to $2^{k-M}(M+1) - 1$, but it does not cover all the integers.

Result (II) stated in Section 1 follows from Theorem 7 and Example, since (1) covers $\alpha + \beta x$ (where $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}^+$ and $x \in \mathbb{Z}$) at least m times if and only if $\left\{\frac{\alpha_s - \alpha}{\beta} + \frac{\beta_s}{\beta}\mathbb{Z}\right\}_{s=1}^{k}$ covers x at least m times, and $2^{k-l}(l+1) \geq 2^{k-M}(M+1)$ if $k \geq M \geq l > 0$. (The case $l = 0$ can be reduced to the case $l = 1$.)

5. Several open problems. Theorem 1 tells us that (2) is a cover of $\mathbb{Z}$ if it covers integers from 1 to

$$\left|\left\{\left\{\sum_{s \in I} \frac{1}{\beta_s}\right\} : I \subseteq \{1, \ldots, k\}\right\}\right| \leq 2^k \leq 2^{n_1 + \ldots + n_k}.$$

This suggests

PROBLEM 1. Can we find a polynomial P with integer coefficients such that a finite system (2) of residue classes forms a cover of $\mathbb{Z}$ whenever it covers all positive integers not exceeding $P(n_1 + \ldots + n_k)$?

In 1973 L. J. Stockmeyer and A. R. Meyer proved that the problem whether there exists an integer not covered by a given (2) is NP-complete. In 1991 S. P. Tung [18] extended this result to algebraic integer rings. If the required P in Problem 1 exists, then there is a polynomial time algorithm to decide whether (2) covers all the integers or not. So a positive answer to Problem 1 would imply that $NP = P$.

By appearances Theorems 2–7 involve no roots of unity. Perhaps vast generalizations of them could be made.

PROBLEM 2. Let $A_1, \ldots, A_k$ be sets of natural numbers having positive densities $d(A_1), \ldots, d(A_k)$ respectively. If no A_s contains $m_s \in \mathbb{Z}^+$ consecutive integers, does $\bigcup_{s=1}^{k} A_s$ have density 1 when it covers $m_1 \ldots m_k$ arbitrarily large consecutive integers? Suppose that $\{A_s\}_{s=1}^{k}$ covers all the natural numbers; does there exist, for any $J \subseteq \{1, \ldots, k\}$, an $I \subseteq \{1, \ldots, k\}$

with $I \neq J$ such that

$$\sum_{s \in I} d(A_s) - \sum_{s \in J} d(A_s) \in \mathbb{Z}?$$

PROBLEM 3.   Let K be an algebraic number field and O_K the ring of algebraic integers in K. Let $a_1, \ldots, a_k \in O_K$ and $A_1, \ldots, A_k$ be ideals of O_K with norms $N(A_1), \ldots, N(A_k)$ respectively. If $\{a_s + A_s\}_{s=1}^k$ forms an exact m-cover of O_K for some $m \in \mathbb{Z}^+$, is it true that for any $\emptyset \neq J \subseteq \{1, \ldots, k\}$ there exists a subset I of $\{1, \ldots, k\}$ with $I \neq J$ such that

$$\sum_{s \in I} \frac{1}{N(A_s)} = \sum_{s \in J} \frac{1}{N(A_s)}?$$

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