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Theta and L-function splittings

by

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Introduction. The base change lift of an automorphic form by means of a theta kernel was first done by Kudla in [2, 3] and Zagier in [6]. Kudla's paper omitted the computation of the Fourier series coefficients; he instead referred to the paper of Niwa [4] on the Shimura lift. Knowledge of these Fourier coefficients lets one write the L-series of the lifted form as a product of the original L-series and its quadratic twist. In this paper the factorization of the L-series is shown directly. Niwa's idea of splitting the theta function lets us explicitly compute the Mellin transform $L(s,\widetilde{f})$ of the lifted form $\widetilde{f}$. It is a Rankin–Selberg convolution of the original form f with a Maass wave form coming from the quadratic extension. The factorization of the L-series then follows as in the work of Doi and Naganuma [1].

To avoid excessive notation, only the simplest case is considered: the lift to $\mathbb{Q}(\sqrt{q})$, with q an odd prime $q \equiv 1 \bmod 4$ such that $h_+(K) = 1$. We use χ to denote the Kronecker symbol $\left(\frac{q}{*}\right)$. We take a cusp form $f(z) = \sum a(n)\exp(2\pi i n z)$ of weight k for $SL(2,\mathbb{Z})$, an eigenfunction of all the Hecke operators. Recall that in Section 3 of [2] Kudla defined the theta kernel

$$\theta(z,z_1,z_2) = y\sum_{l\in L}\chi(m)(-mz_1z_2+\alpha z_1+{}^\sigma\alpha z_2+n)^k e\{(xQ+iyR)[l]\}$$

where

- $z = x+iy$ is in $\mathcal{H}$ and (z_1,z_2) is in $\mathcal{H}^2$,
- the lattice variable l is written as $\left[\begin{smallmatrix}\alpha & n\\ m & -{}^\sigma\alpha\end{smallmatrix}\right]$ with α in $\mathcal{O}$, ${}^\sigma\alpha$ the Galois conjugate, and m,n in $\mathbb{Z}$,
- $Q[l]$ is the indefinite quadratic form $-2\det(l)$,
- each $z_j = u_j+iv_j$ defines an element $g_j = \left[\begin{smallmatrix}\sqrt{v_j} & u_j/\sqrt{v_j}\\ 0 & 1/\sqrt{v_j}\end{smallmatrix}\right]$ in $SL(2,\mathbb{R})$.
- The pair $g=(g_1,g_2)$ acts on the vector space by $g\cdot l = g_2^{-1}lg_1$,
- $R[l]$ is a majorant for Q defined by $\mathrm{tr}({}^t(g\cdot l)g\cdot l)$.

Then the lifting $\widetilde{f}$ is defined by

$$\widetilde{f}(z_1, z_2) = \int_{\mathcal{F}} f(z)\overline{\theta}(z, z_1, z_2) y^k \, \frac{dx\,dy}{y^2},$$

where $\mathcal{F}$ is a fundamental domain for $\Gamma_0(q)\backslash\mathcal{H}$.

Splitting the theta function. Let

$$\theta_{1,j}(z, v) = y^{(1-j)/2} 2^{-j} \sum_{\alpha\in\mathcal{O}} H_j(\sqrt{\pi y}(\alpha v^{1/2} + {}^{\sigma}\alpha v^{-1/2})) \times \exp(2\pi i x N\alpha - \pi y(\alpha^2 v + {}^{\sigma}\alpha^2/v))$$

and

$$\theta_{2,j}(z, v) = y^{(1-j)/2} 2^{-j} \sum_{m,n\in\mathbb{Z}} \chi(m) H_j(\sqrt{\pi y}(m v^{1/2} + n v^{-1/2})) \times \exp(2\pi i x m n - \pi y(v m^2 + n^2/v)).$$

Lemma. *Along the "purely imaginary axis"* $(z_1, z_2) = (iv_1, iv_2)$ *in* $\mathcal{H}^2$,

$$\theta(z, iv_1, iv_2) = (-1)^k \pi^{-k/2} \sum_{2\nu\le k} (-1)^{\nu} \binom{k}{2\nu} \theta_{1,2\nu}\left(z, \frac{v_1}{v_2}\right) \theta_{2,k-2\nu}(z, v_1 v_2).$$

Proof. Along the imaginary axis

$$R[l] = \frac{v_1}{v_2}\alpha^2 + \frac{v_2}{v_1}\,{}^{\sigma}\alpha^2 + v_1 v_2 m^2 + \frac{n^2}{v_1 v_2}$$

and the spherical polynomial term is equal to

$$(-1)^k \left(m(v_1 v_2)^{1/2} + \frac{n}{(v_1 v_2)^{1/2}} + i\alpha\left(\frac{v_1}{v_2}\right)^{1/2} + i\,{}^{\sigma}\alpha\left(\frac{v_2}{v_1}\right)^{1/2}\right)^k.$$

Apply to this the Hermite identity

$$(a + ib)^k = 2^{-k} \sum_{j=0}^{k} \binom{k}{j} H_{k-j}(a) H_j(b) i^j$$

where $H_j(x) = (-1)^j \exp(x^2) \frac{d^j}{dx^j}(\exp(-x^2))$ is the jth Hermite polynomial. Include a factor of $\sqrt{\pi y}$ (which will be needed later) to show that the spherical polynomial term is

$$2^{-k}(-1)^k(\pi y)^{-k/2} \sum_{j=0}^{k} \binom{k}{j} H_{k-j}\left(m(\pi y v_1 v_2)^{1/2} + n\left(\frac{\pi y}{v_1 v_2}\right)^{1/2}\right) \times H_j\left(\alpha\left(\frac{\pi y v_1}{v_2}\right)^{1/2} + {}^{\sigma}\alpha\left(\frac{\pi y v_2}{v_1}\right)^{1/2}\right) i^j.$$

$H_j(x)$ is an odd or even function according to whether j is odd or even. If j is odd, the α and $-\alpha$ terms in the sum defining g_j cancel and $g_j(z)$ is identically zero. Writing $j = 2\nu$ finishes the lemma. ■

The point of this is that the Dirichlet series $L(s,\widetilde{f})$ is given by the Mellin transform

$$L(s,\widetilde{f}) = \int\limits_{(\mathbb{R}^+)^2/U^+} \widetilde{f}(iv_1, iv_2)(v_1v_2)^{s-1}\,dv_1\,dv_2$$
$$= \int\limits_{(\mathbb{R}^+)^2/U^+} \int\limits_{\mathcal{F}} f(z)\overline{\theta}(z, iv_1, iv_2) y^k \frac{dx\,dy}{y^2}(v_1v_2)^{s-k/2-1}\,dv_1\,dv_2.$$

Here U^+ is the group of totally positive units, generated by ε.

Change the variables to $v_1' = v_1/v_2$ and $v_2' = v_1v_2$ (and by abuse of notation go back to writing v_1 and v_2). Then using the splitting of θ, the Mellin transform becomes

$$L(s,\widetilde{f}) = 2^{-1}(-1)^k\pi^{-k/2}\sum_{2\nu\le k}(-1)^\nu\binom{k}{2\nu}$$
$$\times \int\limits_0^\infty \int\limits_{\varepsilon^{-1}}^{\varepsilon} \int\limits_{\mathcal{F}} f(z)\overline{\theta}_{1,2\nu}(z, v_1)\overline{\theta}_{2,k-2\nu}(z, v_2) y^k \frac{dx\,dy}{y^2}\frac{dv_1}{v_1}v_2^{s-k/2}\frac{dv_2}{v_2}.$$

Let

$$g_{2\nu}(z) = \int\limits_{\varepsilon^{-1}}^{\varepsilon} \theta_{1,2\nu}(z, v)\frac{dv}{v} \quad \text{and} \quad E_{2\nu}(z, s, 0) = \int\limits_0^\infty \overline{\theta}_{2,2\nu}(z, v)v^{s-k/2}\frac{dv}{v}.$$

Rearranging the integrals shows that $L(s,\widetilde{f})$ is equal to

$$(1)\qquad \frac{\pi^{-k/2}}{2}\sum_{2\nu\le k}\binom{k}{2\nu}(-1)^{k-\nu}\int\limits_{\mathcal{F}} f(z)\overline{g}_{2\nu}(z)E_{k-2\nu}(z, s, 0)y^k\frac{dx\,dy}{y^2}.$$

Two ugly lemmas. Now two lemmas are required. The first is folklore, the second is sketched in [4].

LEMMA 1. *$g_{2\nu}(z)$ is equal to*

$$y^{1/2-\nu}2^{-2\nu}\int\limits_{\varepsilon^{-1}}^{\varepsilon}\sum_{\alpha\in\mathcal{O}} H_{2\nu}(\sqrt{\pi y}(\alpha v^{1/2} + {}^\sigma\alpha v^{-1/2}))$$
$$\times \exp(2\pi i x N\alpha - \pi y(\alpha^2 v + {}^\sigma\alpha^2/v))\frac{dv}{v}$$

and is a Maass wave form of weight 2ν.

Proof. Computing the integral will show that this is the Fourier expansion of a Maass form in terms of Whittaker functions. (Alternatively, one could use the method of Vignéras [5] to see that the integral is a Maass form, but in the end the Fourier expansion is wanted to apply the Rankin–Selberg method.)

From ([H], Vol. 2, p. 193) $H_{2\nu}(0) = (-1)^\nu 2\nu!/\nu!$ so the $\alpha = 0$ term contributes $2^{-2\nu}y^{1/2-\nu}(-1)^\nu 2\nu!/(\nu! 2\log(\varepsilon))$. For the terms $\alpha \neq 0$ in the sum interchange the sum and the integral and change the variables by $w = \varepsilon^{2n}v$ for $n \in \mathbb{Z}$. This gives

$$y^{1/2-\nu}2^{-2\nu} \sum_{\substack{\alpha\in\mathcal{O}/U^+ \\ \alpha\neq 0}} \int_0^\infty H_{2\nu}(\sqrt{\pi y}(\alpha w^{1/2} + {}^\sigma\alpha w^{-1/2})) \times \exp(-\pi y(\alpha^2 w + {}^\sigma\alpha^2/w))\frac{dw}{w}\exp(2\pi i x N\alpha).$$

To compute the integral of the term corresponding to α in the sum change variables again to let $v = \alpha(w/|N\alpha|)^{1/2}$ to get $2^{-2\nu}y^{1/2-\nu}\exp(2\pi i x N\alpha)$ times

$$2\int_0^\infty H_{2\nu}((\pi y|N\alpha|)^{1/2}(v \pm 1/v))\exp(-\pi y|N\alpha|(v\pm 1/v)^2)\exp(2\pi y N\alpha)\frac{dv}{v}$$

with the $\pm$ chosen according to whether $N\alpha$ is positive or negative. A final change of variables with $t = \log(v)$ gives

$$2\int_{-\infty}^{\infty} H_{2\nu}\left(2(\pi y|N\alpha|)^{1/2}\begin{matrix}\cosh t\\ \sinh t\end{matrix}\right)\exp\left(-4\pi y|N\alpha|\begin{matrix}\cosh^2 t\\ \sinh^2 t\end{matrix}\right)\exp(2\pi y N\alpha)\,dt.$$

For integral ν the parabolic cylinder functions are defined by ([H], Vol. 2, p. 117)

$$D_{2\nu}(z) = 2^{-\nu}\exp(-z^2/4)H_{2\nu}(z/\sqrt{2}).$$

Thus the integral is

$$2^{\nu+1}\int_{-\infty}^{\infty} D_{2\nu}\left(2a\begin{matrix}\cosh t\\ \sinh t\end{matrix}\right)\exp\left(-a^2\begin{matrix}\sinh^2 t\\ \cosh^2 t\end{matrix}\right)dt$$

with $a = (2\pi y|N\alpha|)^{1/2}$. For $N\alpha > 0$ apply ([I], Vol. 2, p. 398, (20)) to see that this is the Whittaker function

$$y^{-\nu}|N\alpha|^{-1/2}W_{\nu,0}(4\pi y|N\alpha|)\exp(2\pi i x N\alpha)$$

when the omitted constants are included.

For $N\alpha < 0$, use the imaginary phase shift

$$\cosh t = -i\sinh(t + i\pi/2) = i\sinh(t - i\pi/2),$$
$$\sinh t = -i\cosh(t + i\pi/2) = i\cosh(t - i\pi/2)$$

to get

$$2^{\nu+1}\int_{-\infty}^{\infty} D_{2\nu}(2a\,i\cosh(t-i\pi/2))\exp(a^2\sinh^2(t\pm i\pi/2))\,dt.$$

(The $\pm$ will be chosen later.)

The identity ([H], Vol. 2, p. 117)

$$D_{2\nu}(z)=(-1)^{\nu}\frac{2\nu!}{\sqrt{2\pi}}(D_{-2\nu-1}(iz)+D_{-2\nu-1}(-iz))$$

gives

$$(-1)^{\nu}2^{\nu+1}\frac{2\nu!}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\{D_{-2\nu-1}(-2a\cosh(t-i\pi/2)) + D_{-2\nu-1}(2a\cosh(t-i\pi/2))\}\exp(a^2\sinh^2(t\pm i\pi/2))\,dt.$$

In the first cylinder function, moving the -1 inside the $\cosh(t-i\pi/2)$ adds $i\pi$ to the argument, giving

$$(-1)^{\nu}2^{\nu+1}\frac{2\nu!}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\{D_{-2\nu-1}(2a\cosh(t+i\pi/2)) + D_{-2\nu-1}(2a\cosh(t-i\pi/2))\}\exp(a^2\sinh^2(t\pm i\pi/2))\,dt.$$

Write this as two integrals, choosing $\sinh^2(t+i\pi/2)$ in the first and $\sinh^2(t-i\pi/2)$ in the second. Since $D_{-2\nu-1}$ is an entire function one can shift the line of integration by $\mp i\pi/2$ to get

$$(-1)^{\nu}2^{\nu+2}\frac{2\nu!}{\sqrt{2\pi}}\int_{-\infty}^{\infty} D_{-2\nu-1}(2a\cosh t)\exp(a^2\sinh^2 t)\,dt.$$

Apply ([I], Vol. 2, p. 398, (21)) to see that this is the Whittaker function

$$(-1)^{\nu}\frac{\Gamma(\nu+1/2)^2}{\pi}y^{-\nu}|N\alpha|^{-1/2}W_{-\nu,0}(4\pi y|N\alpha|)\exp(2\pi ixN\alpha)$$

when the omitted constants are included. Summarizing, this gives

$$g_{2\nu}(z)=2^{1-2\nu}(-1)^{\nu}2\nu!/(\nu!\log(\varepsilon)y^{1/2-\nu})+(-1)^{\nu}y^{-\nu}\frac{\Gamma(\nu+1/2)^2}{\pi} \times\sum_{\substack{\alpha\in\mathcal{O}/U^+\\ N\alpha<0}}|N\alpha|^{-1/2}W_{-\nu,0}(4\pi y|N\alpha|)\exp(2\pi ixN\alpha) + y^{-\nu}\sum_{\substack{\alpha\in\mathcal{O}/U^+\\ N\alpha>0}}|N\alpha|^{-1/2}W_{\nu,0}(4\pi y|N\alpha|)\exp(2\pi ixN\alpha).\ \blacksquare$$

LEMMA 2. *$E_{2\nu}(z,s,0)$ is equal to*

$$y^{1/2-\nu}2^{-2\nu}\int\limits_0^\infty \sum_{m,n\in\mathbb{Z}} \chi(m)H_{2\nu}(\sqrt{\pi y}(mv^{1/2}+nv^{-1/2})) \times \exp\bigg(-2\pi ixmn-\pi y\bigg(vm^2+\frac{n^2}{v}\bigg)\bigg)v^{s-k/2}\frac{dv}{v}$$

and is a (non-holomorphic) Eisenstein series of weight 2ν.

Proof. The Fourier transform $\widehat{f}(t)=\int f(s)\exp(-2\pi ist)\,ds$ of

$$H_{2\nu}(m(\pi yv)^{1/2}+(\pi y/v)^{1/2}s)\exp(-(m(\pi yv)^{1/2}+(\pi y/v)^{1/2}s)^2)$$

is

$$(-1)^\nu 2^{2\nu}\pi^\nu(v/y)^{\nu+1/2}t^{2\nu}\exp(2\pi imvt)\exp(-\pi vt^2/y)$$

by ([I], Vol. 1, p. 39, (9)) and the usual Fourier transform theorems. The Poisson summation formula (using $\{f^\wedge\}^\wedge(s)=f(-s)$ and evaluating at mz) then gives

$$\begin{aligned}\sum_n & H_{2\nu}(m(\pi yv)^{1/2}-n(\pi y/v)^{1/2}) \\ &\times\exp(-(m(\pi yv)^{1/2}-n(\pi y/v)^{1/2})^2+2\pi imnz)\\ &=(-\pi)^\nu 2^{2\nu}(v/y)^{\nu+1/2}\sum_n(mz+n)^{2\nu}\\ &\quad\times\exp\bigg(2\pi imv(mz+n)-\pi\frac{v}{y}(mz+n)^2\bigg)\\ &=(-\pi)^\nu 2^{2\nu}(v/y)^{\nu+1/2}\sum_n(mz+n)^{2\nu}\exp\bigg(-\pi\frac{v}{y}|mz+n|^2\bigg).\end{aligned}$$

Thus $E_{2\nu}(z,s,0)$ is equal to the Mellin transform

$$\begin{aligned}y^{-2\nu}(-\pi)^\nu\int\limits_0^\infty\sum_{m,n}\chi(m)(mz+n)^{2\nu}\exp\bigg(-\pi\frac{v}{y}|mz+n|^2\bigg)v^{s+\nu+(1-k)/2}\frac{dv}{v}\\ =(-1)^\nu\pi^{-s+(k-1)/2}\Gamma(s+\nu+(1-k)/2)\\ \times\sum_{m,n}\chi(m)(m\overline{z}+n)^{-2\nu}\frac{y^{s-\nu+(1-k)/2}}{|mz+n|^{2s-2\nu+1-k}}.\end{aligned}$$ ■

The group $\Gamma_0(q)$ has two cusps, and thus two Eisenstein series. Unfortunately, the above is the one for the cusp at 0, and the one for the cusp at ∞ would be more convenient. This is a result of not making the optimal definition of the theta function above. To fix this, let $\omega_q = \left[\begin{smallmatrix} 0 & -1 \\ q & 0 \end{smallmatrix}\right]$. Since ω_q normalizes $\Gamma_0(q)$, $\omega_q^{-1}\mathcal{F}$ is another fundamental domain. Thus the integral in (1) can be written

$$\int\limits_{\omega_q^{-1}\mathcal{F}} f(\omega_q z)\overline{g}_{2\nu}(\omega_q z)E_{k-2\nu}(\omega_q z, s, 0)y(\omega_q z)^k\,\frac{dx\,dy}{y^2}$$
$$= q^{s+1/2}\int\limits_{\mathcal{F}} f(qz)\overline{g}_{2\nu}(z)E_{k-2\nu}(z,s,\infty)y^k\,\frac{dx\,dy}{y^2}.$$

Here $E_{2\nu}(z,s,\infty)$ is equal to

$$(-1)^{\nu}\pi^{-s+(k-1)/2}\Gamma(s+\nu+(1-k)/2)$$
$$\times\sum_{\substack{m,n\\ n\equiv 0 \bmod q}}\chi(m)(n\overline{z}+m)^{-2\nu}\frac{y^{s-\nu+(1-k)/2}}{|nz+m|^{2s-2\nu+1-k}},$$

i.e., the Eisenstein series at ∞.

To do the Rankin trick write $E_{k-2\nu}(z,s,\infty)$ as

$$(-1)^{k/2-\nu}2\pi^{-s+(k-1)/2}\Gamma(s+1/2-\nu)L(2s-k+1,\chi)$$

times a sum over $\Gamma_\infty\backslash\Gamma_0(q)$ and unfold the integral. This gives

$$L(s,\widetilde{f}) = (-1)^{k/2}\pi^{-s-1/2}q^{s+1/2}L(2s-k+1,\chi)$$
$$\times\sum_{2\nu\le k}\binom{k}{2\nu}\Gamma(s+1/2-\nu)\int\limits_0^\infty\int\limits_0^1 f(qz)\overline{g}_{2\nu}(z)y^{s+\nu+1/2}\,\frac{dx\,dy}{y^2}$$
$$= (-1)^{k/2}\pi^{-s-1/2}q^{s+1/2}L(2s-k+1,\chi)\sum_{2\nu\le k}\binom{k}{2\nu}\Gamma(s+1/2-\nu)$$
$$\times\sum_{n=1}^{\infty}\frac{a(n)t(nq)}{(nq)^{1/2}}\int\limits_0^\infty \exp(-2\pi nqy)\overline{W}_{\nu,0}(4\pi nqy)y^{s-1/2}\,\frac{dy}{y}.$$

Here $t(n)$ is the cardinality of the set $\{\alpha\in\mathcal{O}/U^+ \mid N\alpha = n\}$, so by the Euler product for the Dedekind zeta function $t(nq)=t(n)$. The integral representation of the Whittaker functions shows that $\overline{W}_{\nu,0} = W_{\nu,0}$ and (7.621 (11)) in [G] gives the Mellin transform as a ratio of Gamma functions $\Gamma(s)^2/\Gamma(s+1/2-\nu)$. One can show $\sum_{2\nu\le k}\binom{k}{2\nu} = 2^{k-1}$. Finally, Doi and Naganuma [1] have shown that $L(2s-k+1,\chi)\sum a(n)t(n)n^{-s}$ is equal to $L(s,f)L(s,f\otimes\chi)$. This completes the proof of the

Theorem.

$$L(s,\widetilde{f}) = (-1)^{k/2}2^k q^{1/2}(2\pi)^{-2s}\Gamma(s)^2L(s,f)L(s,f\otimes\chi).$$

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