The intersection of recurrence sequences

by

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1. Introduction. Let $\{u_m\}_{m \in \mathbb{Z}}$ be a linear recurrence sequence, i.e., a sequence satisfying a relation

$$(1.1) \qquad u_{m+k} = \nu_{k-1} u_{m+k-1} + \ldots + \nu_1 u_{m+1} + \nu_0 u_m \qquad (m \in \mathbb{Z})$$

with $k > 0$ and given coefficients ν_i with $\nu_0 \neq 0$. Equations

$$(1.2) \qquad\qquad\qquad u_m = u_n$$

in unknowns $(m, n) \in \mathbb{Z}^2$ have been studied in [2] and [7]. It was shown there that for nondegenerate $\{u_m\}$, (1.2) has only finitely many solutions $m > n \geq 0$. More generally, the equation

$$(1.3) \qquad\qquad\qquad u_m = v_n,$$

where $\{u_m\}$ and $\{v_n\}$ are given recurrence sequences, was treated by Laurent [3] and the current authors [11]. They give a complete qualitative description of the set of solutions $(m, n) \in \mathbb{Z}^2$ of (1.3).

It is the purpose of the present paper to derive quantitative results on equations (1.3). Let

$$(1.4) \qquad P(z) = z^k - \nu_{k-1} z^{k-1} - \ldots - \nu_0 = \prod_{i=1}^{r} (z - \alpha_i)^{\sigma_i}$$

with distinct roots $\alpha_1, \ldots, \alpha_r$ be the companion polynomial of relation (1.1). It is well known that there exist polynomials $f_1, \ldots, f_r$ which are not all zero and have $\deg f_i \leq \sigma_i - 1$ such that

$$(1.5) \qquad\qquad\qquad u_m = \sum_{i=1}^{r} f_i(m) \alpha_i^m.$$

Notice that in view of (1.4) we have in particular

$$(1.6) \qquad\qquad\qquad \deg f_i < k \qquad (i = 1, \ldots, r).$$

The sequence $\{u_m\}$ is called *nondegenerate* if none of the quotients α_i/α_j for $i \neq j$ is a root of unity.

To unify the notation in the sequel, we will consider instead of (1.5) the function

$$(1.7) \qquad F(x) = \sum_{i=0}^{r} f_i(x)\alpha_i^x$$

of polynomial–exponential type, where α_0 is a root of unity, where α_i/α_j for $i \neq j$ is not a root of unity, and where the f_i are polynomials with

$$(1.8) \qquad \deg f_i < k \quad (0 \leq i \leq r)$$

and with

$$(1.9) \qquad f_i \neq 0 \quad \text{for } i = 1, \ldots, r.$$

If the companion polynomial (1.4) of $\{u_m\}$ does not have a zero which is a root of unity, then we put $f_0(x) \equiv 0$ and $\alpha_0 = 1$.

Similarly, we write $v_m = G(m)$ with

$$(1.10) \qquad G(y) = \sum_{i=0}^{r'} g_i(y)\beta_i^y,$$

where again we suppose that the g_i are polynomials with

$$(1.11) \qquad \deg g_i < k \quad (i = 0, \ldots, r'),$$

$$(1.12) \qquad g_i \neq 0 \quad \text{for } i = 1, \ldots, r'$$

and where we assume, moreover, that $\beta_i \neq 0$, that β_0 is a root of unity, and β_i/β_j for $i \neq j$ is not a root of unity.

We will suppose throughout that both $r, r' \geq 1$. Thus, equation (1.3) becomes

$$(1.13) \qquad F(x) = G(y),$$

to be solved in integers x, y.

In this paper we will study (1.13) assuming that F and G are defined over the algebraic numbers. So, let K be a number field of degree

$$(1.14) \qquad [K : \mathbb{Q}] = d$$

containing $\alpha_0, \ldots, \alpha_r$, $\beta_0, \ldots, \beta_{r'}$. We assume, moreover, that the leading coefficients and the zeros of the polynomials f_i and g_i are contained in K.

We recall the following definitions from [11].

DEFINITION. F and G are called *related* if

(i) $r = r'$,
(ii) either $f_0 \equiv g_0 \equiv 0$ or $f_0 g_0 \not\equiv 0$,

(iii) there is a reordering of $\beta_1, \ldots, \beta_r$ such that

$$(1.15) \qquad \alpha_i^p = \beta_i^q \quad (i = 1, \ldots, r)$$

with certain nonzero integers p, q.

Now, suppose that F and G are related with (1.15) and that r is even. F and G are called *doubly related* if after reordering we have both (1.15) and

$$(1.16) \qquad \alpha_i^{p'} = \beta_{i+1}^{q'}, \quad \alpha_{i+1}^{p'} = \beta_i^{q'} \quad \text{for } 1 \le i < r, \ i \text{ odd},$$

with certain nonzero integers p', q'.

In the sequel, when F and G are related or doubly related, we will assume that the reorderings guaranteeing (1.15), or (1.15) and (1.16), have been applied.

It was shown in [11] that the related pair F, G is doubly related if and only if

$$(1.17) \qquad \alpha_i \alpha_{i+1} \text{ and } \beta_i \beta_{i+1} \text{ for } 0 \le i < r, \ i \text{ odd, are roots of unity.}$$

Moreover, if p, q and p', q' are as in (1.15) and (1.16), then $p/q = -p'/q'$. There cannot be a third permutation with a property like (1.15) or (1.16).

A pair F, G that is related but not doubly related is called *simply related*. Let us mention in this context that relatedness as well as double relatedness, in view of Lemma 6.3 below, are decidable properties.

We denote by $M(K)$ the set of absolute values of K and by $M_\infty(K)$ the subset of archimedean absolute values in $M(K)$. Let S be the subset of $M(K)$ consisting of $M_\infty(K)$ and those absolute values $\| \ \|_v$ in $M(K)$ for which $\|\alpha_i\|_v \ne 1$ or $\|\beta_j\|_v \ne 1$ for some i with $1 \le i \le r$ or some j with $1 \le j \le r'$. It is clear that the cardinality s of S is finite and has

$$(1.18) \qquad s \le d + \omega,$$

where ω denotes the number of distinct prime ideals occurring in the decomposition of the fractional ideals (α_i), (β_j) in K.

THEOREM 1. *Suppose that F and G are not related. Then equation* (1.13) *has not more than*

$$(1.19) \qquad 2^{s^7 2^{45d!(k^2(r+r'+2))!}}$$

solutions $(x, y) \in \mathbb{Z}^2$.

Now, suppose that F and G are related. In [11] (Proposition 1) we proved the following:

When F and G are simply related, then all but finitely many solutions of (1.13) *satisfy the system of equations*

$$(1.20) \qquad f_i(x)\alpha_i^x = g_i(y)\beta_i^y \quad (i = 0, \ldots, r).$$

When F and G are doubly related, then all but finitely many solutions of (1.13) *satisfy either* (1.20) *or the system*

(1.21a) $f_i(x)\alpha_i^x = g_{i+1}(y)\beta_{i+1}^y, \quad f_{i+1}(x)\alpha_{i+1}^x = g_i(y)\beta_i^y \quad (1 \le i < r, \ i \ odd),$

(1.21b) $$f_0(x)\alpha_0^x = g_0(y)\beta_0^y.$$

If F and G are simply related, we write $\mathfrak{S}$ for the set of solutions $(x, y) \in \mathbb{Z}^2$ of (1.13) that do not satisfy the system (1.20).

If F and G are doubly related, we let $\mathfrak{S}$ be the set of solutions of (1.13) which satisfy neither (1.20) nor (1.21a,b).

THEOREM 2. *Suppose that F and G are related. Then $\mathfrak{S}$ has cardinality*

(1.22) $$|\mathfrak{S}| \le 2^{s^7} 2^{45d!(2k^2(r+1))!}.$$

R e m a r k. If in (1.7) and (1.10), $f_0 = g_0 = 0$, then in (1.22), $r + 1$ may be replaced by r. The significant feature in Theorems 1 and 2 is that the bounds (1.19) and (1.22) are uniform, as they involve only the degree d of the field K, the bound k for the degrees of the polynomials f_i and g_j, the numbers r and r' of characteristic roots and the number s of absolute values in S. No particular care was taken in optimizing the actual shape of the bounds; we rather tried to avoid painstaking estimates.

For F and G related, we still want some information about the solutions $(x, y) \in \mathbb{Z}^2$ of either (1.20) or of (1.20) and (1.21a,b).

Again following [11], we call the ordered pair F, G *exceptional* if

(i) F and G are simply related,

(ii) there is a natural number $N > 1$ which is an integral power of each α_i and each β_i with $1 \le i \le r$,

(iii) either $|\alpha_i| > 1$ for $1 \le i \le r$, or $|\alpha_i| < 1$ for $1 \le i \le r$,

(iv) f_0 and g_0 are constant,

(v) each g_i is constant and for $1 \le i \le r$, $f_i(x) = a_i(x - A)^{l_i}$ where A is rational and $l_i > 0$.

We remark that by Lemma 6.3 below, exceptionality is decidable. Notice that if F, G is exceptional, then by (1.15) either $|\beta_i| > 1$ for $1 \le i \le r$ or $|\beta_i| < 1$ for $1 \le i \le r$. Thus, in view of (1.17), if the pair F, G is doubly related neither F, G nor G, F is exceptional.

If F and G are related, then as α_0 and β_0 are roots of unity, we may extend (1.15) also to $i = 0$ (by replacing p, q by suitable multiples, if necessary). So we will suppose in the sequel that we have

(1.23) $$\alpha_i^p = \beta_i^q \quad \text{for } i = 0, \dots, r$$

with nonzero integers p, q. And in fact, we will assume that $p > 0$ in (1.23), and p is chosen minimal with this property.

Similarly, when F and G are doubly related we will assume instead of (1.16) that

$$(1.24) \quad \alpha_i^{p'} = \beta_{i+1}^{q'}, \quad \alpha_{i+1}^{p'} = \beta_i^{q'} \quad \text{for } i = 1, \ldots, r, \; i \text{ odd},$$

$$\text{and} \quad \alpha_0^{p'} = \beta_0^{q'},$$

with nonzero integers p', q', where again $p' > 0$, p' minimal.

The systems (1.20) or (1.21a,b) are of exactly the same type. Thus it suffices to deal with (1.20).

Suppose that the polynomials f_i in (1.7) have leading coefficients a_i and zeros $A_1^{(i)}, \ldots, A_{l_i}^{(i)}$ $(i = 1, \ldots, r)$, and similarly define b_j and $B_1^{(j)}, \ldots, B_{m_j}^{(j)}$ with respect to the polynomials g_j in (1.10). Write

$$(1.25) \qquad H(F) = \max\{h(a_i), h(\alpha_i), h(A_j^{(i)}) \text{ over } 0 \le i \le r, \; 1 \le j \le l_i\},$$

where h is the absolute height whose definition will be detailed in Section 5. Similarly define $H(G)$ with respect to G.

Let H be a quantity having

$$H \ge \max\{3, H(F), H(G)\}.$$

We write $c(d, k, H)$ for an unspecified but *effective* constant that depends only upon d, k and H. So $c(d, k, H)$ at different instances may have different values.

In Sections 8–11 we will detail some arguments that may give the reader an idea about explicit versions of these constants.

THEOREM 3. *Suppose that F and G are related, but neither F, G nor G, F is exceptional. Then it is decidable whether* (1.20) *admits infinitely many solutions* $(x, y) \in \mathbb{Z}^2$ *or not.*

(a) *If* (1.20) *has only finitely many solutions, then all but at most*

$$(1.26) \qquad\qquad M_1 := 2^{s^7 2^{540 d!(2k)!}}$$

solutions satisfy

$$(1.27) \qquad\qquad \max(|x|, |y|) \le c(d, k, H),$$

where $c(d, k, H)$ is effectively computable.

(b) *If* (1.20) *has infinitely many solutions, there is a unique linear one-parameter family of solutions*

$$(1.28) \qquad x(t) = pt + p', \quad y(t) = qt + q' \quad (t \in \mathbb{Z})$$

with $0 \le p' < p$ such that all but at most M_1 solutions lie in this family or satisfy (1.27).

The family (1.28) *is effectively computable.*

The point of the theorem is that the solutions with (1.27) as well as the family (1.28) may be effectively computed, while on the other hand M_1 is independent of H.

R e m a r k. The family (1.28) occurs if and only if we have the polynomial identities

$$(1.29) \qquad f_i(pX + p')\alpha_i^{p'} \equiv g_i(qX + q')\beta_i^{q'} \qquad (0 \le i \le r).$$

THEOREM 4. *Suppose that the pair* F, G *is exceptional. Then it is decidable whether* (1.20) *admits infinitely many solutions* $(x, y) \in \mathbb{Z}^2$ *or not.*

(a) *If* (1.20) *has only finitely many solutions, then all but at most*

$$(1.30) \qquad M_2 := (r - 1)2^{2^{1080}s^7}$$

solutions satisfy

$$(1.31) \qquad \max(|x|, |y|) \le c(d, k, H).$$

(b) *If* (1.20) *has infinitely many solutions, then there is a finite number of one-parameter exponential families of solutions of the type*

$$(1.32) \qquad x_j(t) = pE_j R^t + A, \quad y_j(t) = qE_j R^t + Ft + G_j \quad (t \in \mathbb{Z}, \ t \ge 0)$$

with $1 \le j \le c(d, k, r, H)$ *such that all but at most* M_2 *solutions lie in one of these families or satisfy* (1.31). *The parameters* p, q, E_j, R, A, F, G_j *determining the families are effectively computable. Moreover, in* (1.32), R *is an integral power of each* $\alpha_1, \ldots, \alpha_r$ *and* F *is a nonzero integer that does not depend upon the particular family.*

Theorems 1–4 imply in particular that for equations $u_m = v_n$ we have an algorithm that allows us to determine effectively all solutions, except possibly a finite set whose cardinality is uniformly bounded.

A particular instance of our results concerns equations $u_m = u_n$ or more generally

$$(1.33) \qquad au_m = bu_n$$

with $a, b \in K^*$, where $\{u_m\}_{m \in \mathbb{Z}}$ is a nondegenerate linear recurrence sequence as in (1.1). We suppose that relation (1.1) is minimal, i.e., that u_m does not satisfy a relation of smaller order. Moreover, we assume that $k > 1$ and that at least one of the characteristic roots of the polynomial in (1.4) is not a root of unity. Using the representation (1.5), we may write $u_m = F(m)$ with F as in (1.7), and here by (1.4), if $f_0 \equiv 0$ then $r \le k$ and otherwise $r + 1 \le k$. We suppose that $\{u_m\}$, i.e., $F(m)$, is defined over K. It is clear that F is related with itself. We call $\{u_m\}$ *symmetric* if F is doubly related with itself.

The relations (1.15) or (1.23) now simply reduce to $\alpha_i^1 = \alpha_i^1$ $(i = 0, \ldots, r)$, and if F is symmetric, then since in (1.15), (1.16) we have $p/q =$

$-p'/q'$, the relations (1.16) or (1.24) become

$$(1.34) \qquad \alpha_i^w = \alpha_{i+1}^{-w} \quad \text{for } 1 \leq i < r, \ i \text{ odd},$$

where $w > 0$. In fact, it follows from Lemma 6.1 below that here the minimal w has

$$(1.35) \qquad 0 < w \leq 2d^2.$$

Let $H_1 = \max\{3, H(F)\}$.

COROLLARY 1. (i) *Suppose that $\{u_m\}$ is not symmetric. Then all but at most*

$$(1.36) \qquad M_3 := 2^{s^7 2^{46d!(2k^3)!}}$$

solutions $(m, n) \in \mathbb{Z}^2$ of the equation

$$(1.37) \qquad u_m = u_n$$

with $m \neq n$ satisfy

$$(1.38) \qquad \max\{|m|, |n|\} \leq c(d, k, H_1),$$

where $c(d, k, H_1)$ is effective.

(ii) *If $\{u_m\}$ is symmetric, then in addition to solutions as in (i), there may be a one-parameter family of solutions*

$$(1.39) \qquad m(t) = wt + w', \quad n(t) = -wt + w'' \quad (t \in \mathbb{Z})$$

where w is an integer with $0 < w \leq 2d^2$, and w' has $0 \leq w' < w$. The family (1.39) is effectively computable.

We next treat the slightly more general equation

$$(1.40) \qquad au_m = bu_n.$$

Let $H_2 = \max\{3, H(F), h(b/a)\}$.

COROLLARY 2. (i) *Suppose that $b/a \neq 1$ is a root of unity. Then, apart from at most*

$$(1.41) \qquad M_4 := 2^{s^7 2^{46d!(2k^3)!}}$$

solutions, and apart from a possible one-parameter family as in (1.39), the solutions $(m, n) \in \mathbb{Z}^2$ of (1.40) satisfy

$$(1.42) \qquad \max\{|m|, |n|\} \leq c(d, k, H_2).$$

Again the family is effectively computable and the constant in (1.42) is effective. The family (1.39) may only occur if $\{u_m\}$ is symmetric.

(ii) *Suppose that a/b is not a root of unity. Then all but at most M_4 solutions of (1.40) have (1.42).*

A qualitative version of Corollaries 1 and 2 was proved by Laurent [4] (Théorème 2).

2. Polynomial–exponential equations. We consider equations

$$(2.1) \qquad \sum_{i=1}^{h} P_i(\mathbf{x})\boldsymbol{\alpha}_i^{\mathbf{x}} = 0$$

in variables $\mathbf{x} = (x_1, \ldots, x_N) \in \mathbb{Z}^N$, where the P_i are nonzero polynomials with coefficients in K of total degree $\leq \delta$, and where $\boldsymbol{\alpha}_i^{\mathbf{x}} = \alpha_{i1}^{x_1} \ldots \alpha_{iN}^{x_N}$. Here we suppose that the α_{ij} are elements in K^*. The letter S will indicate a finite set $S \subset M(K)$ which contains $M_\infty(K)$ as well as the nonarchimedean absolute values v of K for which $|\alpha_{ij}|_v \neq 1$ for some pair i, j ($1 \leq i \leq h$, $1 \leq j \leq N$). We set $s = \operatorname{card} S$.

When $\mathcal{P}$ is a partition of $\{1, \ldots, h\}$ and π is a subset of $\{1, \ldots, h\}$, we write $\pi \in \mathcal{P}$ if π is among the subsets belonging to $\mathcal{P}$. Consider the splitting of equation (2.1) into the system

$$(2.1\mathcal{P}) \qquad \sum_{i \in \pi} P_i(\mathbf{x})\boldsymbol{\alpha}_i^{\mathbf{x}} = 0 \qquad (\pi \in \mathcal{P}).$$

We denote by $\mathfrak{S}(\mathcal{P})$ the set of solutions $\mathbf{x} \in \mathbb{Z}^N$ of $(2.1\mathcal{P})$ which do not satisfy a system $(2.1\mathcal{Q})$ for a proper refinement $\mathcal{Q}$ of $\mathcal{P}$. It is clear that any solution $\mathbf{x} \in \mathbb{Z}^N$ of (2.1) is contained in some set $\mathfrak{S}(\mathcal{P})$ for a suitable $\mathcal{P}$ (which is not necessarily unique). Thus, to give an upper bound for the number of solutions of (2.1), it suffices to give an upper bound for the cardinalities of the sets $\mathfrak{S}(\mathcal{P})$ where $\mathcal{P}$ runs through the partitions of $\{1, \ldots, h\}$.

Given $\mathcal{P}$ and elements $i, j \in \{1, \ldots, h\}$, we write $i \overset{\mathcal{P}}{\sim} j$ if i, j belong to the same subset $\pi \in \mathcal{P}$. We denote by $G(\mathcal{P})$ the subgroup of $\mathbb{Z}^N$ consisting of points $\mathbf{x}$ having

$$(2.2) \qquad \boldsymbol{\alpha}_i^{\mathbf{x}} = \boldsymbol{\alpha}_j^{\mathbf{x}} \qquad \text{for every } i, j \text{ with } i \overset{\mathcal{P}}{\sim} j.$$

The following theorem is the main result of Schlickewei and Schmidt [10].

THEOREM A. *Suppose that $G(\mathcal{P}) = \{\mathbf{0}\}$. Then*

$$(2.3) \qquad |\mathfrak{S}(\mathcal{P})| < 2^{20N^4 + Ns^7 2^{43d!(Dh)!}},$$

where $D = \binom{N+\delta}{N}$.

We mention that Theorem A is a consequence of the Subspace Theorem in diophantine approximation.

3. Groups $G(\mathcal{P})$. To prove our theorems we will apply Theorem A of Section 2.

It is clear that equation (1.13) is a special instance of (2.1), where in view of (1.7) and (1.10) we have $h = r + r' + 2$, $N = 2$ and $\delta < k$. We will symbolize the $r + r' + 2$ summands of equation (1.13) as

$$0_x, 1_x, \ldots, r_x, \; 0_y, 1_y, \ldots, r'_y.$$

Given a partition $\mathcal{P}$ of $\{0_x, \ldots, r_x,\ 0_y, \ldots, r'_y\}$, the relations (2.2) are

$$\tag{3.1} \alpha_i^x = \alpha_j^x \qquad \text{for all } i, j \text{ with } i_x \overset{\mathcal{P}}{\sim} j_x,$$

$$\tag{3.2} \beta_i^y = \beta_j^y \qquad \text{for all } i, j \text{ with } i_y \overset{\mathcal{P}}{\sim} j_y,$$

$$\tag{3.3} \alpha_i^x = \beta_j^y \qquad \text{for all } i, j \text{ with } i_x \overset{\mathcal{P}}{\sim} j_y.$$

We say that $\mathcal{P}$ *contains an essential singleton* if $\mathcal{P}$ contains a one-element subset $\{i_x\}$ or $\{i_y\}$ and if, moreover, in the case $\{i_x\} = \{0_x\}$ or $\{i_y\} = \{0_y\}$ we have $f_0(x) \not\equiv 0$ or $g_0(x) \not\equiv 0$, respectively.

LEMMA 3.1. *Suppose that $\mathcal{P}$ contains an essential singleton. Then*

$$\tag{3.4} |\mathfrak{S}(\mathcal{P})| \leq 2^{s^7 2^{44d!(k(r+r'+2))!}}.$$

Proof. Suppose without loss of generality that $\{i_y\} \in \mathcal{P}$. Then $(1.13\mathcal{P})$ contains the equation $g_i(y)\beta_i^y = 0$. As g_i is a nonzero polynomial of degree $\leq k$, this equation has not more than k solutions, say $y_1, \ldots, y_k \in \mathbb{Z}$. Given y, we write $G(y) = c$ and equation (1.13) becomes

$$\tag{3.5} f_0(x)\alpha_0^x + f_1(x)\alpha_1^x + \ldots + f_r(x)\alpha_r^x - c \cdot 1^x = 0.$$

Consider partitions $\mathcal{Q}$ of the set $\{0, 1, \ldots, r, r+1\}$, where $r+1$ symbolizes the summand $-c \cdot 1^x$ in (3.5).

Either, $\mathcal{Q}$ contains a singleton i with $1 \leq i \leq r$. Then $f_i(x)\alpha_i^x = 0$, and there are not more than k solutions $x \in \mathbb{Z}$. Consequently, in this case

$$\tag{3.6} |\mathfrak{S}(\mathcal{P})| \leq k^2.$$

Or, for each i $(1 \leq i \leq r)$ there exists j with $0 \leq j \leq r+1$ and $j \neq i$ such that $i \sim j$. But then $G(\mathcal{Q})$ is contained in the set of $x \in \mathbb{Z}$ having

$$\tag{3.7} \alpha_i^x = \alpha_j^x.$$

As we assume that $F(x)$ is nondegenerate, (3.7) has only the trivial solution $x = 0$. Hence $G(\mathcal{Q}) = \{0\}$. Thus we may apply Theorem A to (3.5), and consequently, $(3.5\mathcal{Q})$ has not more than

$$\tag{3.8} 2^{20+s^7 2^{43d!(k(r+2))!}}$$

solutions.

Allowing a factor k for the number of possible values y and a factor $2^{(r+2)^2}$ for the number of possible partitions $\mathcal{Q}$ we may conclude from (3.8) that

$$\tag{3.9} |\mathfrak{S}(\mathcal{P})| \leq k \cdot 2^{(r+2)^2} \cdot 2^{20+s^7 2^{43d!(k(r+2))!}} \leq 2^{s^7 2^{44d!(k(r+r'+2))!}}.$$

It is clear that we obtain the same estimate if $\mathcal{P}$ contains an essential singleton $\{i_x\}$, and in view of (3.6) the lemma follows.

LEMMA 3.2. *Let $\mathcal{P}$ be a partition of $\{0_x, \ldots, r_x, \ 0_y, \ldots, r'_y\}$ with $G(\mathcal{P}) = \{\mathbf{0}\}$. Then*

$$(3.10) \qquad |\mathfrak{S}(\mathcal{P})| \leq 2^{s^7 2^{44d!(k^2(r+r'+2))!}}.$$

Proof. We apply Theorem A with $h = r + r' + 2$, $N = 2$ and $\delta < k$ to obtain, from (2.3),

$$|\mathfrak{S}(\mathcal{P})| < 2^{320 + 2s^7 2^{43d!(k^2(r+r'+2))!}} < 2^{s^7 2^{44d!(k^2(r+r'+2))!}}.$$

LEMMA 3.3. *Suppose that F and G are not related. Let $\mathcal{P}$ be a partition of $\{0_x, 1_x, \ldots, r_x, \ 0_y, 1_y, \ldots, r'_y\}$. Then either $\mathcal{P}$ contains an essential singleton or $G(\mathcal{P}) = \{(0,0)\}$.*

Proof. See Laurent [3], Lemme 2, and [11], §5.

LEMMA 3.4. *Suppose that F and G are related and $f_0(x)g_0(x) \not\equiv 0$. Let $\mathcal{P}$ be a partition of $\{0_x, \ldots, r_x, \ 0_y, \ldots, r_y\}$ that does not contain an essential singleton and such that $G(\mathcal{P}) \neq \{(0,0)\}$.*

(i) *If F and G are simply related, then*

$$(3.11) \qquad \mathcal{P} = \{0_x, 0_y\}, \{1_x, 1_y\}, \ldots, \{r_x, r_y\}.$$

(ii) *If F and G are doubly related, then either $\mathcal{P}$ is as in (3.11) or*

$$(3.12) \quad \mathcal{P} = \{0_x, 0_y\}, \{1_x, 2_y\}, \{2_x, 1_y\}, \ldots, \{(r-1)_x, r_y\}, \{r_x, (r-1)_y\}.$$

Proof. See [3], Lemme 2, and [11], §5.

4. Proof of Theorems 1 and 2. As for Theorem 1, in view of Lemma 3.3 we have only partitions $\mathcal{P}$ of $\{0_x, \ldots, r_x, 0_y, \ldots, r'_y\}$ that have $G(\mathcal{P}) = \{(0,0)\}$ or contain an essential singleton. In Theorem 2 the situation is the same by the definition of $\mathfrak{S}$. Therefore, we may apply Lemmata 3.1 and 3.2. Comparing (3.4) and (3.10) we see that for any partition $\mathcal{P}$ under consideration we have

$$|\mathfrak{S}(\mathcal{P})| \leq 2^{s^7 2^{44d!(k^2(r+r'+2))!}}.$$

As the number of partitions of $\{0_x, \ldots, r_x, 0_y, \ldots, r'_y\}$ to be considered does not exceed $2^{2(r+r'+2)}$, we get

$$(4.1) \qquad |\mathfrak{S}| \leq 2^{s^7 2^{45d!(k^2(r+r'+2))!}},$$

and Theorem 1 follows. If F and G are related, then $r = r'$ and hence (4.1) implies Theorem 2 as well.

5. Heights. Recall that K is a number field of degree d. We denote by $M(K)$ an indexing set for the absolute values of K. Thus given $v \in$

$M(K), |\ |_v$ is an extension of either the standard absolute value on $\mathbb{Q}$ or of a p-adic absolute value. Moreover, for $\alpha \in K$ and $v \in M(K)$ we define

$$\|\alpha\|_v = |\alpha|_v^{d_v/d},$$

where d_v denotes the local degree. Given an element $\alpha \in K^*$, we put

$$(5.1) \qquad h(\alpha) = \prod_{v \in M(K)} \max\{1, \|\alpha\|_v\}.$$

If $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ is a vector in K^n, we define $\|\boldsymbol{\alpha}\|_v = \max\{\|\alpha_1\|_v, \ldots \ldots, \|\alpha_n\|_v\}$ and put

$$(5.2) \qquad H(\boldsymbol{\alpha}) = \prod_{v \in M(K)} \|\boldsymbol{\alpha}\|_v.$$

Thus, for $\alpha \in K^*$ we have $h(\alpha) = H(1, \alpha)$. Given a subset S of $M(K)$, we put

$$(5.3) \qquad H_S(\boldsymbol{\alpha}) = \prod_{v \in S} \|\boldsymbol{\alpha}\|_v.$$

LEMMA 5.1. *Given* $\alpha_1, \ldots, \alpha_l \in K^*$ *we have*

$$(5.4) \qquad h(\alpha_1 \ldots \alpha_l) \le h(\alpha_1) \ldots h(\alpha_l),$$

$$(5.5) \qquad h(\alpha_1 + \ldots + \alpha_l) \le l h(\alpha_1) \ldots h(\alpha_l).$$

Moreover, for $\alpha \in K^*$ *let* C *be a denominator of* α, *i.e., a natural number such that* $C\alpha$ *is an algebraic integer. Then* C *may be chosen such that*

$$(5.6) \qquad h(C) \le h(\alpha)^d.$$

This is well known. A proof may be found, e.g., in Schmidt [12].

LEMMA 5.2 (Dobrowolski [1]). *Suppose that* $A \in K^*$ *is not a root of unity. Then*

$$(5.7) \qquad h(A) > 1 + \frac{1}{20d^3}.$$

LEMMA 5.3. *Let* x' *and* y' *be integers. Let* A' *and* B' *be elements in* K^* *such that*

$$(5.8) \qquad x' - y' - A' + B' \neq 0.$$

Suppose that H' *is a quantity with*

$$(5.9) \qquad H' \ge \max\{h(A'), h(B')\}.$$

Suppose moreover that

$$(5.10) \qquad \max\{|x'|, |y'|\} \ge 16 H'^{d+3}.$$

Then

$$(5.11) \qquad H((x' - A', y' - B', x' - y' - A' + B')) \ge \frac{\max\{|x'|, |y'|\}^{1/2}}{\max\{1, |x' - y'|\}}.$$

Proof. We may assume that $|x'| \geq |y'|$ and get by (5.2), (5.8),

$$H((x' - A', y' - B', x' - y' - A' + B'))$$

$$= \prod_{v \in M(K)} \max\{\|x' - A'\|_v, \|y' - B'\|_v, \|x' - y' - A' + B'\|_v\}$$

$$\geq \prod_{v|\infty} \|x' - A'\|_v \prod_{v \nmid \infty} \|x' - y' - A' + B'\|_v$$

$$= \prod_{v|\infty} \|x' - A'\|_v \|x' - y' - A' + B'\|_v^{-1}.$$

On the other hand, by (5.1) and (5.5),

$$\prod_{v|\infty} \|x' - y' - A' + B'\|_v \leq \prod_{v \in M(K)} \max\{1, \|x' - y' - A' + B'\|_v\}$$

$$= h(x' - y' - A' + B')$$

$$\leq 3 \max\{1, |x' - y'|\} h(A') h(B').$$

We denote by $\sigma_1, \ldots, \sigma_d$ the isomorphic embeddings of K in $\mathbb{C}$. Then, in view of (5.10),

$$\prod_{v|\infty} \|x' - A'\|_v = \Big(\prod_{i=1}^{d} |x' - \sigma_i A'|\Big)^{1/d} \geq \prod_{i=1}^{d} (|x'| - |\sigma_i A'|)^{1/d}$$

$$\geq \prod_{i=1}^{d} (|x'| - h(A')^d)^{1/d} = |x'| - h(A')^d \geq \frac{3}{4}|x'|.$$

Altogether, using (5.9) and (5.10) we get

$$H((x' - A', y' - B', x' - y' - A' + B'))$$

$$\geq \tfrac{3}{4}|x'|(3 \max\{1, |x' - y'|\} h(A') h(B'))^{-1}$$

$$\geq \frac{|x'|^{1/2}}{\max\{1, |x' - y'|\}} = \frac{\max\{|x'|, |y'|\}^{1/2}}{\max\{1, |x' - y'|\}}.$$

Let

(5.12) $$f^*(x) = a^* \prod_{i=1}^{l} (x - A_i^*), \quad a^* \neq 0,$$

and

(5.13) $$g^*(x) = b^* \prod_{j=1}^{m} (x - B_j^*), \quad b^* \neq 0,$$

be polynomials with $a^*, A_1^*, \ldots, A_l^*, b^*, B_1^*, \ldots, B_m^* \in K$. Suppose that

(5.14) $$\max\{l, m\} < k.$$

Let $\alpha, \beta \in K^*$ be elements with

(5.15) $\|\beta\|_v = \|\alpha\|_v = 1$ for each $v \notin S$,

where S is as in Section 1. Thus α, β are "S-units".

LEMMA 5.4. *Let* $f(x), g(x), \alpha, \beta$ *be as in* (5.12)–(5.15). *Suppose that*

(5.16) $l > 0, \quad m > 0.$

Let H^* *be a quantity with*

(5.17) $H^* \geq \max\{h(a^*), h(A_1^*), \ldots, h(A_l^*), h(b^*), h(B_1^*), \ldots, h(B_m^*)\}.$

Let x, y, v, w *be integers such that*

(5.18) $f^*(x)\alpha^v = g^*(y)\beta^w,$

(5.19) $f^*(x)g^*(y) \neq 0,$

(5.20) $x - A_i^* \neq y - B_j^*$ *for each pair* i, j $(1 \leq i \leq l, \ 1 \leq j \leq m).$

Then there exists a pair i_0, j_0 *with* $1 \leq i_0 \leq l, \ 1 \leq j_0 \leq m$ *such that*

(5.21) $\displaystyle\prod_{v \in S} (\|x - A_{i_0}^*\|_v \|y - B_{j_0}^*\|_v \|x - y - A_{i_0}^* + B_{j_0}^*\|_v)$

$$\leq 3^{4k^2} H^{*8k^2} \max\{1, |x - y|\}^{4k^2 - 1}.$$

Proof. (5.15), (5.18) and (5.19) imply

(5.22) $\displaystyle\prod_{v \in S} \|f^*(x)\|_v = \prod_{v \in S} \|g^*(y)\|_v = \prod_{v \notin S} \|f^*(x)\|_v^{-1} = \prod_{v \notin S} \|g^*(y)\|_v^{-1},$

(5.23) $\|f^*(x)\|_v = \|g^*(y)\|_v$ for each $v \notin S$.

Given $v \in M(K)$, we denote by λ_v and μ_v respectively subscripts having

$$\|x - A_{\lambda_v}^*\|_v = \min_{1 \leq \lambda \leq l} \|x - A_\lambda^*\|_v, \quad \|y - B_{\mu_v}^*\|_v = \min_{1 \leq \mu \leq m} \|y - B_\mu^*\|_v.$$

We partition the set $M(K) \setminus S$ into two subsets S_1, S_2; here S_1 consists of those v for which $\|y - B_{\mu_v}^*\|_v \geq \|x - A_{\lambda_v}^*\|_v$ and S_2 of those v for which $\|x - A_{\lambda_v}^*\|_v > \|y - B_{\mu_v}^*\|_v$. Then by (5.23),

(5.24) $\|f^*(x)\|_v \geq \|b^*\|_v \|y - B_{\mu_v}^*\|_v^m$ for each $v \in S_1$,

(5.25) $\|f^*(x)\|_v \geq \|a^*\|_v \|x - A_{\lambda_v}^*\|_v^l$ for each $v \in S_2$.

On the other hand, for each $v \notin S$ we have

$$\|x - y - A_{\lambda_v}^* + B_{\mu_v}^*\|_v \leq \max\{\|x - A_{\lambda_v}^*\|_v, \|y - B_{\mu_v}^*\|_v\}.$$

Therefore (5.24) and (5.25) respectively yield

(5.26) $\|f^*(x)\|_v \geq \|b^*\|_v \|x - y - A_{\lambda_v}^* + B_{\mu_v}^*\|_v^m$ for each $v \in S_1$,

(5.27) $\|f^*(x)\|_v \geq \|a^*\|_v \|x - y - A_{\lambda_v}^* + B_{\mu_v}^*\|_v^l$ for each $v \in S_2$.

Let $S_{1,\lambda,\mu}$ be the subset of S_1 for which (5.26) holds with $(\lambda_v, \mu_v) = (\lambda, \mu)$. Then

$$\prod_{v \in S_{1,\lambda,\mu}} \|f^*(x)\|_v^{-1} \leq \prod_{v \in S_{1,\lambda,\mu}} \|b^*\|_v^{-1} \|x - y - A_\lambda^* + B_\mu^*\|_v^{-m}$$

$$= \prod_{v \notin S_{1,\lambda,\mu}} \|b^*\|_v \|x - y - A_\lambda^* + B_\mu^*\|_v^{m}$$

$$\leq \prod_{v \notin S_{1,\lambda,\mu}} \max\{1, \|b^*\|_v\} \max\{1, \|x - y - A_\lambda^* + B_\mu^*\|\}^{m}$$

$$\leq h(b^*) h(x - y - A_\lambda^* + B_\mu^*)^{m}.$$

Taking the product over $1 \leq \lambda \leq l$ and $1 \leq \mu \leq m$ and using the same argument for (5.27), we get

$$\prod_{v \notin S} \|f^*(x)\|_v^{-1} \leq h(a^*)^{l+m} h(b^*)^{l+m} \prod_{\lambda=1}^{l} \prod_{\mu=1}^{m} h(x - y - A_\lambda^* + B_\mu^*)^{l+m},$$

and thus by (5.17) and Lemma 5.1,

$$\prod_{v \notin S} \|f^*(x)\|_v^{-1} \leq 3^{lm(l+m)} h(x - y)^{lm(l+m)} H^{*2(lm+1)(l+m)}.$$

Thus by (5.12) and (5.22) there exists i_0 with $1 \leq i_0 \leq l$ such that

$$\prod_{v \in S} \|x - A_{i_0}^*\|_v \leq \left(\prod_{v \notin S} \|f^*(x)\|_v^{-1} \right)^{1/l} \left(\prod_{v \in S} \|a^*\|_v \right)^{1/l}$$

$$\leq 3^{m(l+m)} H^{*2m(l+m)+l^{-1}(2l+2m+1)} \max\{1, |x - y|\}^{m(l+m)}.$$

Similarly, we find j_0 with $1 \leq j_0 \leq m$ such that

$$\prod_{v \in S} \|y - B_{j_0}^*\|_v \leq 3^{l(l+m)} H^{*2l(l+m)+m^{-1}(2l+2m+1)} \max\{1, |x - y|\}^{l(l+m)}.$$

Finally, by (5.14) we get

$$\prod_{v \in S} \|x - A_{i_0}^*\|_v \|y - B_{j_0}^*\|_v \|x - y - A_{i_0}^* + B_{j_0}^*\|_v$$

$$\leq h(x - y - A_{i_0}^* + B_{j_0}^*) \prod_{v \in S} (\|x - A_{i_0}^*\|_v \|y - B_{j_0}^*\|_v)$$

$$\leq 3^{(m+l)^2+1} H^{2(m+l)^2+(m+l)(2l+2m+1)/(ml)+2} \max\{1, |x - y|\}^{(l+m)^2+1}$$

$$< 3^{4k^2} H^{*8k^2} \max\{1, |x - y|\}^{4k^2-1},$$

as asserted.

6. Multiplicative relations

LEMMA 6.1. *Let K be a number field of degree d. Denote by w the number of roots of unity in K. Then*

$$(6.1) \qquad\qquad w \leq 2d^2.$$

P r o o f. The group of roots of unity in K is cyclic of order w. It is generated by a primitive wth root of unity ζ, which is a root of the wth cyclotomic polynomial. Thus $\varphi(w) = \deg \zeta \leq d$, where φ is Euler's function. On the other hand, $\varphi(w) \geq (w/2)^{1/2}$, and (6.1) follows.

LEMMA 6.2. *Suppose again that K has degree d. Suppose that $\alpha \in K^*$ is such that there exists a natural number u with $\alpha^u \in \mathbb{Q}$. Then in fact there exists such a u having*

$$(6.2) \qquad\qquad u \leq 2d^3.$$

P r o o f. We have $\alpha^u = q$ for some $q \in \mathbb{Q}^*$ and therefore $\alpha = \zeta|q|^{1/u}$, where $|q|^{1/u}$ is the positive uth root of $|q|$ and ζ is a $2u$th root of unity. So α is a root of the polynomial

$$x^{2u} - q^2 = \prod_\zeta (x - \zeta|q|^{1/u})$$

where ζ runs through the $2u$th roots of unity. On the other hand, α is in K. Therefore the minimal polynomial of α over $\mathbb{Q}$, say $p(x)$, is of degree say $e \leq d$ and $p(x) \mid (x^{2u} - q^2)$. Hence $p(x) = \prod_{i=1}^e (x - \zeta_i|q|^{1/u})$, where $\zeta_1, \ldots, \zeta_e$ are certain $2u$th roots of unity. We may infer that $|q|^{e/u} \prod_{i=1}^e \zeta_i \in \mathbb{Q}$. This implies that $\prod_{i=1}^e \zeta_i$ is real, therefore ± 1, and therefore $|q|^{e/u} \in \mathbb{Q}$. On the other hand, $\alpha = \zeta_i|q|^{1/u}$ for some i. It follows that $\zeta_i^e \in K$, and hence $\zeta_i^{ew} = 1$. But this implies that

$$\alpha^{ew} = \zeta_i^{ew}|q|^{ew/u} = |q|^{ew/u} \in \mathbb{Q}.$$

Consequently, $\alpha^{ew} \in \mathbb{Q}$. In view of Lemma 6.1 and the hypotheses we have

$$we \leq 2d^3,$$

and the assertion follows.

LEMMA 6.3. *Let α and β be elements in K^*, and neither a root of unity. Let H be a quantity with*

$$(6.3) \qquad\qquad H \geq \max\{h(\alpha), h(\beta)\}.$$

Suppose that there exist nonzero integers z_1, z_2 with

$$(6.4) \qquad\qquad \alpha^{z_1}\beta^{z_2} = 1.$$

Then there exist such integers having

$$(6.5) \qquad\qquad \max\{|z_1|, |z_2|\} \leq 60d^6 \log H.$$

P r o o f. This follows at once from Loxton and van der Poorten [5] (Theorem 3). In fact, they proved that there exist nonzero integers z_i with (6.4) such that

$$\max\{|z_1|, |z_2|\} \leq w \cdot d \cdot \log H \cdot \lambda^{-1},$$

where w is the number of roots of unity in K and where λ is the logarithm of the right hand side in formula (5.7) of Dobrowolski's result as quoted in Lemma 5.2. Thus Lemma 6.1 implies the assertion.

7. Linear equations in S-integers. Let S be a finite subset of cardinality $|S| = s$ of $M(K)$ as in Section 1. We suppose that S contains the set of archimedean prime divisors $M_\infty(K)$ of K. An element $x \in K$ is called an *S-integer* if $\|x\|_v \leq 1$ for each $v \notin S$. We consider the equation

$$(7.1) \qquad\qquad x_1 + \ldots + x_n = 0$$

to be solved in S-integers $x_i \in K$.

Given a vector $\mathbf{x} = (x_1, \ldots, x_n) \in K^n$ we define the *S-height* by

$$H_S(\mathbf{x}) = \prod_{v \in S} \|\mathbf{x}\|_v$$

with $\|\mathbf{x}\|_v$ as in Section 5.

We will need the following result of Schlickewei [9] (Theorem 1.4).

LEMMA 7.1. *Suppose that $n \geq 3$. Let $\delta > 0$. Then the set of solutions of equation (7.1) in S-integers $x_1, \ldots, x_n$ satisfying*

$$(7.2) \qquad\qquad \prod_{v \in S} \|x_1\|_v \ldots \|x_n\|_v < H_S(\mathbf{x})^{1-\delta}$$

is contained in the union of not more than

$$(7.3) \qquad\qquad t = 2n^s [(4sd!)^{2^{37(n-1)d!}s^6\delta^{-2}}]$$

proper subspaces $U_1, \ldots, U_t$ of the $(n-1)$-dimensional linear space U defined by equation (7.1).

8. The equation $f(x)\alpha^x = g(y)\beta^y$. Let

$$(8.1) \qquad\qquad f(x) = a \prod_{i=1}^{l} (x - A_i) \qquad \text{with } a \neq 0$$

and

$$(8.2) \qquad\qquad g(x) = b \prod_{j=1}^{m} (x - B_j) \qquad \text{with } b \neq 0$$

be polynomials with $a, A_1, \ldots, A_l, b, B_1, \ldots, B_m \in K$. We suppose that

$$(8.3) \qquad \max\{l, m\} < k.$$

Assume that $\alpha, \beta \in K^*$ are not roots of unity but are multiplicatively dependent. Let $(p, q) \in \mathbb{Z}^2$, with $p > 0$ minimal, such that

$$(8.4) \qquad \alpha^p = \beta^q \quad (= \delta, \text{ say}).$$

We want to study the solutions $(x, y) \in \mathbb{Z}^2$ of the equation

$$(8.5) \qquad f(x)\alpha^x = g(y)\beta^y.$$

In a qualitative sense such equations have been studied in [2] and in [11]. Here we ask for some quantitative information about the solutions.

Throughout, we assume that H is a quantity satisfying

$$(8.6) \quad H \geq \{3, h(\alpha), h(\beta), h(a), h(b), h(A_1), \ldots, h(A_l), h(B_1), \ldots, h(B_m)\}.$$

Given $(x, y) \in \mathbb{Z}^2$ we write

$$(8.7) \qquad x = px' + p', \quad y = qy' + q',$$

where $x', y' \in \mathbb{Z}$ and $0 \leq p' < p$, $0 \leq q' < q$. (Notice that p' and q' are not fixed, but will depend upon x and y respectively.)

LEMMA 8.1. *The solutions* $(x, y) \in \mathbb{Z}^2$ *of* (8.5) *with*

$$(8.8) \qquad \max\{|x|, |y|\} \geq \exp(2^5 d^6 \log^2 H)$$

satisfy

$$(8.9) \qquad |qx - py| \leq 2^{19} dk^{15} \log^2 H \log(\max\{|qx|, |py|\}).$$

P r o o f. Using (8.4) and (8.7), we may write (8.5) as

$$\delta^{x'-y'} = \frac{g(y)}{f(x)} \, \alpha^{-p'} \beta^{q'}.$$

(We remark in this context that (8.8) implies $f(x) \neq 0$.) Therefore by Lemma 5.1, (8.1) and (8.2),

$$h(\delta)^{|x'-y'|} \leq h(a)h(b)\Big(\prod_{j=1}^{m} h(y - B_j) \prod_{i=1}^{l} h(x - A_i)\Big)h(\alpha)^{|p'|}h(\beta)^{|q'|},$$

and by (8.6) this is

$$\leq H^2 \cdot 2^m \max\{1, |y|\}^m H^m \cdot 2^l \max\{1, |x|\}^l H^l H^{|p|+|q|}$$
$$\leq 2^{2k} H^{2k+|p|+|q|} \max\{|x|, |y|\}^{2k}.$$

But by Lemma 6.3, we have

$$(8.10) \qquad \max\{|p|, |q|\} \leq 60 d^6 \log H.$$

Combination of Lemma 5.2 and (8.8) yields

$$|x' - y'| \cdot \frac{1}{30d^3}$$
$$\leq 2k \log 2 + (2k + 120d^6 \log H) \log H + 2k \log(\max\{|x|, |y|\})$$
$$\leq 4k \log(\max\{|x|, |y|\}).$$

Finally, by (8.10) this implies

$$|qx - py| \leq |pqx' - pqy'| + |qp'| + |pq'|$$
$$\leq 3600d^{12} \log^2 H \cdot 120d^3 k \log(\max\{|x|, |y|\}) + 7200d^{12} \log^2 H$$
$$\leq 2^{19}d^{15}k \log^2 H \log(\max\{|x|, |y|\})$$
$$\leq 2^{19}d^{15}k \log^2 H \log(\max\{|qx|, |py|\}),$$

as asserted.

9. The equation $f(x)\alpha^x = g(y)\beta^y$**, continued.** We assume (8.1)–(8.7) throughout. Recall that our number field K has degree d. Let S be the finite subset of $M(K)$ of Section 1 with $|S| = s$ and assume that $\|\alpha\|_v = \|\beta\|_v = 1$ for $v \notin S$.

LEMMA 9.1. *Let* c_1, c_2, c_3 *be integers with* $c_1 c_2 \neq 0$ *and* $c_1/c_2 \neq q/p$. *Then the number of common solutions* $(x, y) \in \mathbb{Z}^2$ *of the two equations*

$$(9.1) \qquad\qquad c_1 x = c_2 y + c_3, \qquad f(x)\alpha^x = g(y)\beta^y$$

does not exceed

$$(9.2) \qquad\qquad 2^{s^7 2^{44d!(2k)!}}.$$

P r o o f. We assume without loss of generality that $(c_1, c_2) = 1$. Then we may parametrize the solutions of the linear equation as $x = x_0 + c_2 t$, $y = y_0 + c_1 t$ $(t \in \mathbb{Z})$, where x_0 and y_0 are fixed and the second equation in (9.1) becomes

$$(9.3) \qquad f(c_2 t + x_0)\alpha^{x_0}(\alpha^{c_2})^t = g(c_1 t + y_0)\beta^{y_0}(\beta^{c_1})^t \qquad (t \in \mathbb{Z}).$$

Equation (9.3) is of the type covered by Theorem A of Section 2. We have to consider two partitions. The first corresponds to the vanishing of each side in (9.3) and gives $f(c_2 t + x_0) = 0$. Since $f \not\equiv 0$, $\deg f < k$, this has less than k solutions.

The second partition of (9.3) has no vanishing term. The corresponding group $G(\mathcal{P})$ is defined by

$$(9.4) \qquad\qquad (\alpha^{c_2})^z = (\beta^{c_1})^z.$$

As α and β are not roots of unity and $c_1/c_2 \neq q/p$, (8.4) implies that (9.4) has only the trivial solution $z = 0$. By Theorem A, this partition gives not

more than

$$2^{20+s^7 2^{43d!(2k)!}}$$

solutions. Since this, plus k, does not exceed (9.2), the lemma follows.

LEMMA 9.2. *Suppose that in* (8.1), (8.2) *we have*

(9.5) $$\min\{l, m\} > 0.$$

Let H be as in (8.6). *Then the number of solutions $(x, y) \in \mathbb{Z}^2$ of equation* (8.5) *satisfying*

(9.6) $\quad qx - qA_i \neq py - pB_j \quad$ *for each pair i, j $(1 \leq i \leq l,\ 1 \leq j \leq m)$*

and

(9.7) $$\max\{|x|, |y|\} > \exp(2^{10} k^4 d^6 \log^2 H)$$

does not exceed

(9.8) $$2^{s^7 2^{45d!(2k)!}} - 2d^2 k^3.$$

P r o o f. We first remark that (9.7) implies that we are only considering solutions (x, y) with $f(x)g(y) \neq 0$. Given a solution (x, y), we consider the product

(9.9) $$\prod_{i=1}^{l}(x - A_i) \prod_{j=1}^{m}(y - B_j)$$

corresponding to $f(x)g(y)$ (but omitting the leading coefficients). We partition the solutions (x, y) into classes $\mathcal{L} = \mathcal{L}(i, j)$ $(1 \leq i \leq l,\ 1 \leq j \leq m)$ as follows: (x, y) belongs to $\mathcal{L}(i, j)$ if the term $(x - A_i)(y - B_j)$ in (9.9) is small in the sense of (5.21) of Lemma 5.4. Two different classes may overlap, but this is of no importance. We will give an upper bound for the number of solutions in each single class. As the total number of classes is

(9.10) $$lm < k^2,$$

our assertion (9.8) will eventually follow.

We now restrict ourselves to solutions x, y in a fixed class say $\mathcal{L}(i_0, j_0)$, and so we may assume in the sequel that Lemma 5.4 holds true for the pair (i_0, j_0). To simplify the notation we will omit subscripts in what follows, and we will write A, B instead of A_{i_0}, B_{j_0}. Recall (8.7), i.e.,

$$x = px' + p', \quad y = qy' + q' \quad (0 \leq p' < p,\ 0 \leq q' < |q|).$$

Let D be a common denominator of A and B. By Lemma 5.1, D may be chosen to satisfy

(9.11) $$D \leq H^{2d}.$$

Put

$$(9.12) \qquad x_1 = Dqx - DqA, \quad x_2 = -(Dpy - DpB),$$
$$x_3 = -(Dqx - Dpy - DqA + DpB).$$

The definition of D implies that x_1, x_2, x_3 are algebraic integers, and hence in particular they are S-integers. Moreover,

$$(9.13) \qquad x_1 + x_2 + x_3 = 0.$$

We want to apply Lemma 7.1 to equation (9.13). To do so, we have to verify hypothesis (7.2). For this purpose we apply Lemma 5.4. We rewrite equation (8.5) as

$$(9.14) \quad aD^{-l}q^{-l} \prod_{i=1}^{l} (Dqx - DqA_i)\alpha^x = bD^{-m}p^{-m} \prod_{j=1}^{m} (Dpy - DpB_j)\beta^y$$

and apply Lemma 5.4 with

$$f^*(Dqx) = a^* \prod_{i=1}^{l} (Dqx - A_i^*), \quad g^*(Dpy) = b^* \prod_{j=1}^{m} (Dpy - B_j^*),$$

where $a^* = aD^{-l}q^{-l}$, $A_i^* = DqA_i$, $b^* = bD^{-m}p^{-m}$, $B_j^* = DpB_j$. It is clear that $f^*(Dqx) = f(x)$, $g^*(Dpy) = g(y)$ and so (8.5) becomes $f^*(Dqx)\alpha^x = g^*(Dpy)\beta^y$. Hypothesis (5.20) of Lemma 5.4 is now hypothesis (9.6).

The conclusion of Lemma 5.4 will be true with

$$H^* = \max\{h(a^*), h(A_i^*), h(b^*), h(B_j^*)\}.$$

By Lemma 6.3 we have (cf. (8.6))

$$(9.15) \qquad \max\{|p|, |q|\} \le 60d^6 \log H.$$

Combining (9.15) and (9.11), by Lemma 5.1 we obtain

$$H^* \le (60d^6 \log H)^k H^{1+2kd} \le H^{5kd}.$$

By Lemma 5.4 we may conclude (using (9.11)) that the x_i in (9.12), (9.13) satisfy

$$(9.16) \quad \prod_{v \in S} \|x_1\|_v \|x_2\|_v \|x_3\|_v \le 3^{4k^2} H^{40k^3 d} \max\{1, |Dqx - Dpy|\}^{4k^2 - 1}$$

$$\le 3^{4k^2} H^{44k^3 d} \max\{1, |qx - py|\}^{4k^2 - 1}.$$

On the other hand, by Lemma 5.3 and (9.7) we get

$$(9.17) \quad H(x_1, x_2, x_3)$$

$$= H(Dqx - DqA, Dpy - DpB, Dqx - Dpy - DqA + DpB)$$

$$= H(qx - qA, py - pB, qx - py - qA + pB) \ge \frac{\max\{|qx|, |py|\}^{1/2}}{\max\{1, |qx - py|\}}$$

provided that

(9.18) $$\max\{|qx|, |py|\} \geq 16 \max\{h(qA), h(pB)\}^{d+3}.$$

By (9.15) we have

$$16h(qA)^{d+3} \leq 2^4 (2^6 d^6 \log H)^{d+3} H^{d+3} \leq (2^3 dH)^{6d+18}$$

and similarly

$$16h(pB)^{d+3} \leq (2^3 dH)^{6d+18}.$$

Therefore (9.18) will be satisfied if $\max\{|x|, |y|\} \geq (2^3 dH)^{6d+18}$, but this is amply guaranteed by (9.7).

We want to obtain (7.2) with $\delta = 1/2$. Since $H_S(x_1, x_2, x_3) \geq H(x_1, x_2, x_3)$, and by (9.16), (9.17), this will be true if we have

(9.19) $$\max\{|qx|, |py|\} \geq 3^{16k^2} H^{176k^3 d} \max\{1, |qx - py|\}^{16k^2}.$$

We are going to use Lemma 8.1, which is allowed since (9.7) implies (8.8). We obtain

$$\max\{1, |qx - py|\} \cdot 3 \cdot H^{11kd} \leq 2^{21} kd^{15} \log^2 H \cdot H^{11kd} \log(\max\{|qx|, |py|\}),$$

and hence (9.19) will be satisfied if we have

(9.20) $$2^{21} kd^{15} H^{11kd} \log^2 H \log(\max\{|qx|, |py|\}) \leq (\max\{|qx|, |py|\})^{1/16k^2}.$$

But it is easily checked that (9.7) implies (9.20).

So hypothesis (7.2) of Lemma 7.1 is indeed satisfied with $\delta = 1/2$. We may conclude that the solutions x_1, x_2, x_3 of (9.12), (9.13) satisfy one of

(9.21) $$2 \cdot 3^s (4sd!)^{2^{74d!} s^6 \cdot 4}$$

linear relations, each of which may be taken as

(9.22) $$c_1 x_1 + c_2 x_2 = 0$$

with $c_1, c_2 \in K$, $(c_1, c_2) \neq (0, 0)$.

Suppose first that in (9.22) we have $c_1 = 0$. Then (9.12) implies $y = B$ and therefore $g(B) = 0$, which is impossible by (9.7). Similarly, we may conclude that $c_2 \neq 0$.

Next assume that $c_1 = c_2$. Then (9.22) entails $qx - qA - py + pB = 0$, which is excluded by (9.6).

Therefore, we may assume that in (9.22), $c_1 c_2 \neq 0$ and $c_1 \neq c_2$. In view of (9.12), (9.22) is a relation between x and y of the shape

(9.23) $$d_1 x = d_2 y + d_3,$$

and here $d_1 d_2 \neq 0$ and $d_1/d_2 \neq q/p$. Now, if a relation (9.23) involves two different solutions $(x^{(1)}, y^{(1)})$, $(x^{(2)}, y^{(2)})$, then (d_1, d_2, d_3) is proportional to

a vector with integral components. Thus we may apply Lemma 9.1. Consequently, each relation (9.23) leads to not more than

$$2^{s^7} 2^{44d!(2k)!}$$

solutions (x, y). Taking into account the number of possible relations (9.23) as estimated in (9.21) and allowing a factor k^2 from (9.10) for the number of classes $\mathcal{L}$, we finally get the bound

$$k^2 \cdot 2 \cdot 3^s (4sd!)^{2^{74d!} s^6 \cdot 4} 2^{s^7} 2^{44d!(2k)!} < 2^{s^7} 2^{45d!(2k)!} - 2d^2 k^3.$$

The lemma follows.

LEMMA 9.3. *Let f and g be as in (8.1)–(8.3). Suppose that*

$$(9.24) \qquad\qquad \min\{l, m\} > 0.$$

Let i, j with $1 \le i \le l$, $1 \le j \le m$ be fixed. Consider the system of two equations

$$(9.25(i,j)) \qquad\qquad qx - qA_i = py - pB_j$$

where q and p are as in (8.4), and

$$(9.26) \qquad\qquad f(x)\alpha^x = g(y)\beta^y$$

in $(x, y) \in \mathbb{Z}^2$. Set

$$(9.27) \qquad\qquad M = 2d^2 k.$$

We distinguish two cases.

(a) *There exist no integers p' and q' having*

$$(9.28) \qquad\qquad 0 \le p' < p, \qquad q' = B_j + \frac{q}{p}(p' - A_i)$$

such that we have the polynomial identity

$$(9.29) \qquad\qquad f(pt + p')\alpha^{p'} \equiv g(qt + q')\beta^{q'}.$$

Then $(9.25(i,j))$ and (9.26) have at most M common solutions.

(b) *There exist unique integers p', q' with (9.28) and (9.29). Then the common solutions of $(9.25(i,j))$ and (9.26) consist of the linear one-parameter family*

$$(9.30) \qquad\qquad x = pt + p', \quad y = qt + q' \quad (t \in \mathbb{Z}),$$

plus at most M further solutions.

R e m a r k. By Lemma 6.3, p and q in (8.4) satisfy the estimate $\max\{p, |q|\} \le 60d^6 \log H$. Thus by (9.28) it is decidable whether we get a family (9.30) or not. In fact, if there is a solution family (9.30), then its data p, q, p', q' are effectively computable (at least in principle).

Proof of Lemma 9.3. Let (x, y) be a solution of $(9.25(i, j))$. Then writing $x = pt + p'$ with $0 \le p' < p$ we get $y = qt + q'$ with q' as in (9.28). So the solutions of $(9.25(i, j))$ in fact split into families

$$\mathcal{F}_{p'}: \quad x = pt + p', \quad y = qt + q', \quad 0 \le p' < p,$$

as in (9.30).

Given a family $\mathcal{F}_{p'}$, in view of (8.4) equation (9.26) becomes

$$(9.31) \qquad f(pt + p')\alpha^{p'} = g(qt + q')\beta^{q'}.$$

Now, if we had two families $\mathcal{F}_{p'_1}, \mathcal{F}_{p'_2}$ for which (9.31) is an identity, then comparing the leading coefficients we get $\alpha^{p'_1}/\beta^{q'_1} = \alpha^{p'_2}/\beta^{q'_2}$. Hence $\alpha^{p'_1 - p'_2} = \beta^{q'_1 - q'_2}$ and by the minimality of (p, q), $(p'_1 - p'_2, q'_1 - q'_2)$ is a multiple of (p, q). In conjunction with $|p'_1 - p'_2| < p$ this gives $p'_1 = p'_2$, whence $q'_1 = q'_2$. Therefore, if there exist integers p', q' with (9.28) and (9.29), then they are uniquely determined.

It remains to be shown that the union of the families $\mathcal{F}_{p'}$ for which we do not have an identity (9.29) contains not more than M solutions (x, y), with M as in (9.27). In fact, the solutions (x, y) of $(9.25(i, j))$ satisfy $y = (q/p)x + c$ with $c = B_j - (q/p)A_i$. Thus (9.26) becomes

$$(9.32) \qquad f(x)\alpha^x = g\left(\frac{q}{p}x + c\right)\beta^{(q/p)x + c}.$$

Notice that in (9.32) the exponent $(q/p)x + c$ of β is an integer; therefore the quotient $\beta^{(q/p)x + c}/\alpha^x$ lies in K and so is a pth root of $\beta^{qx + pc}/\alpha^{px}$ in K. But by (8.4), $\beta^{qx} = \alpha^{px}$. Thus the quotient is a pth root of β^{pc} in K. Let β^c be such a pth root. Then (9.32) splits into equations of the shape

$$(9.33) \qquad f(x) = g\left(\frac{q}{p}x + c\right)\zeta\beta^c,$$

where ζ runs through the pth roots of unity in K. Given ζ, either (9.33) holds identically in x or, since $\max\{\deg f, \deg g\} < k$, it has not more than k solutions. Since the number of roots of unity in K is $\le 2d^2$, solutions x where (9.33) is not an identity are contained in a set of cardinality $< 2d^2 k$, as asserted in (9.27).

Comparing the leading coefficients in (9.33), it follows, since β is not a root of unity, that there exists at most one root of unity ζ such that (9.33) holds identically in x. It is clear that only such a ζ can give rise to a one-parameter family $\mathcal{F}_{p'}$ as in (9.30) having (9.29), and the lemma follows.

10. Equations $f(x)\alpha^x = b\beta^y$. In Section 9 we have treated equation (8.5) under the assumption that both f and g have positive degrees.

In this section we study equation (8.5) assuming that

$$(10.1) \qquad g(y) \equiv b,$$

i.e., that $m = 0$. We assume again that we have (8.4) and that (p, q) is minimal. For the sake of completeness in our applications we state

LEMMA 10.1. *Let a and b be elements of K^*. Suppose that $\alpha, \beta \in K^*$ are not roots of unity. Then the equation*

$$(10.2) \qquad a\alpha^x = b\beta^y$$

either has no solution $(x, y) \in \mathbb{Z}^2$ at all, or the set of solutions consists of the one-parameter family

$$(10.3) \qquad x = pt + p', \quad y = qt + q' \quad (t \in \mathbb{Z})$$

where $(p', q') \in \mathbb{Z}^2$ is the unique solution of the equation $a\alpha^{p'} = b\beta^{q'}$ with $0 \le p' < p$, $q' \in \mathbb{Z}$. Moreover, if $H \ge \max\{h(\alpha), h(\beta), h(a), h(b)\}$, then

$$p' \le 60d^6 \log H \quad and \quad |q'| \le 2^{11}d^9 \log H.$$

Proof. Only the assertion on p' and q' needs some comments. By Lemma 6.3, we have $p' \le 60d^6 \log H$. But, moreover, $\beta^{q'} = (a/b)\alpha^{p'}$ implies $h(\beta)^{|q'|} \le h(a)h(b)h(\alpha)^{p'}$, hence by Lemma 5.2,

$$|q'| \frac{1}{30d^3} \le (p' + 2) \log H,$$

and the assertion follows.

LEMMA 10.2. *Let $f(x)$ and $g(x)$ be given by (8.1) and (10.1) respectively. Suppose that f has at least two distinct zeros. Let H be a quantity with*

$$(10.4) \qquad H \ge \max\{2, h(a), h(A_1), \dots, h(A_l), h(b)\}.$$

Then the number of solutions $(x, y) \in \mathbb{Z}^2$ of equation (8.5) having

$$(10.5) \qquad |x| \ge H^{2d(k+2)^2}$$

does not exceed

$$(10.6) \qquad 2^{s^7 2^{80d!}}.$$

Proof. Equation (8.5) now reads $a(x - A_1) \dots (x - A_l) = b\beta^y \alpha^{-x}$. Let D be a common denominator of $a, A_1, \dots, A_l, b$. By Lemma 5.1, D may be chosen with

$$(10.7) \qquad D \le H^{d(l+2)}.$$

We have

$$(10.8) \qquad Da(Dx - DA_1) \dots (Dx - DA_l) = D^{l+1} b\beta^y \alpha^{-x}.$$

Notice that here $Dx - DA_i \neq 0$ for each i $(1 \leq i \leq l)$. Now (10.8) implies

$$(10.9) \qquad \prod_{v \in S} \|Da\|_v \|Dx - DA_1\|_v \ldots \|Dx - DA_l\|_v = \prod_{v \in S} \|D^{l+1}b\|_v.$$

By the definition of D, each factor on the left hand side of (10.9) is ≥ 1. We suppose without loss of generality that $A_1 \neq A_2$ and obtain from (10.9),

$$\prod_{v \in S} \|Dx - DA_1\|_v \|Dx - DA_2\|_v \leq \prod_{v \in S} \|D^{l+1}b\|_v.$$

Thus, we may conclude that

$$(10.10) \qquad \prod_{v \in S} \|Dx - DA_1\|_v \|Dx - DA_2\|_v \|DA_1 - DA_2\|_v$$

$$\leq \prod_{v \in S} \|D^{l+1}b\|_v \|DA_1 - DA_2\|_v$$

$$\leq D^{l+2} h(b) \cdot 2h(A_1)h(A_2) < H^{d(l+3)^2 - 2}.$$

On the other hand, we get

$$(10.11) \qquad H((Dx - DA_1, Dx - DA_2, DA_1 - DA_2))$$

$$= H((x - A_1, x - A_2, A_1 - A_2))$$

$$\geq \prod_{v \mid \infty} \|x - A_1\|_v \prod_{v \nmid \infty} \|A_1 - A_2\|_v$$

$$= \prod_{v \mid \infty} \|x - A_1\|_v \|A_1 - A_2\|_v^{-1}.$$

Let $\sigma_1, \ldots, \sigma_d$ be the isomorphic embeddings of K in $\mathbb{C}$. Then (10.4) and (10.5) entail

$$\prod_{v \mid \infty} \|x - A_1\|_v = \left(\prod_{i=1}^{d} |x - \sigma_i A_1| \right)^{1/d} \geq \prod_{i=1}^{d} (|x| - |\sigma_i A_1|)^{1/d}$$

$$\geq \prod_{i=1}^{d} (|x| - H^d)^{1/d} = |x| - H^d \geq \frac{|x|}{2}.$$

Moreover, we have

$$\prod_{v \mid \infty} \|A_1 - A_2\|_v \leq 2h(A_1)h(A_2) \leq 2H^2.$$

Therefore, (10.11) implies

$$(10.12) \qquad H((Dx - DA_1, Dx - DA_2, DA_1 - DA_2)) \geq \frac{|x|}{4} H^{-2}.$$

Comparing (10.12) with (10.10) we obtain using (10.5),

$$(10.13) \quad \prod_{v \in S} \|Dx - DA_1\|_v \|Dx - DA_2\|_v \|DA_1 - DA_2\|_v$$

$$< H^{d(l+3)^2 - 2} \leq \frac{|x|^{1/2}}{2} H^{-1}$$

$$\leq H((Dx - DA_1, Dx - DA_2, DA_1 - DA_2))^{1/2}$$

$$\leq H_S((Dx - DA_1, Dx - DA_2, DA_1 - DA_2))^{1/2}.$$

We apply Lemma 7.1 to the equation $x_1 + x_2 + x_3 = 0$ with

$$(10.14) \qquad x_1 = Dx - DA_1, \quad x_2 = -Dx + DA_2, \quad x_3 = DA_1 - DA_2.$$

In view of (10.13), hypothesis (7.2) is satisfied with $\delta = 1/2$. The conclusion is that the solutions x_1, x_2, x_3 satisfy one of

$$(10.15) \qquad\qquad 2 \cdot 2^s (4sd!)^{2^{37 \cdot 2 \cdot d!} s^6 \cdot 4}$$

linear relations, each of which may be taken as

$$(10.16) \qquad\qquad c_1 x_1 + c_2 x_2 = 0,$$

with $c_1, c_2 \in K$, $(c_1, c_2) \neq (0, 0)$. If here $c_1 = c_2$, then by (10.14) we get $A_1 = A_2$, a contradiction. Therefore each relation (10.16) implies an equation of the shape $x + c_3 = 0$, and hence determines x uniquely.

On the other hand, given x there is at most one value y such that $f(x)\alpha^x = b\beta^y$. Thus (10.6) follows from (10.15).

LEMMA 10.3. *Suppose that* $f(x) = a(x - A)^l$ *and* $g(x) = b$, *with* $a, b, A \in K$, $ab \neq 0$. *Let* H *be a quantity satisfying*

$$(10.17) \qquad\qquad H \geq \max\{2, h(a), h(A), h(b)\}.$$

Suppose that

$$(10.18) \qquad\qquad\qquad A \notin \mathbb{Q}.$$

Then the number of solutions $(x, y) \in \mathbb{Z}^2$ *of the equation*

$$(10.19) \qquad\qquad\qquad f(x)\alpha^x = g(y)\beta^y$$

(i.e., of (8.5)) with

$$(10.20) \qquad\qquad\qquad |x| \geq H^{2d^2(k+2)^2}$$

does not exceed

$$(10.21) \qquad\qquad\qquad 2^{s^7 2^{80d!}}.$$

P r o o f. Let $\sigma_1, \ldots, \sigma_d$ be the isomorphic embeddings of our field K in an algebraic closure. Then (10.19) implies

$$\prod_{i=1}^{d} \sigma_i(a) \prod_{i=1}^{d} (x - \sigma_i(A))^l \prod_{i=1}^{d} \sigma_i(\alpha)^x = \prod_{i=1}^{d} \sigma_i(b) \prod_{i=1}^{d} \sigma_i(\beta)^y.$$

Write $\prod_{i=1}^{d} \sigma_i(a) = C$, $\sigma_i(A) = A_i$ $(i = 1, \ldots, d)$, $\prod_{i=1}^{d} \sigma_i(b) = B$, $\prod_{i=1}^{d} \sigma_i(\alpha) = \alpha'$, $\prod_{i=1}^{d} \sigma_i(\beta) = \beta'$. Then we get

$$(10.22) \qquad C \prod_{i=1}^{d} (x - A_i)^l \alpha'^x = B \beta'^y.$$

Since by (10.18), $A \notin \mathbb{Q}$, we may suppose that $A_1 \neq A_2$. So (10.22) is an equation of the type studied in Lemma 10.2.

The only problem with (10.22) is that the zeros A_i of the polynomial on the left hand side in general will not lie in K but in the normal closure of K. Conceivably this could have some impact in the application of Lemma 7.1. However, the proof of Lemma 7.1 (cf. [9], Section 2, and [8], Section 2) is such that in fact the result in Lemma 7.1 covers even the solutions in the normal closure of K (this is the reason that (7.3) involves the parameter $d!$ instead of simply the degree d of K). Now, analyzing the proof of Lemma 10.2 we see easily that in our context everything goes through if we replace the parameter H by H^d. Thus hypothesis (10.5) becomes (10.20). With this new hypothesis the conclusion remains the same, and (10.21) follows from (10.6).

We may summarize the results of this section as follows.

LEMMA 10.4. *Let $f(x)$ be given by (8.1) and $g(x) \equiv b$ with $b \neq 0$. Suppose that $\deg f > 0$ and that f is not of the shape $a(x - A)^l$ with $A \in \mathbb{Q}$. Let H be a quantity with*

$$(10.23) \qquad H \geq \max\{3, h(a), h(A_1), \ldots, h(A_l), h(b), h(\alpha)\}.$$

Suppose that $\alpha \in K^*$ is not a root of unity. Then the number of solutions $(x, y) \in \mathbb{Z}^2$ of the equation

$$(10.24) \qquad f(x)\alpha^x = g(y)\beta^y$$

with

$$(10.25) \qquad |x| \geq H^{2d^2(k+2)^2}$$

does not exceed

$$(10.26) \qquad 2^{s^7} 2^{80d!}.$$

Moreover, solutions $(x, y) \in \mathbb{Z}^2$ where (10.25) is violated have

$$(10.27) \qquad |y| \leq H^{27d^2 k^2}.$$

P r o o f. It suffices to prove (10.27). We get

$$\beta^y = \frac{a}{b}(x - A_1)\ldots(x - A_l)\alpha^x,$$

and hence by Lemma 5.1,

$$h(\beta)^{|y|} \le H^2 \cdot 2^l |x|^l H^{l+|x|}.$$

Thus Lemma 5.2 and (10.25) imply

$$\frac{1}{30d^3}|y| \le (l + 2 + |x|)\log H + l\log|x| + l\log 2$$

$$< (k + 2 + H^{2d^2(k+2)^2})H + k \cdot 2d^2(k+2)^2 H + kH,$$

and (10.27) follows with a simple computation.

11. Equations $f(x)\alpha^x = b\beta^y$, continued. In this section, we study the special case of equation (8.5) given by

(11.1) $$a(x - A)^l \alpha^x = b\beta^y$$

where $a, b \in K^*$, $A \in \mathbb{Q}$, $\alpha, \beta \in K^*$, α and β are not roots of unity and where $l > 0$.

We suppose, moreover, that we have (8.4), i.e.,

(11.2) $$\alpha^p = \beta^q \quad (= \delta)$$

and that here $p > 0$ is minimal.

LEMMA 11.1. *Suppose that for all integers $u \ne 0$ we have $\alpha^u \notin \mathbb{Q}$. Then* (11.1) *has at most two solutions $(x, y) \in \mathbb{Z}^2$.*

P r o o f. As β is not a root of unity, it is clear that given $x \in \mathbb{Z}$, there is at most one $y \in \mathbb{Z}$ such that (x, y) is a solution of (11.1). For a solution (x, y) the relations (11.1), (11.2) imply

$$a^{pq}(x - A)^{pql}\delta^{qx} = b^{pq}\delta^{py}.$$

Write $w = py - qx$. Then we get

(11.3) $$a^{pq}(x - A)^{pql} = b^{pq}\delta^w.$$

Now, suppose that we have two solutions $(x_1, w_1), (x_2, w_2) \in \mathbb{Z}^2$ of (11.3). Then

(11.4) $$\frac{(x_1 - A)^{pql}}{(x_2 - A)^{pql}} = \delta^{w_1 - w_2}.$$

Our hypothesis implies that $\delta^u \in \mathbb{Q}$ for $u \in \mathbb{Z}$ is only possible for $u = 0$. Thus by (11.4), we have $w_1 = w_2$ and consequently $(x_1 - A)^{pql} = (x_2 - A)^{pql}$. It follows that either $x_1 = x_2$ or $x_1 + x_2 = 2A$. Thus given a solution (x_1, w_1), every solution (x, w) has $x = x_1$ or $x = 2A - x_1$.

R e m a r k. The hypothesis of Lemma 11.1 may be effectively verified. In fact, by Lemma 6.2, it suffices to check the powers α^u with

$$(11.5) \qquad\qquad 1 \le u \le 2d^3.$$

In the sequel we will suppose that some nonzero integral power of α lies in $\mathbb{Q}$. Then naturally the same holds true for $\delta = \alpha^p$.

Define u_1 as the least positive integer such that

$$(11.6) \qquad\qquad \delta^{u_1} \in \mathbb{Q}^{pql},$$

and define $Q \in \mathbb{Q}$ by

$$(11.7) \qquad\qquad Q^{pql} = \delta^{u_1}.$$

Since $\delta = \alpha^p$, (11.7) together with (11.5), (8.6) and (6.5) implies

$$(11.8) \qquad\qquad h(Q) \le \exp(120d^9 \log^2 H).$$

LEMMA 11.2. *Suppose that $Q \notin \mathbb{Z}$. Let H be a quantity with*

$$(11.9) \qquad\qquad H \ge \max\{3, h(a), h(b), h(A), h(\alpha), h(\beta)\}.$$

Then the solutions $(x, y) \in \mathbb{Z}^2$ of (11.1) satisfy

$$(11.10) \qquad\qquad py - qx < 2^{22} d^{24} k \log^4 H.$$

Similarly, if $Q^{-1} \notin \mathbb{Z}$, then the solutions $(x, y) \in \mathbb{Z}^2$ of (11.1) satisfy

$$(11.11) \qquad\qquad qx - py < 2^{22} d^{24} k \log^4 H.$$

P r o o f. Put $py - qx = w$; then (11.1) implies

$$(11.12) \qquad\qquad a^{pq}(x - A)^{pql} = b^{pq} \delta^w.$$

Write

$$(11.13) \qquad\qquad w = u_1 z + u_0$$

with u_1 as in (11.6), $z \in \mathbb{Z}$ and $0 \le u_0 < u_1$. We infer from (11.5) and (6.5) that

$$(11.14) \qquad\qquad u_1 \le 2d^3 |pq| l < 7200 k d^{15} \log^2 H.$$

Now combination of (11.6), (11.7) and (11.12) yields

$$(11.15) \qquad\qquad (b/a)^{pq} \delta^{u_0} = ((x - A)Q^{-z})^{pql} \in \mathbb{Q}^{|pql|}.$$

If there exists u_0 in the range $0 \le u_0 < u_1$ with $(b/a)^{pq} \delta^{u_0} \in \mathbb{Q}^{|pql|}$, i.e. with (11.15), then by the definition of u_1, such a u_0 is uniquely determined. (If there is no u_0 with this property, then (11.1) has no solution $(x, y) \in \mathbb{Z}^2$ at all.) It is clear by (11.4) that there exists an algorithm to decide whether u_0 exists and to determine u_0 effectively, if it exists.

Now suppose that u_0 is as in (11.15). Define the rational number r by

$$(11.16) \qquad\qquad r^{pql} = (b/a)^{pq} \delta^{u_0}.$$

Then by (11.14) we get

$$(11.17) \qquad h(r) \le h(b)h(a)h(\delta)^{2d^3}.$$

Combination of (11.15) and (11.16) yields

$$(11.18) \qquad x = A \pm rQ^z \quad \text{with } x, z \in \mathbb{Z}.$$

Suppose that $Q \notin \mathbb{Z}$. Then for $z \in \mathbb{Z}$ with $2^z > h(A)h(r)$, (11.18) entails that $x \notin \mathbb{Z}$. Thus in (11.18) we have necessarily

$$z < 2(\log h(A) + \log h(r)).$$

Using the definition of z in (11.13) together with (11.14) and (11.17) we obtain

$$(11.19) \qquad w = u_1 z + u_0 < 2d^3|pq|l(2(\log h(A) + \log h(r)) + 1)$$
$$\le 2^{22}d^{24}k\log^4 H,$$

and this is (11.10).

(11.11) follows in the same way.

LEMMA 11.3. *Suppose that* $Q \notin \mathbb{Z}$. *Let* H *be as in* (11.9). *Then the solutions* $(x, y) \in \mathbb{Z}^2$ *of* (11.1) *with*

$$(11.20) \qquad py - qx \ge 0$$

satisfy

$$(11.21) \qquad |x| \le \exp(2^{13}d^{18}\log^4 H), \qquad |y| \le \exp(2^{14}d^{18}\log^4 H).$$

Similarly, if $Q^{-1} \notin \mathbb{Z}$, *then the solutions* $(x, y) \in \mathbb{Z}^2$ *of* (11.1) *with*

$$(11.22) \qquad py - qx \le 0$$

satisfy (11.21).

Proof. We have (11.3), i.e.,

$$a^{pq}(x - A)^{pql} = b^{pq}\delta^{py-qx}.$$

So if (11.20) holds true, we have

$$h(x - A) \le (h(b)h(a)h(\delta)^{(py-qx)/|pq|})^{1/l}$$

and therefore since $py - qx = w$ and by (11.19),

$$|x| \le (h(b)h(a)h(\delta)^{(py-qx)/|pq|})^{1/l} + h(A)$$
$$\le H^2\log h(\delta)\exp(4d^2(\log h(A) + \log h(r))) + h(A).$$

Recall that $\delta = \alpha^p$, so that by (6.5), $\log h(\delta) \le 60d^6\log^2 H$, and from (11.17) we get

$$(11.23) \qquad \log h(r) \le 2\log H + 120d^9\log^2 H < 2^7d^9\log^2 H,$$

which finally implies that $|x| \leq \exp(2^{13} d^{18} \log^4 H)$. But then (11.10) and (11.20) in conjunction with (9.15) give $|y| \leq \exp(2^{14} d^{18} \log^4 H)$, and so (11.21) follows.

The combination (11.22), (11.21) may be shown in the same way.

To finish our study of equation (11.1), we still have to deal with the cases when Q in (11.7) satisfies

$$(11.24) \qquad Q \in \mathbb{Z} \quad \text{or} \quad Q^{-1} \in \mathbb{Z}.$$

It turns out that this is the most complicated case. In [11] we called an equation

$$f(x)\alpha^x = g(y)\beta^y$$

exceptional of type 1 if f has exactly one root A (of arbitrary multiplicity) and if this root lies in $\mathbb{Q}$, if g is a nonzero constant and $\alpha, \beta \in K^*$ are not roots of unity but such that there exist nonzero integers p, q with $\alpha^p = \beta^q = \delta$, and such that moreover some power δ^u with $u \in \mathbb{Z} \backslash \{0\}$ lies in $\mathbb{Z}$. We called the equation *exceptional of type* 2 if the roles of f and g are interchanged.

So let us now deal with the exceptional equation of type 1. Suppose $Q \in \mathbb{Z}$. Equation (11.1) gives (11.18), i.e.,

$$(11.25) \qquad x = A \pm rQ^z,$$

and we ask for solutions $(x, z) \in \mathbb{Z}^2$. Put

$$(11.26) \qquad A = B/D, \quad r = C/D \quad \text{with } D > 0$$

and with $B, C, D \in \mathbb{Z}$, $(B, C, D) = 1$. Then $D \leq h(A)h(r)$ and thus by (11.23) and (6.5),

$$(11.27) \qquad D \leq \exp(2^8 d^9 \log^2 H).$$

Then x given by (11.25) lies in $\mathbb{Z}$ if

$$(11.28) \qquad B \pm CQ^z \equiv 0 \pmod{D}.$$

Put $D = D_1 D_2$, where D_1 is made up of primes dividing Q and D_2 of other primes. Then for z with $2^z > D_1$ we get $Q^z \equiv 0 \pmod{D_1}$, and in view of (11.28) this entails

LEMMA 11.4. *Suppose that $Q \in \mathbb{Z}$ and that $B \not\equiv 0 \pmod{D_1}$. Then the solutions $(x, y) \in \mathbb{Z}^2$ of (11.1) satisfy*

$$(11.29) \qquad |x| \leq \exp(2^{17} d^{18} \log^4 H), \quad |y| \leq \exp(2^{18} d^{18} \log^4 H).$$

P r o o f. By Lemma 11.3 we may suppose that $py - qx > 0$. Recall the definition of z in (11.13). We have $u_1 z + u_0 = py - qx$. Our hypothesis implies that only such z are admitted for which $2^z \leq D_1$. We infer from (11.27) that

necessarily $z \leq 2^9 d^9 \log^2 H$. Combining this with (11.14), we get

$$py - qx \leq 2^{10} d^{12} |pq| l \log^2 H.$$

The proof may now be finished in exactly the same way as in Lemma 11.3.

It is clear that we get a result analogous to Lemma 11.4 if $Q^{-1} \in \mathbb{Z}$.

We now suppose that B in (11.28) has $B \equiv 0 \pmod{D_1}$. We detail only the case when in (11.28) we have the $+$ sign. By Lemma 11.3, it suffices to find the solutions $z \in \mathbb{Z}$, $z \geq 0$, of the congruence

$$(11.30) \qquad\qquad B + CQ^z \equiv 0 \pmod{D},$$

since for $z < 0$ we have $py - qx \leq 0$ and that case is treated in (11.22), (11.21). Values z with $2^z \leq D_1$ lead to solutions x, y of (11.1) with (11.29) (this follows at once from the proof of Lemma 11.4).

If (11.30) has a solution z with $2^z > D_1$ at all, then it has such a solution z^+ in the range

$$(11.31) \qquad\qquad \frac{\log D_1}{\log 2} < z^+ \leq \frac{\log D_1}{\log 2} + D_2.$$

Let z^+ be minimal with (11.31) such that (11.30) is satisfied for $z = z^+$. By (11.27), z^+ is effectively computable. By definition of D_2, we have $(Q, D_2) = 1$. Let $v > 0$ be the least solution of $Q^v \equiv 1 \pmod{D_2}$. Again by (11.27) we have

$$(11.32) \qquad\qquad 0 < v \leq \varphi(D_2) \leq \exp(2^8 d^9 \log^2 H).$$

It is now clear that the solutions z of (11.30) with $z > (\log D_1)/\log 2$ are given by

$$(11.33) \qquad\qquad z = z^+ + vs \quad (s \in \mathbb{Z},\ s \geq 0).$$

In view of (11.18) we therefore obtain

$$(11.34) \qquad x = rQ^{z^+}(Q^v)^s + A = E'R_0^s + A \quad (s \in \mathbb{Z},\ s \geq 0),$$

where $E' = rQ^{z^+}$ and $R_0 = Q^v$. Now by (11.3) we get

$$(11.35) \quad py = w + qx = u_1 z + u_0 + qx = qE'R_0^s + u_1 vs + u_1 z^+ + u_0 + qA$$
$$= qE'R_0^s + F's + G',$$

say, with $F' = u_1 v$ and $G' = u_1 z^+ + u_0 + qA$. But y needs to be an integer. So the question is: for which s do we have

$$(11.36) \qquad\qquad qE'R_0^s + F's + G' \equiv 0 \pmod{p}?$$

Write $p = p_1 p_2$ where p_1 is a product of primes dividing R_0, and p_2 is a product of other primes. If s is so large that $2^s > h(r)p$, then the rational number $qE'R_0^s$ will have a numerator $\equiv 0 \pmod{p_1}$. In view of (11.23)

and (6.5) this will be satisfied if

$$(11.37) \qquad s > 2^8 d^9 \log^2 H.$$

Solutions where (11.36) is violated lead to $z = z^+ + vs \le \exp(2^9 d^9 \log^2 H)$ by (11.27), (11.32) and (11.37), and this implies an effective bound for the corresponding solutions x and y of (11.1) (cf. the proof of Lemma 11.3). So it suffices to solve (11.36) for $s \ge s_0$ with $s_0 = [2^8 d^9 \log^2 H + 1]$, and consequently we have to solve simultaneously

$$(11.38) \qquad F's + F's_0 + G' \equiv 0 \pmod{p_1}$$

and

$$(11.39) \qquad qE'R_0^{s_0} R_0^s + F's + F's_0 + G' \equiv 0 \pmod{p_2}$$

with $s \in \mathbb{Z}$, $s \ge 0$.

If the congruence (11.38) is solvable at all, then the solutions s run through certain residue classes $\mod p_1$, and since we have effective bounds for all parameters in (11.38), we may compute these classes effectively. Now, given such a residue class $s = p_1 t + p_1'$ ($t \in \mathbb{Z}$, $t \ge 0$), we have to solve (11.39) for s in this class, i.e.,

$$(11.40) \qquad qE'R_0^{s_0+p_1'}(R_0^{p_1})^t + F'p_1 t + F'(s_0 + p_1') + G' \equiv 0 \pmod{p_2}$$
$$(t \in \mathbb{Z}, \ t \ge 0).$$

Since $(R_0, p_2) = 1$, the residue class of $(R_0^{p_1})^t \pmod{p_2}$ depends only upon the residue class of $t \mod \varphi(p_2)$ and the solutions t of (11.40) lie in suitable residue classes $\mod p_2 \varphi(p_2)$ (if there are solutions t at all). Again since we have effective bounds for all the parameters in (11.40), it is decidable whether (11.40) admits solutions, and in the affirmative case we may determine effectively all residue classes.

Collecting our results, we see that the solutions s of (11.36) consist of certain residue classes $\mod p_1 p_2 \varphi(p_2)$ and these classes are effectively computable. Let $Pt + P'$ ($t \in \mathbb{Z}$, $t \ge 0$) be such a residue class. Then by (11.34) we get $x(t) = rQ^{z^+} Q^{vP'}(Q^{vP})^t + A$, and so

$$(11.41) \qquad x(t) = EpR^t + A \qquad (t \in \mathbb{Z}, \ t \ge 0),$$

where

$$(11.42) \qquad E = rQ^{z^+ + vP'}p^{-1}, \quad R = Q^{vP}.$$

Moreover, we obtain

$$py = qx + w = qx + u_1 z + u_0 = qx + u_1 vs + u_1 z^+ + u_0$$
$$= qx + u_1 vPt + u_1 vP' + u_1 z^+ + u_0,$$

and hence by (11.41),

$$(11.43) \qquad y(t) = EqR^t + Ft + G \qquad (t \in \mathbb{Z}, \ t \ge 0),$$

where

(11.44) $$F = u_1 v P, \quad G = qA + u_1 v P' + u_1 z^+ + u_0.$$

All this was when $Q \in \mathbb{Z}$ and we have the $+$ sign in (11.28). If we have the $-$ sign in (11.28), then defining z^- with respect to the congruence $B - CQ^z \equiv 0$ (mod D) in the same way as we got z^+, all the above still holds true mutatis mutandis.

If, however, $Q^{-1} \in \mathbb{Z}$, then in the above construction we have to study solutions x, y with $0 > w = py - qx = u_1 z + u_0$, and we have to focus on z with $z < 0$. Again we obtain exponential families $x(t), y(t)$ of the type (11.41), (11.43). We only have to replace z^+ by $-z^+$ or z^- by $-z^-$ respectively, and v by $-v$. We remark that in (11.41) and (11.43), the quantities R and F do not depend upon the particular residue class $\bmod P$ and that, moreover, we have $F \neq 0$.

We summarize what we have shown.

LEMMA 11.5. *Suppose that equation* (11.1) *is exceptional of type* 1. *Then apart from finitely many solutions* x, y *which may be effectively computed, the solutions* (x, y) *of* (11.1) *make up a finite number* (*possibly zero*) *of one-parameter exponential families, each of type* (11.41), (11.43). *There exists an algorithm to decide whether such exponential families do occur at all. If such families do occur, then they are effectively computable.*

12. Summary on equations $f(x)\alpha^x = g(y)\beta^y$. We collect the results on equation (8.5) proved in the preceding sections. We assume throughout that (8.1)–(8.6) are valid.

PROPOSITION 12.1. *Suppose that* (8.5) *is not exceptional. Then there exists an algorithm to decide whether* (8.5) *has infinitely or only finitely many solutions.*

(a) *If* (8.5) *has only finitely many solutions, then all but at most*

(12.1) $$M = 2^{s^7 2^{45 d! (2k)!}}$$

solutions have

(12.2) $$\max(|x|, |y|) \leq c(d, k, H)$$

with effectively computable $c(d, k, H)$.

(b) *If* (8.5) *has infinitely many solutions, then there is a unique linear one-parameter family*

(12.3) $$x(t) = pt + p', \quad y(t) = qt + q' \quad (t \in \mathbb{Z})$$

with $0 \leq p' < p$, *such that all but at most* M *solutions lie in this family or have* (12.2). *The family* (12.3) *and the bound* $c(d, k, H)$ *are effectively computable.*

P r o o f. The case when both f and g have positive degree follows from Lemmata 9.2 and 9.3. The bound in (12.1) is the sum of the bounds in (9.8) and k^2 times the bound in (9.27).

Lemma 10.1 covers the case when both f and g are constants. The remaining cases are treated in Lemmata 10.4, 11.1 and 11.3.

We remark that the bound $2^{s^7 2^{80d!}}$ in Lemma 10.4 is smaller than the bound in (12.1) since $k \geq 1$.

PROPOSITION 12.2. *Suppose that* (8.5) *is exceptional of type* 1. *Then there exists an algorithm to decide whether* (8.5) *has infinitely many or only finitely many solutions. Moreover*:

(a) *If* (8.5) *has only finitely many solutions, then each solution satisfies*

$$(12.4) \qquad \max(|x|, |y|) \leq c(d, k, H).$$

(b) *If* (8.5) *has infinitely many solutions, then they make up finitely many one-parameter exponential families, each of the type*

$$(12.5) \quad x_j(t) = pE_j R^t + A, \quad y_j(t) = qE_j R^t + Ft + G_j \quad (t \in \mathbb{Z}, \ t \geq 0)$$

$(1 \leq j \leq c(d, k, H))$, *and possibly some further solutions with* (12.4).

Here the families (12.5) *are effectively computable. Moreover, E_j, G_j are rational numbers, F is a nonzero integer that does not depend upon the particular family, and $R = \alpha^u \in \mathbb{Z}$ for some $u \in \mathbb{Z} \backslash \{0\}$. The constant in* (12.4) *is effective.*

P r o o f. The assertion follows at once from Lemmata 11.3 and 11.4 and from formulas (11.41)–(11.44) in the proof of Lemma 11.4.

13. Intersections of solution families. We consider a system of equations

$$(13.1) \qquad f_1(x)\alpha_1^x = g_1(y)\beta_1^y,$$
$$\qquad\qquad\qquad\qquad\qquad (x, y) \in \mathbb{Z}^2.$$
$$(13.2) \qquad f_2(x)\alpha_2^x = g_2(y)\beta_2^y,$$

Here we suppose that f_1, f_2, g_1, g_2 are nonzero polynomials as in (8.1), (8.2), of respective degrees l_1, l_2, m_1, m_2, with

$$(13.3) \qquad \max\{l_1, l_2, m_1, m_2\} < k.$$

We assume that $\alpha_1, \alpha_2, \beta_1, \beta_2 \in K^*$, that α_2 and β_2 are not roots of unity and that

$$(13.4) \qquad \alpha_i^p = \beta_i^q \quad \text{for } i = 1, 2,$$

where p and q are nonzero integers such that $p > 0$ is minimal with a relation (13.4). Our hypotheses are such that at least equation (13.2) is of the shape

studied in Sections 8–12. In studying simultaneously (13.1) and (13.2) we distinguish the following cases.

(i) α_1 and β_1 are not roots of unity;
(ii) α_1 and β_1 are roots of unity.

We first treat alternative (i).
We will make throughout the hypothesis

(H)   Both (13.1) and (13.2) admit infinitely many solutions.

In view of Propositions 12.1 and 12.2 this will be no restriction.

If we choose (p_1, q_1) and (p_2, q_2) in $\mathbb{Z}^2$ minimal such that $\alpha_i^{p_i} = \beta_i^{q_i}$, then the pair (p, q) in (13.4) will be such that p is the least common multiple of p_1 and p_2.

LEMMA 13.1. *Suppose that neither* (13.1) *nor* (13.2) *is exceptional and that we have* (H). *Then the intersection of the linear families*

(13.5) $\qquad x_1(t_1) = p_1 t_1 + p_1', \quad y_1(t_1) = q_1 t_1 + q_1' \quad (t_1 \in \mathbb{Z})$

and

(13.6) $\qquad x_2(t_2) = p_2 t_2 + p_2', \quad y_2(t_2) = q_2 t_2 + q_2' \quad (t_2 \in \mathbb{Z})$

from (12.3) *either is empty or equals*

$$x(t) = pt + p', \quad y(t) = qt + q' \quad (t \in \mathbb{Z}),$$

where p is the least common multiple of p_1 and p_2 and where $0 \leq p' < p$ is uniquely determined.

P r o o f. Suppose that the intersection of the families (13.5) and (13.6) is not empty. Let $t_1^{(0)}, t_2^{(0)} \in \mathbb{Z}$ be parameters having

$$p_1 t_1^{(0)} + p_1' = p_2 t_2^{(0)} + p_2', \quad q_1 t_1^{(0)} + q_1' = q_2 t_2^{(0)} + q_2'.$$

Then the intersection consists of points

$$x(t_1) = p_1 t_1 + p_1 t_1^{(0)} + p_1', \quad y(t_1) = q_1 t_1 + q_1 t_1^{(0)} + q_1' \quad (t_1 \in \mathbb{Z}),$$

where t_1 satisfies the equation $p_1 t_1 = p_2 t_2$ with $t_1, t_2 \in \mathbb{Z}$. Here t_1 may be written as $t_1 = p_2 t / (p_1, p_2)$ with $t \in \mathbb{Z}$ and the assertion follows.

LEMMA 13.2. *Suppose that one of* (13.1), (13.2) *is exceptional and the other is not. Suppose, moreover, that we have hypothesis* (H). *Then the intersection of the linear family* (12.3) *with the union of the exponential families* (12.5) *contains at most two points* (x, y).

P r o o f. Suppose without loss of generality that the linear family is

$$x_1(t_1) = p_1 t_1 + p_1', \quad y_1(t_1) = q_1 t_1 + q_1'.$$

Then we get $py_1(t_1) - qx_1(t_1) = pq_1' - qp_1'$. As for the exponential families, let us go back in the proof of Lemma 11.5.

The exponential families of the lemma are contained in the family

$$(13.7) \quad x(s) = E'R_0^s + A, \quad y(s) = \frac{p}{q}E'R_0^s + \frac{F'}{p}s + G' \quad (s \in \mathbb{Z},\ s \geq 0)$$

in (11.34) and (11.35), and possibly a second family of the same type corresponding to the $-$ sign in (11.28). (Recall that in (13.7), $y(s)$ is not necessarily an integer.)

Now, $F' \neq 0$ in (13.7) and we get $py(s) - qx(s) = F's + pG' - qA'$, and the intersection with the linear family leads to parameters s with $F's + pG' - qA' = pq_1' - qp_1'$. Hence we get at most one such parameter s. As we have to consider at most two families of type (13.7), the assertion follows.

LEMMA 13.3. *Suppose that one of* (13.1), (13.2) *is exceptional of type* 1 *and the other is exceptional of type* 2. *Suppose that hypothesis* (H) *holds true. Let* L_1 *be the union of the exponential families* (12.5) *coming from* (13.1), *and similarly* L_2 *from* (13.2). *Then*

$$(13.8) \qquad\qquad |L_1 \cap L_2| \leq 2^{2^{52} s^7}.$$

Proof. We suppose without loss of generality that (13.1) is of type 1 and (13.2) of type 2. As in the proof of Lemma 13.2, we go back to (11.34), (11.35).

Therefore L_1 is contained in the union of two exponential families $x_1(s_1)$, $y_1(s_1)$ where

$$x_1(s_1) = E_1 R_1^{s_1} + A_1 \quad (s_1 \in \mathbb{Z})$$

with $E_1 \in \mathbb{Q}^*$, $A_1 \in \mathbb{Q}$ and $R_1 \in \mathbb{Z}$, $|R_1| > 1$. Similarly L_2 is contained in the union of two exponential families $x_2(s_2), y_2(s_2)$, where

$$x_2(s_2) = E_2 R_2^{s_2} + Fs + G$$

with $E_2 \in \mathbb{Q}^*$, $R_2 \in \mathbb{Z}$, $|R_2| > 1$, $F \in \mathbb{Q}^*$, $G \in \mathbb{Q}$. Moreover, as R_1 is an integral power of α_1 and R_2 an integral power of α_2, R_1 and R_2 are both S-units. The equation $x_1(s_1) = x_2(s_2)$ leads to an equation

$$(13.9) \qquad\qquad E_1 R_1^{s_1} = E_2 R_2^{s_2} + (Fs_2 + C) \cdot 1^{s_2},$$

and hence is of the type covered by Theorem 1 (notice that the left hand side and the right hand side in (13.9) are not *related*). The conclusion is that (13.9) has not more than

$$2^{2^{51} s^7}$$

solutions $(s_1, s_2) \in \mathbb{Z}^2$. Allowing a factor 4 for the two alternatives we possibly have in L_1 as well as in L_2, we finally get the bound $2^{2^{52} s^7}$, as asserted.

LEMMA 13.4. *Suppose that both* (13.1) *and* (13.2) *are exceptional of type* 1. *Suppose, moreover, that* $|\alpha_1| > 1$ *and* $|\alpha_2| < 1$. *Assume that hypothesis* (H) *holds true. Then the intersection of the set of exponential families* L_1 *corresponding to* (13.1) *and the set of exponential families* L_2 *corresponding to* (13.2) *is empty.*

P r o o f. Let $(x_1(s_1), y_1(s_1))$ be an exponential family belonging to α_1 and $(x_2(s_2), y_2(s_2))$ be an exponential family belonging to α_2. The construction of the families in Section 11 is such that (cf. Lemma 11.3) for $s_i \in \mathbb{Z}$, $s_i \geq 0$, we have $py_1(s_1) - qx_1(s_1) > 0$ and $py_2(s_2) - qx_2(s_2) < 0$. The assertion follows.

LEMMA 13.5. *Suppose that both* (13.1) *and* (13.2) *are exceptional of type* 1. *Suppose, moreover, that* $|\alpha_1| > 1$ *and* $|\alpha_2| > 1$. *Assume that hypothesis* (H) *holds true. Let* L_1 *be the union of the exponential families from* (13.1) *and* L_2 *the union of the exponential families from* (13.2). *Then it is decidable whether* $L_1 \cap L_2$ *is infinite or not.* $L_1 \cap L_2$ *is infinite if and only if* f_1 *and* f_2 *have the same rational zero and* $\alpha_1^{u_1} = \alpha_2^{u_2}$ *for some nonzero integers* u_1, u_2. *Moreover*:

(a) *If* $L_1 \cap L_2$ *is finite, then it is of cardinality at most*

$$(13.10) \qquad\qquad M := 2^{2^{1080 s^7}}.$$

(b) *If* $L_1 \cap L_2$ *is infinite, then it consists of exponential families* $x_j(t)$, $y_j(t)$ $(t \in \mathbb{Z}, t \geq 0)$ $(1 \leq j \leq c(d, k, H))$ *of the type* (12.5), *plus at most* M *further elements. These families are effectively computable. Here the parameter* R *of the families is an integral power of both* α_1 *and* α_2.

P r o o f. To get in (13.10) a bound M that is independent of heights, it is suitable not to use directly the different exponential families we have in Proposition 12.2, but to go back to the proof of Lemma 11.5, as we have already done in proving Lemmata 13.2 and 13.3. We got our exponential families (12.5) by intersecting (11.34) and (11.35), i.e.,

$$(13.11) \quad x(s) = E'R_0^s + A, \quad y(s) = \frac{q}{p}E'R_0^s + \frac{F'}{p}s + \frac{G'}{p} \quad (s \in \mathbb{Z},\ s \geq 0)$$

with $\mathbb{Z}^2$. (There may be two such families, if in (11.25) the $-$ sign occurs as well.)

To keep $L_1 \cap L_2$ under control, we first study the intersection of two families (13.11) (never mind whether the points are in $\mathbb{Z}^2$ or not) and only when we have this intersection under control, will we ask for points in the intersection whose components are integers.

We have to solve a system

(13.12) $$x_1(s_1) = x_2(s_2), \qquad y_1(s_1) = y_2(s_2)$$

where $x_i(s_i)$, $y_i(s_i)$ is of the type (13.11). The first equation in (13.12) yields a relation

$$E^{(1)} R_0^{(1)s_1} + A^{(1)} = E^{(2)} R_0^{(2)s_2} + A^{(2)},$$

say. We may apply Theorems 1 and 2 and conclude that apart from possible solutions in a set of cardinality $\leq 2^{2^{1080} s^7}$, this implies

(13.13) $$E^{(1)} R_0^{(1)s_1} = E^{(2)} R_0^{(2)s_2}, \qquad A^{(1)} = A^{(2)},$$

$$\text{where } s_1, s_2 \in \mathbb{Z}, \; s_1, s_2 \geq 0.$$

We recall from (11.1) and (11.34) that $A^{(1)}, A^{(2)}$ are the zeros of f_1 and f_2 respectively. As $R^{(1)}$ and $R^{(2)}$ are integral powers of α_1, α_2 respectively, $R^{(1)}$ and $R^{(2)}$ are bounded in terms of d and H. So, either we have a relation

(13.14) $$R^{(1)w_1} = R^{(2)w_2}$$

with $w_1, w_2 \in \mathbb{N}$ and bounded in terms of d and H, or (13.13) has not more than one solution. (The bound in (13.10) is so generous that it easily takes care also of such a single additional solution.) If (13.14) is solvable, then the solutions (s_1, s_2) of (13.13) are of the shape

$$s_1 = w_1 t + s_1^{(0)}, \quad s_2 = w_2 t + s_2^{(0)} \quad (t \in \mathbb{Z}, \; t \geq 0),$$

where w_1, w_2 is a minimal solution of (13.14) and where $s_1^{(0)}, s_2^{(0)}$ is a minimal solution of (13.13). It is clear that $w_1, w_2, s_1^{(0)}, s_2^{(0)}$ are computable. We next have to check the second equation in (13.12). In view of (13.13), and with our values of s_1, s_2, this reduces to

$$F^{(1)}(w_1 t + s_1^{(0)}) + G^{(1)} = F^{(2)}(w_2 t + s_2^{(0)}) + G^{(2)} \quad (t \in \mathbb{Z}, \; t \geq 0),$$

and hence it either has at most one solution t or it is an identity in t.

In the latter case, we have to apply the considerations of Section 11 to the family

(13.15)
$$x(t) = E^{(1)} R_0^{(1)s_1^{(0)}} (R_0^{(1)w_1})^t + A^{(1)},$$

$$y(t) = \frac{q}{p} E^{(1)} R_0^{(1)s_1^{(0)}} (R_0^{(1)w_1})^t + \frac{F^{(1)}}{p} w_1 t + \frac{F^{(1)} s_1^{(0)} + G^{(1)}}{p}$$

$$(t \in \mathbb{Z}, \; t \geq 0)$$

to guarantee that we get only values $y(t) \in \mathbb{Z}$. So we follow the considerations after (11.36) with E' replaced by $E^{(1)} R_0^{(1)s_1^{(0)}}$, R_0 replaced by $R_0^{(1)w_1}$, F'

replaced by $F^{(1)}w_1$, and G' replaced by $F^{(1)}s_1^{(0)} + G^{(1)}$. Again we see that either we get no solutions t at all, or we get solutions t that make up certain residue classes. The details may be left to the reader.

Since $R^{(1)}$ is an integral power of α_1 and $R^{(2)}$ is an integral power of α_2, it also follows from (13.14) that the parameter $R_0 = R_0^{(1)w_1}$ occurring in the final families we get is an integral power of both α_1 and α_2.

We now treat alternative (ii), where α_1 and β_1 are roots of unity.

LEMMA 13.6. *Suppose that*

$$(13.16) \qquad\qquad \max\{\deg f_1, \deg g_1\} > 0.$$

Suppose, moreover, that (13.2) *is exceptional and admits a set L_2 of exponential families* (12.5) *among its solutions. Then the intersection of L_2 with the set of solutions (x, y) of* (13.1) *has cardinality not exceeding*

$$(13.17) \qquad\qquad 4k^2.$$

P r o o f. L_2 is contained in two families $x_i(s), y_i(s)$, $i = 1, 2$, each of the shape (11.34), (11.35), i.e.,

$$(13.18) \qquad x(s) = ER^s + A, \quad y(s) = \frac{q}{p}ER^s + Fs + G \quad (s \in \mathbb{Z})$$

with $|R| > 1$, $R \in \mathbb{Z}$, $F \neq 0$, $E, A, F, G \in \mathbb{Q}$. Since α_1 and β_1 are roots of unity, (13.1) implies

$$(13.19) \qquad\qquad f_1(x(s))\bar{f}_1(x(s)) = g_1(y(s))\bar{g}_1(y(s)).$$

Combination of (13.18) and (13.19) gives an equation of the type

$$(13.20) \quad P_0(s)R^{(2k-2)s} + P_1(s)R^{(2k-3)s} + \ldots + P_{2k-3}(s)R^s + P_{2k-2}(s) = 0,$$

where the P_i in (13.20) are polynomials with real coefficients and of degree $\leq i$ ($0 \leq i \leq 2k - 2$). Since $F \neq 0$ in (13.18), and by (13.16), we may conclude that not all polynomials P_i in (13.20) are identically zero.

But then an easy application of Rolle's Theorem (cf. Pólya–Szegö [6], p. 48, Aufgabe 75) shows that (13.20) has less than $2k^2$ solutions $s \in \mathbb{R}$. Allowing a factor two for the possible second family $x_2(s), y_2(s)$ making up L_2, we get the assertion.

LEMMA 13.7. *Suppose that in* (13.1) *we have $f_1(x) = a$, $g_1(x) = b$. Suppose, moreover, that* (13.2) *is exceptional and admits a set L_2 of exponential families of type* (12.5) *among its solutions. Then the intersection of L_2 with the set of solutions (x, y) of* (13.1) *consists of a finite set which is effectively computable and possibly a finite set of exponential families again of type* (12.5), *and these families are effectively computable.*

P r o o f. Let w be the smallest natural number such that

$$(13.21) \qquad\qquad \alpha_1^w = \beta_1^w.$$

By Lemma 6.1, $w \leq 2d^2$. Write $x = wx' + w_x$ and $y = wy' + w_y$ with $0 \leq w_x < w$ and $0 \leq w_y < w$. Thus the problem consists in determining the intersections of exponential families $x(s), y(s)$ with residue classes $x \equiv w_x$ (mod w), $y \equiv w_y$ (mod w). As was shown in Section 11, this is effectively solvable and leads to a finite set and possibly some exponential families. (Since $w \leq 2d^2$, everything is under control.)

LEMMA 13.8. *Suppose that* (13.2) *is not exceptional, but has a linear one-parameter family* (12.3) *among its solutions. Then the intersection of this family with the set of solutions of* (13.1) *consists of a set of cardinality not exceeding*

$$(13.22) \qquad\qquad 2d^2 k$$

and possibly a (*unique*) *linear family*

$$(13.23) \qquad x(t) = pt + p', \quad y(t) = qt + q' \quad (t \in \mathbb{Z}),$$

where $0 \leq p' < p$.

P r o o f. Let (p_2, q_2) be a minimal pair with $\alpha_2^{p_2} = \beta_2^{q_2}$ and $p_2 > 0$. Then (13.2) gives the unique linear family

$$(13.24) \qquad x(t) = p_2 t + p_2', \quad y(t) = q_2 t + q_2' \quad (t \in \mathbb{Z})$$

with $0 \leq p_2' < p_2$. Pick $w \in \mathbb{N}$ minimal with $\alpha_1^w = \beta_1^w$. By Lemma 6.1, we have $w \leq 2d^2$. Therefore, the pair p, q in (13.4) has $p = vp_2$, $q = vq_2$ with $v \in \mathbb{N}$, $v \leq 2d^2$.

The family (13.24) splits into v families

$$(13.25) \qquad x(t) = pt + p', \quad y = qt + q' \quad (t \in \mathbb{Z})$$

with $0 \leq p' < p$ (in fact, $p' = p_2' + ip_2$ and $q' = q_2' + iq_2$ for some i with $0 \leq i < v$).

Now, given a family (13.25), equation (13.1) becomes

$$(13.26) \qquad f_1(pt + p')\alpha_1^{p'} = g_1(qt + q')\beta_1^{q'}.$$

This is a polynomial equation in t. Thus either (13.26) is an identity in t or it has less than k solutions t. The minimality of the pair (p, q) in (13.4) implies that at most one of the families (13.25) may give an identity in (13.26). As $v \leq 2d^2$, we obtain (13.22).

14. Proof of Theorems 3 and 4 and of the corollaries. Let us begin with Theorem 4. We proceed by induction on r. If $r = 1$, the assertion follows from Proposition 12.2 and Lemma 13.7. Now suppose that $r > 1$ and that the assertion holds true for $r - 1$. The theorem then follows from Lemma 13.5(b), in fact (13.10) implies the bound (1.30).

Now let us turn to Theorem 3. If $r = 1$, we have two possibilities.

Either $f_1(x)\alpha_1^x = g_1(y)\beta_1^y$ is not exceptional. Then by Proposition 12.1 a set with $2^{s^7 2^{45d!(2k)!}}$ elements will suffice, plus additionally $2d^2k$ elements from Lemma 13.8, and the theorem follows.

Or $f_1(x)\alpha_1^x = g_1(y)\beta_1^y$ is exceptional but not both f_0 and g_0 are constant. Then by Lemma 13.6 a set M_1 with $\leq 4k^2$ elements suffices.

Now suppose that $r > 1$. As F, G is not exceptional, we may infer that there is a pair i, j ($0 \leq i < j \leq r$) such that the system

$$(14.1) \qquad f_i(x)\alpha_i^x = g_i(y)\beta_i^y, \qquad f_j(x)\alpha_j^x = g_j(y)\beta_j^y$$

is not exceptional (in an obvious sense). So, for the set of solutions of (14.1) we may apply one of Lemmata 13.1–13.6, 13.8. The assertion follows at once as for the exceptional set of cardinality $\leq M_1$ the worst cases arise if one equation in (14.1) is not exceptional and the other involves roots of unity, or if both equations are as in Lemma 13.5. We may conclude that an exceptional set of cardinality

$$\leq \max\{2^{2^{1080}s^7}, 2^{s^7 2^{45d!(2k)!}} + 2d^2k\}$$

will suffice. The theorem follows.

As for Corollary 1, we clearly get the one-parameter family of solutions $m = t$, $n = t$ ($t \in \mathbb{Z}$). If $\{u_m\}$ is symmetric, then we have relations (1.34). Hence $\alpha_i\alpha_{i+1}$ is a root of unity in K and so by Lemma 6.1 we have $w \leq 2d^2$, which gives the possible second family (1.39) by Theorem 3. We still have to prove the bound (1.36).

We apply Theorems 2 and 3. By Theorem 2 the number of solutions (1.37) that do not satisfy (1.20) or (1.21a,b) has cardinality

$$(14.2) \qquad\qquad \leq 2^{s^7 2^{45d!(2k^3)!}}.$$

From Theorem 3, applied to (1.20), and also to (1.21a,b) if $\{u_m\}$ is symmetric, we get in addition a set of cardinality

$$(14.3) \qquad\qquad \leq 2\max\{2^{2^{1080}s^7}, \; 2^{s^7 2^{46d!(2k)!}}\}.$$

As we assume $k > 1$, the sum of the bounds in (14.2), (14.3) does not exceed

$$2^{s^7 2^{46d!(2k^3)!}},$$

and Corollary 1 follows.

Finally, let us deal with Corollary 2. The linear family corresponding to equations (1.20), by Theorem 3 now has the shape $m(t) = t$, $n(t) = t + t_0$ ($t \in \mathbb{Z}$). It implies the polynomial identities

$$f_i(t) \equiv \frac{b}{a} f_i(t + t_0)\alpha_i^{t_0} \qquad (i = 0, \ldots, r).$$

Hence we get $(b/a)\alpha_1^{t_0} = 1$ and the identity $f_1(t) \equiv f_1(t + t_0)$. Now either $\deg f_1 > 0$ and $t_0 = 0$ or $\deg f_1 = 0$. In the latter case, as $k > 1$, we have at least one further root α_i with $i \neq 1$ and $(b/a)\alpha_i^{t_0} = 1$. But then by nondegeneracy $t_0 = 0$, and so in all cases $b/a = 1$, which contradicts our hypothesis.

Now assume that $\{u_m\}$ is symmetric. As above in Corollary 1, we possibly get a one-parameter family $m(t) = wt + t'$, $n(t) = -wt + t''$ $(t \in \mathbb{Z})$ with $0 < w \leq 2d^2$, which induces the polynomial identities in t,

$$
(14.4) \qquad
\begin{aligned}
f_i(wt + t')\alpha_i^{t'} &= \frac{b}{a} f_{i+1}(-wt + t'')\alpha_{i+1}^{t''} \\
f_{i+1}(wt + t')\alpha_{i+1}^{t'} &= \frac{b}{a} f_i(-wt + t'')\alpha_i''
\end{aligned}
\qquad (1 \leq i < r, \; i \text{ odd}).
$$

Write $l = \deg f_i = \deg f_{i+1}$. Let a_i and a_{i+1} be the leading coefficients of f_i and f_{i+1} respectively.

Comparing the leading coefficients in (14.4) we get

$$
a_i w^l \alpha_i^{t'} = (-1)^l \frac{b}{a} a_{i+1} w^l \alpha_{i+1}^{t''}, \quad a_{i+1} w^l \alpha_{i+1}^{t'} = (-1)^l \frac{b}{a} a_i w^l \alpha_i^{t''},
$$

hence $(\alpha_i \alpha_{i+1})^{t' - t''} = (b/a)^2$. It follows that b/a is a root of unity. The estimate (1.41) follows in the same way as (1.36) in Corollary 1.

References

[1] E. Dobrowolski, *On a question of Lehmer and the number of irreducible factors of a polynomial*, Acta Arith. 34 (1979), 391–401.

[2] J. H. Evertse, *On sums of S-units and linear recurrences*, Compositio Math. 53 (1984), 225–244.

[3] M. Laurent, *Équations exponentielles polynômes et suites récurrentes linéaires*, Astérisque 147-148 (1987), 121–139.

[4] —, *Équations exponentielles-polynômes et suites récurrentes linéaires, II*, J. Number Theory 31 (1989), 121–139.

[5] J. H. Loxton and A. J. van der Poorten, *Multiplicative dependence in number fields*, Acta Arith. 42 (1983), 291–302.

[6] G. Pólya und G. Szegö, *Aufgaben und Lehrsätze aus der Analysis, II*, 4. Auflage, Springer, Berlin, 1971.

[7] A. J. van der Poorten and H. P. Schlickewei, *The growth condition for recurrence sequences*, Macquarie Univ. Math. Rep. 82–00041, North Ryde, N.S.W., 1982.

[8] H. P. Schlickewei, *S-unit equations over number fields*, Invent. Math. 102 (1990), 95–107.

[9] —, *Multiplicities of algebraic linear recurrences*, Acta Math. 170 (1993), 151–180.

[10] H. P. Schlickewei and W. M. Schmidt, *On polynomial-exponential equations*, Math. Ann. 296 (1993), 339–361.

[11] —, —, *Linear equations in members of recurrence sequences*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 20 (1993), 219–246.

[12]   W. M. Schmidt, *Diophantine Approximations and Diophantine Equations*, Lecture Notes in Math. 1467, Springer, 1990.

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