A class of transcendental numbers having explicit g-adic and Jacobi–Perron expansions of arbitrary dimension

by

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1. Introduction. Davison [6] showed the identity

$$(1.1) \qquad \psi := \sum_{n=1}^{\infty} 2^{-\lfloor \alpha n \rfloor}$$

$$= [0; 2^0, 2^1, 2^1, 2^2, 2^3, 2^5, 2^8, 2^{13}, \ldots] \qquad (\alpha = (1 + \sqrt{5})/2),$$

and that the number ψ is transcendental. Here, $\lfloor x \rfloor$ denotes the integral part of a real number x, the right-hand side indicates a simple continued fraction, where the powers of 2 appearing in the partial denominators are Fibonacci numbers ([1]). The binary expansion of ψ can be described by the fixed point of a substitution. For this purpose, we introduce some definitions.

K^* denotes the set of all finite words over an alphabet $K = \{a, b, c, \ldots, d\}$, i.e., K^* is the free monoid generated by K with the operation of concatenation and the empty word λ as its unit. K^{∞} denotes the set of all ω-words $w_1 w_2 w_3 \ldots (w_n \in K)$.

A *substitution* σ (over K) is a monoid endomorphism σ on K^* extended to K^{∞}, defined by

$$\sigma(w) = \sigma(w_1)\sigma(w_2)\sigma(w_3) \ldots \qquad \text{for } w = w_1 w_2 w_3 \ldots \in K^{\infty}.$$

A *fixed point* of σ is a ω-word $w \in K^{\infty}$ such that $\sigma(w) = w$. Any substitution σ over K of the form

$$\sigma(a) = au \quad (u \neq \lambda), \qquad \sigma(x) \neq \lambda \quad (\forall x \in K)$$

has the unique fixed point w prefixed by a, namely, $w = au\sigma(u)\sigma^2(u) \ldots$ Here, the product $\tau\sigma$ denotes the composition of τ and σ, and σ^n indicates the n-fold iteration $\sigma^{n-1} \circ \sigma$ $(n \geq 1)$ with $\sigma^0(u) = u$ $(u \in K^* \cup K^{\infty})$.

([1]) Cf. Böhmer [3], Mahler [11], Danilov [5], Adams–Davison [1], Bundschuh [4], Nishioka–Shiokawa–Tamura [13].

The base-2 expansion of the number ψ is given by

$$\psi = 0.1011010110110101101011011010110110\ldots = 0.w_1 w_2 w_3 \ldots,$$

where the word $w = w_1 w_2 w_3 \ldots \in \{0,1\}^\infty$ is the unique fixed point of the substitution σ over $\{0,1\}$ defined by $\sigma(1) = 10$, $\sigma(0) = 1$.

We gave in [19] an analogue of (1.1), where the continued fraction expansion is replaced by the Jacobi–Perron algorithm of dimension 2:

$$\begin{pmatrix} 0.10101011010101010101011010\ldots \\ 0.11101111110111110111111110\ldots \end{pmatrix}$$
$$= \begin{bmatrix} 0; & 2^0, & 2^0, & 2^1, & 2^1, & 2^2, & 2^4, & 2^7, & 2^{13}, & \ldots, & 2^{f_n}, & \ldots \\ 0; & 2^0, & 2^1, & 2^2, & 2^3, & 2^6, & 2^{11}, & 2^{20}, & 2^{37}, & \ldots, & 2^{g_n}, & \ldots \end{bmatrix},$$

where the left-hand side denotes the vector of two real numbers $0.\tau(w)$ and $0.\upsilon(w)$ having the sequence $\tau(w)$ and $\upsilon(w)$ as their digits in the binary expansion respectively with the fixed point $w = abacabaabacab\ldots$ of a substitution over $\{a,b,c\}$ defined by

$$\sigma(a) = ab, \qquad \sigma(b) = ac, \qquad \sigma(c) = a,$$

and the coding

$$\tau(a) := 1, \quad \tau(b) = \tau(c) := 0, \quad \upsilon(a) = \upsilon(b) := 1, \quad \upsilon(c) := 0;$$

the right-hand side denotes the Jacobi–Perron algorithm, and f_n, g_n denote linear recurrence sequences with their characteristic polynomial $x^3 - x^2 - x - 1$.

Instead of the fixed point w, we can state theorems similar to the results in [19, 20] for the fixed point of the substitution σ over $\{a_1, a_2, \ldots, a_{s+1}\}$ ($s \geq 1$) defined by

$$(1.2) \qquad \sigma(a_j) := a_1^{k_{s-j+1}} a_{j+1} \quad (1 \leq j \leq s), \qquad \sigma(a_{s+1}) := a_1$$

with $k_s \geq k_{s-1} \geq \ldots \geq k_1 \geq 1$ ($k_j \in \mathbb{Z}$). Namely, we shall give a vector of s real numbers $\Psi^{(i)}$ ($1 \leq i \leq s$) with its explicit expression in the Jacobi–Perron algorithm of dimension s such that each expansion of $\Psi^{(i)}$ in base g can be described by the fixed point of σ. We shall also show the transcendence, and linear independence of the numbers $\Psi^{(i)}$ in Theorem 1. We have a more general transcendence result in Theorem 3. For certain functions connected with the values $\Psi^{(i)}$, we can show the similar results to those in Theorem 1 in the algorithm defined by Parusnikov, which is the counterpart of the Jacobi–Perron algorithm for functions, see Theorem 2. Theorems 1–3 will be stated in Section 3. For the sake of completeness, the Jacobi–Perron algorithms used in Theorems 1 and 2 will be introduced in Section 2. We shall give the proofs of Theorems 1–3 in Section 4. We shall

describe the fixed point of σ in terms of summation sequences defined in Section 5 (Theorem 4). The identity (1.1) will be found in Section 5 as a consequence of Theorem 1 in the case of $s = 1$.

2. Jacobi–Perron and Jacobi–Perron–Parusnikov algorithms. In this section, we define two kinds of continued fraction expansions of dimension s ($s \geq 1$) due to Jacobi and Parusnikov (cf. [2, 14, 12]). We use the following notation:

$\mathbb{K} = \mathbb{C}((z^{-1}))$, the field of formal Laurent series with complex coefficients. $\mathbb{K}$ is a metric space with the distance function $\|\varPhi - \varPsi\|$ ($\varPhi, \varPsi \in \mathbb{K}$), where $\| \, \|$ is the usual non-archimedian norm defined by $\|\varPhi\| = e^{-k}$ for $\varPhi = \sum_{m=k}^{\infty} c_m z^{-m} \in \mathbb{K}$ with $c_k \neq 0$, $k \in \mathbb{Z}$, and $\|0\| = 0$.

$[\varPhi] :=$ the polynomial part of $\varPhi \in \mathbb{K}$, i.e., $[\varPhi] = \sum_{m=k}^{0} c_m z^{-m}$ for $\varPhi = \sum_{m=k}^{\infty} c_m z^{-m}$.

$\langle \varPhi \rangle := \varPhi - [\varPhi]$.

$[\boldsymbol{\varPhi}] := {}^t([\varPhi_1], \ldots, [\varPhi_s])$, $\langle \boldsymbol{\varPhi} \rangle := {}^t(\langle \varPhi_1 \rangle, \ldots, \langle \varPhi_s \rangle)$ for $\boldsymbol{\varPhi} = {}^t(\varPhi_1, \ldots, \varPhi_s)$ $\in \mathbb{K}^s$, where ${}^t(\varPhi_1, \ldots, \varPhi_s)$ indicates the transpose of $(\varPhi_1, \ldots, \varPhi_s)$.

$T : \mathbb{K}^s \to \mathbb{K}^s$ denotes the map defined by

$$ T({}^t(\varPhi_1, \ldots, \varPhi_s)) := {}^t(1/\varPhi_s, \varPhi_1/\varPhi_s, \ldots, \varPhi_{s-1}/\varPhi_s). $$

To be precise, T is not a map on $\mathbb{K}^s$. For brevity, in what follows, we shall simply write $f : A \to B$ for a "map" f with some exceptional elements $x \in A$ for which f is not defined. We also write

$$ T({}^t(\varPhi_1, \ldots, \varPhi_s)) = \frac{1}{{}^t(\varPhi_1, \ldots, \varPhi_s)}. $$

Now we define, following Parusnikov [14], the Jacobi–Perron algorithm for $\boldsymbol{\varPhi} = {}^t(\varPhi^{(1)}, \ldots, \varPhi^{(s)}) \in \mathbb{K}^s$. Let $\mathbf{b}_0 := [\boldsymbol{\varPhi}]$, $\boldsymbol{\varPhi}_0 = {}^t(\varPhi_0^{(1)}, \ldots, \varPhi_0^{(s)}) :=$ $\langle \boldsymbol{\varPhi} \rangle$. If $\varPhi_0^{(1)} \neq 0$, then, noting that $T^{-1}(\boldsymbol{\varPhi}_0) = {}^t(\varPhi_0^{(2)}/\varPhi_0^{(1)}, \ldots, \varPhi_0^{(s)}/\varPhi_0^{(1)}, 1/\varPhi_0^{(1)})$, we can write

$$ \boldsymbol{\varPhi} = \mathbf{b}_0 + \boldsymbol{\varPhi}_0 = \mathbf{b}_0 + \frac{1}{T^{-1}(\boldsymbol{\varPhi}_0)}. $$

Let $\mathbf{b}_1 := [T^{-1}(\boldsymbol{\varPhi}_0)]$, $\boldsymbol{\varPhi}_1 = {}^t(\varPhi_1^{(1)}, \ldots, \varPhi_1^{(s)}) := \langle T^{-1}(\boldsymbol{\varPhi}_0) \rangle$. If $\varPhi_1^{(1)} \neq 0$, then

$$ \boldsymbol{\varPhi} = \mathbf{b}_0 + \cfrac{1}{\mathbf{b}_1 + \cfrac{1}{T^{-1}(\boldsymbol{\varPhi}_1)}}. $$

Continuing the process, we get

$$(2.1) \qquad \boldsymbol{\Phi} = \mathbf{b}_0 + \cfrac{1}{\mathbf{b}_1 + \cfrac{1}{\mathbf{b}_2 + \cfrac{\ddots}{\quad + \cfrac{1}{\mathbf{b}_m + \boldsymbol{\Phi}_m}}}},$$

provided that $\Phi_n^{(1)} \neq 0$ for all $0 \leq n \leq m - 1$, where

$$\boldsymbol{\Phi}_n := {}^t(\Phi_n^{(1)}, \ldots, \Phi_n^{(s)}) = \langle T^{-1}(S^{n-1}(\langle \boldsymbol{\Phi} \rangle)) \rangle$$

and

$$(2.2) \qquad \mathbf{b}_n := [T^{-1}(S^{n-1}(\langle \boldsymbol{\Phi} \rangle))] \quad (n \geq 1), \qquad \mathbf{b}_0 := [\boldsymbol{\Phi}],$$

and S^n is the n-fold iteration of

$$S(\boldsymbol{\Phi}) := \langle T^{-1}(\boldsymbol{\Phi}) \rangle.$$

If $\Phi_m^{(1)} = 0$, then the algorithm terminates. If $\Phi_n^{(1)} \neq 0$ for all $n \geq 0$, then

$$\boldsymbol{\pi}_n = \boldsymbol{\pi}_n(\boldsymbol{\Phi}) := \mathbf{b}_0 + \cfrac{1}{\mathbf{b}_1 + \cfrac{1}{\mathbf{b}_2 + \cfrac{\ddots}{\quad + \cfrac{1}{\mathbf{b}_n}}}},$$

which will be denoted by $[\mathbf{b}_0; \mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n]$, converges componentwise to $\boldsymbol{\Phi}$ as $n \to \infty$ with respect to the metric induced by the norm $\| \ \|$ (cf. [14]). Hence we can write

$$(2.3) \qquad \boldsymbol{\Phi} = [\mathbf{b}_0; \mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n, \ldots].$$

In what follows, we also write

$$\boldsymbol{\pi}_n = \begin{bmatrix} b_0^{(1)}; & b_1^{(1)}, & b_2^{(1)}, & \ldots, & b_n^{(1)} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ b_0^{(s)}; & b_1^{(s)}, & b_2^{(s)}, & \ldots, & b_n^{(s)} \end{bmatrix},$$

$$\boldsymbol{\Phi} = \begin{bmatrix} b_0^{(1)}; & b_1^{(1)}, & b_2^{(1)}, & \ldots, & b_n^{(1)}, & \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ b_0^{(s)}; & b_1^{(s)}, & b_2^{(s)}, & \ldots, & b_n^{(s)}, & \ldots \end{bmatrix}$$

for $\mathbf{b}_n = {}^t(b_n^{(1)}, \ldots, b_n^{(s)})$. The algorithm given by (2.2) will be called the *Jacobi–Perron–Parusnikov* (abbr. JPP) *algorithm* (of dimension s). Independently of the algorithm (2.2), we can consider the expression (2.3) for a given sequence $\mathbf{b}_n \in \mathbb{K}^s$ provided that its nth convergent $\boldsymbol{\pi}_n$ is well-defined

except for a finite number of n and converges to some element in $\mathbb{K}^s$. The JPP expression (2.3) will be called *admissible* if it is derived from the algorithm (2.2). The admissible expression satisfies

$$
(2.4) \qquad
\begin{aligned}
b_n^{(i)} &\in \mathbb{C}[z] \quad (1 \le i \le s,\ n \ge 0), \\
\deg b_n^{(i)} &< \deg b_n^{(s)} \quad (1 \le i < s,\ n \ge 1),
\end{aligned}
$$

where $\deg 0 := -\infty < 0$.

If we take the field $\mathbb{R}$, and the integral part $\lfloor \Phi \rfloor$ of $\Phi \in \mathbb{R}$ instead of $\mathbb{K}$, and the polynomial part $[\Phi]$ of $\Phi \in \mathbb{K}$, respectively, we have an algorithm which is the simplest one among the Jacobi–Perron algorithms (cf. [2], p. 49). This algorithm will be simply referred to as the *Jacobi–Perron* (abbr. JP) *algorithm*. When (2.3) is admissible in the JP algorithm, we have

$$
(2.5) \qquad
\begin{aligned}
b_n^{(i)} &\in \mathbb{Z} \quad (1 \le i \le s,\ n \ge 0), \\
0 \le b_n^{(i)} &\le b_n^{(s)} \neq 0 \quad (1 \le i < s,\ n \ge 1).
\end{aligned}
$$

3. Main results. In what follows, s, k_i $(0 \le i \le s)$ denote fixed integers with $s \ge 1$, $1 = k_0 \le k_1 \le \ldots \le k_s$, and σ indicates the substitution over $K := \{a_1, a_2, \ldots, a_{s+1}\}$ defined by (1.2). We denote by $\omega = \omega_1 \omega_2 \ldots \omega_n \ldots \in K^\infty$ $(\omega_n \in K)$ the fixed point of σ, and $\{f_n\}_{n=-s,-s+1,-s+2,\ldots}$ stands for the linear recurrence sequence with a characteristic polynomial $f(x) := x^{s+1} - k_s x^s - \ldots - k_1 x - k_0$ and an initial condition given by

$$
(3.1) \qquad
\begin{pmatrix} f_0 \\ f_{-1} \\ \cdot \\ \cdot \\ \cdot \\ f_{-s} \end{pmatrix}
:=
\begin{pmatrix}
1 & & & & \\
k_1 & 1 & & 0 & \\
k_2 & k_1 & \ddots & & \\
\vdots & \vdots & & 1 & \\
k_s & k_{s-1} & \ldots & k_1 & 1
\end{pmatrix}^{-1}
\begin{pmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix}.
$$

THEOREM 1. *Let* $\Psi^{(i)} = \Psi^{(i)}(g)$ $(1 \le i \le s)$ *be the numbers defined by the JP expression*

$$
(3.2) \qquad
\begin{pmatrix} \Psi^{(1)} \\ \Psi^{(2)} \\ \cdot \\ \cdot \\ \cdot \\ \Psi^{(s-1)} \\ \Psi^{(s)} \end{pmatrix}
:=
\begin{bmatrix}
0; & 1, & 1, & 1, & \ldots, & 1, & b_{s+1}^{(1)}, & \ldots \\
0; & 1, & 1, & 1, & \ldots, & b_s^{(2)}, & b_{s+1}^{(2)}, & \ldots \\
\cdot & \cdot & \cdot & \cdot & \ldots, & b_s^{(3)}, & b_{s+1}^{(3)}, & \ldots \\
\cdot & \cdot & \cdot & 1, & \ldots, & \cdot & \cdot & \ldots \\
\cdot & 1, & 1, & 1, & \ldots, & \cdot & \cdot & \ldots \\
0; & 1, & 1, & b_3^{(s-1)}, & \ldots, & b_s^{(s-1)}, & b_{s+1}^{(s-1)}, & \ldots \\
0; & 1, & b_2^{(s)}, & b_3^{(s)}, & \ldots, & b_s^{(s)}, & b_{s+1}^{(s)}, & \ldots
\end{bmatrix},
$$

where

$$b_n^{(1)} := g^{f_{n-s-1}} \sum_{h=0}^{k_1-1} g^{f_{n-s}h}, \quad n \geq s+1,$$

$$b_n^{(2)} := g^{f_{n-s-1}+k_1 f_{n-s}} \sum_{h=0}^{k_2-1} g^{f_{n-s+1}h}, \quad n \geq s,$$

(3.3) ....

$$b_n^{(s-1)} := g^{f_{n-s-1}+k_1 f_{n-s}+\ldots+k_{s-2} f_{n-3}} \sum_{h=0}^{k_{s-1}-1} g^{f_{n-2}h}, \quad n \geq 3,$$

$$b_n^{(s)} := g^{f_{n-s-1}+k_1 f_{n-s}+\ldots+k_{s-2} f_{n-3}+k_{s-1} f_{n-2}} \sum_{h=0}^{k_s-1} g^{f_{n-1}h}, \quad n \geq 2,$$

with $g \in \mathbb{N}$. Then we have the following assertions:

(i) *If $g \geq 2$, then g-adic expansion of $\Psi^{(i)}(g)$ is given by*

$$\Psi^{(i)}(g) = 0.\tau_i(\omega) \quad (1 \leq i \leq s),$$

where τ_i are codings defined by

$$\tau_i(a_j) = g-1 \quad (1 \leq j \leq i), \quad \tau_i(a_j) = 0 \quad (i+1 \leq j \leq s+1).$$

(ii) *If $g \geq 2$, then $\Psi^{(i)}(g)$ $(1 \leq i \leq s)$ is a transcendental number.*

(iii) *The $s+1$ numbers $1, \Psi^{(i)}(g)$ $(1 \leq i \leq s)$ are linearly independent over $\mathbb{Q}$ for all $g \in \mathbb{N}$.*

(iv) *The right-hand side of (3.2) is an admissible expression in the JP algorithm.*

R e m a r k 1. $\Psi^{(i)}(1)$ $(1 \leq i \leq s)$ are numbers belonging to a field $\mathbb{Q}(\alpha)$, where $\alpha > 1$ is an algebraic number with $f(x)$ as its minimal polynomial (see Lemmas 10 and 11).

THEOREM 2. *Let $\Phi^{(i)} = \Phi^{(i)}(z)$ be the analytic functions on the region $|z| > 1$ defined by*

$$\Phi^{(i)} := \sum_{n \in \mathbb{N}, \, \omega_n = a_i} z^{-n} \quad (1 \leq i \leq s+1),$$

and let $\Xi^{(i)} = \Xi^{(i)}(z)$ be the analytic functions defined by

$$\Xi^{(i)} := \sum_{j=2}^{i+1} \Phi^{(j)}/\Phi^{(1)} \quad (1 \leq i \leq s).$$

Then we have the following statements:

(i) *The $\Phi^{(i)}$ $(1 \leq i \leq s+1)$ are transcendental functions over the field $\mathbb{C}(z)$.*

(ii) *The $\Phi^{(i)}$ $(1 \le i \le s+1) \in \mathbb{K}$ are linearly independent over $\mathbb{C}(z)$.*

(iii) *The admissible expression in the JPP algorithm for ${}^t(\Xi^{(1)}, \Xi^{(2)}, \ldots \ldots, \Xi^{(s)})$ is given by*

$$
(3.4) \quad
\begin{bmatrix}
0; & 1, & 1, & \ldots, & 1, & b_{s+1}^{(1)}, & b_{s+2}^{(1)}, & \ldots \\
0; & 1, & 1, & \ldots, & b_s^{(2)}, & b_{s+1}^{(2)}, & b_{s+2}^{(2)}, & \ldots \\
\cdot & \cdot & \cdot & \ldots, & b_s^{(3)}, & b_{s+1}^{(3)}, & b_{s+2}^{(3)}, & \ldots \\
\cdot & \cdot & 1, & \ldots, & \cdot & \cdot & \cdot & \ldots \\
\cdot & 1, & 1, & \ldots, & \cdot & \cdot & \cdot & \ldots \\
0; & 1, & b_3^{(s-1)}, & \ldots, & b_s^{(s-1)}, & b_{s+1}^{(s-1)}, & b_{s+2}^{(s-1)}, & \ldots \\
0; & b_2^{(s)}, & b_3^{(s)}, & \ldots, & b_s^{(s)}, & b_{s+1}^{(s)}, & b_{s+2}^{(s)}, & \ldots
\end{bmatrix},
$$

where $b_j^{(i)} = b_j^{(i)}(z)$ are polynomials of z in place of g in (3.3).

R e m a r k 2. If in the expression (3.2) g is replaced by a variable z, then (3.2) changes into a JPP expression, while it is not admissible in the JPP algorithm.

We denote by $|w|_x$ $(w \in K^*,\ x \in K)$ the number of occurrences of the symbol x in the word w.

THEOREM 3. *Let $g \ge 2$ be an integer and let τ be a morphism*

$$\tau(a_i) \in \{0, 1, \ldots, g-1\}^* - \{\lambda\} \quad (1 \le i \le s+1)$$

such that

$$\operatorname{rank} U(\tau) \ge 2, \quad U(\tau) := (|\tau(a_i)|_j)_{1 \le i \le s+1,\, 0 \le j \le g-1}.$$

Then the number defined by the g-adic expansion $0.\tau(\omega)$ is transcendental.

4. The proofs of the main results. We first prove Theorem 1(i) and Theorem 2(iii) except for the admissibility of (3.4), which will be shown in the last paragraph of the section.

Let us denote by B_n and P_n the matrices defined by

$$
(4.1) \quad
B_n = \left(\begin{array}{c|c} {}^t\mathbf{0}_s & 1 \\ \hline E_s & \mathbf{b}_n \end{array} \right) \in \mathrm{SL}(s+1; \mathbb{K}),
$$

$$
P_n = B_1 B_2 \ldots B_n \quad (n \ge 1), \quad P_0 = E,
$$

for given $\mathbf{b}_n = {}^t(b_n^{(1)}, \ldots, b_n^{(s)}) \in \mathbb{K}^s$ $(n \ge 0)$, where E_j denotes the $j \times j$ unit matrix, E stands for E_{s+1}, and $\mathbf{0}_j = {}^t(0, \ldots, 0) \in \mathbb{K}^j$. We can set

$$(4.2) \quad P_n = (p_j^{(i)})_{1 \le i \le s+1,\, n-s \le j \le n} \in \mathrm{SL}(s+1; \mathbb{K}).$$

Then we have the following well-known formula in Lemma 1 below (cf. [2], Chap. 1, §3). For the sake of completeness, we give a short proof of Lemma 1 following Nikishin and Sorokin [12].

LEMMA 1. *Let $\mathbf{\Phi} \in \mathbb{K}^s$ have the expression (2.1) with $\mathbf{\Phi}_m = {}^t(\Phi_m^{(1)}, \ldots$*
$\ldots, \Phi_m^{(s)})$, $\mathbf{b}_n = {}^t(b_n^{(1)}, \ldots, b_n^{(s)}) \in \mathbb{K}^s$ $(0 \le n \le m)$ such that $\mathbf{b}_0 = \mathbf{0}_s$. Then

$$
\mathbf{\Phi} = \begin{pmatrix}
\left(p_m^{(1)} + \sum_{j=1}^{s} \Phi_m^{(j)} p_{m-s+j-1}^{(1)}\right) \Big/ \left(p_m^{(s+1)} + \sum_{j=1}^{s} \Phi_m^{(j)} p_{m-s+j-1}^{(s+1)}\right) \\
\left(p_m^{(2)} + \sum_{j=1}^{s} \Phi_m^{(j)} p_{m-s+j-1}^{(2)}\right) \Big/ \left(p_m^{(s+1)} + \sum_{j=1}^{s} \Phi_m^{(j)} p_{m-s+j-1}^{(s+1)}\right) \\
\vdots \\
\left(p_m^{(s)} + \sum_{j=1}^{s} \Phi_m^{(j)} p_{m-s+j-1}^{(s)}\right) \Big/ \left(p_m^{(s+1)} + \sum_{j=1}^{s} \Phi_m^{(j)} p_{m-s+j-1}^{(s+1)}\right)
\end{pmatrix},
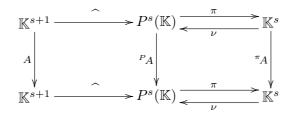$$

in particular,

$$
\boldsymbol{\pi}_m(\mathbf{\Phi}) = {}^t(p_m^{(1)}/p_m^{(s+1)}, \ldots, p_m^{(s)}/p_m^{(s+1)}) \quad (m \ge 0).
$$

P r o o f. Let $P^s(\mathbb{K}) = (\mathbb{K}^{s+1} \setminus \{\mathbf{0}_{s+1}\})/\sim$ be the s-dimensional projective space over $\mathbb{K}$ as usual. We denote by $\mathbf{\Phi}\hat{\ }$ the element of $P^s(\mathbb{K})$ which contains $\mathbf{\Phi} \in \mathbb{K}^{s+1}$. By $\nu : \mathbb{K}^s \to P^s(\mathbb{K})$ and $\pi : P^s(\mathbb{K}) \to \mathbb{K}^s$ we denote the inclusion map and the projection defined by

$$
\nu({}^t(\Phi_1, \ldots, \Phi_s)) := {}^t(1, \Phi_1, \ldots, \Phi_s)\hat{\ },
$$
$$
\pi({}^t(\Phi_0, \Phi_1, \ldots, \Phi_s)\hat{\ }) := {}^t(\Phi_1/\Phi_0, \ldots, \Phi_s/\Phi_0),
$$

respectively. $A : \mathbb{K}^{s+1} \to \mathbb{K}^{s+1}$ indicates the linear map over $\mathbb{K}$ for a given matrix $A \in \mathrm{GL}(s+1; \mathbb{K})$ as usual. Then we can define the maps ${}^P A$ and ${}^\pi A$ which make the following diagram commutative:

$$
\begin{array}{ccccc}
\mathbb{K}^{s+1} & \xrightarrow{\hat{\ }} & P^s(\mathbb{K}) & \underset{\nu}{\overset{\pi}{\rightleftarrows}} & \mathbb{K}^s \\
\downarrow{\scriptstyle A} & & \downarrow{\scriptstyle {}^P A} & & \downarrow{\scriptstyle {}^\pi A} \\
\mathbb{K}^{s+1} & \xrightarrow{\hat{\ }} & P^s(\mathbb{K}) & \underset{\nu}{\overset{\pi}{\rightleftarrows}} & \mathbb{K}^s
\end{array}
$$

Then we get ${}^\pi B(\boldsymbol{\theta}) = \pi^P B \nu(\boldsymbol{\theta}) := \pi({}^P B(\nu(\boldsymbol{\theta}))) = \mathbf{b} + T(\boldsymbol{\theta})$ $(\boldsymbol{\theta}, \mathbf{b} \in \mathbb{K}^s)$ for

$$
B = \left(\begin{array}{c|c} {}^t\mathbf{0}_s & 1 \\ \hline E_s & \mathbf{b} \end{array}\right) \in \mathrm{SL}(s+1; \mathbb{K}).
$$

Therefore, by the commutativity of the diagram, it follows from (2.1) that $\mathbf{\Phi} = {}^\pi B_0 {}^\pi B_1 \ldots {}^\pi B_{m-1}(\mathbf{b}_m + \mathbf{\Phi}_m) = {}^\pi(B_0 P_{m-1})(\mathbf{b}_m + \mathbf{\Phi}_m)$, which implies the lemma. ∎

LEMMA 2. *Let* $\{f_n^{(i)}\}_{n=-s,-s+1,-s+2,\ldots}$ $(1 \leq i \leq s+1)$ *be linear recurrence sequences with the $f(x)$ as their characteristic polynomial with initial conditions*

$$(4.3) \quad \begin{pmatrix} f_{-s}^{(1)} & \cdots & f_{-1}^{(1)} & f_0^{(1)} \\ f_{-s}^{(2)} & \cdots & f_{-1}^{(2)} & f_0^{(2)} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ f_{-s}^{(s+1)} & \cdots & f_{-1}^{(s+1)} & f_0^{(s+1)} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & & \\ k_1 & 1 & & 0 & \\ k_2 & k_1 & \ddots & & \\ \vdots & \vdots & & 1 & \\ k_s & k_{s-1} & \cdot & \cdot & k_1 & 1 \end{pmatrix}^{-1} .$$

Then the equalities $f_n^{(i)} = |\sigma^{n-1}(a_1)|_{a_i}$ $(1 \leq i \leq s+1)$ hold for all $n \in \mathbb{N}$.

P r o o f. We put

$$u_n^{(i)} := \sigma^{n-1}(a_i) \in K^+ := K^* - \{\lambda\} \quad (1 \leq i \leq s+1), \quad u_n := u_n^{(1)},$$

$$g_n^{(i)} := |u_n|_{a_i} \quad (1 \leq i \leq s+1).$$

Then we have by definition

$$(4.4) \qquad \begin{aligned} u_n^{(i)} &= u_{n-1}^{(1)k_{s-i+1}} u_{n-1}^{(i+1)} \quad (1 \leq i \leq s), \quad n \geq 2, \\ u_n^{(s+1)} &= u_{n-1}^{(1)}, \quad n \geq 2. \end{aligned}$$

Hence, we get

$$(4.5) \qquad u_n = u_{n-1}^{k_s} u_{n-2}^{k_{s-1}} \ldots u_{n-s}^{k_1} u_{n-s-1}, \quad n \geq s+2.$$

Taking $|\ |_{a_i}$ of both sides of this equality, we see that $\{g_n^{(i)}\}_{n=1,2,\ldots}$ $(1 \leq i \leq s+1)$ are linear recurrence sequences with $f(x)$ as their characteristic polynomial.

We shall check the initial conditions of the sequence $g_n^{(i)}$. In view of (1.2), we have

$$(4.6) \qquad \begin{aligned} u_1 &= a_1, \\ u_2 &= u_1^{k_s} a_2, \\ u_3 &= u_2^{k_s} u_1^{k_{s-1}} a_3, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ u_{s+1} &= u_s^{k_s} u_{s-1}^{k_{s-1}} \ldots u_1^{k_1} a_{s+1}. \end{aligned}$$

We set

$$
A := \begin{pmatrix} 0 & \cdot & \cdot & \cdot & 0 & 1 \\ & & & & & k_1 \\ & & & & & k_2 \\ & & E_s & & & \vdots \\ & & & & & k_s \end{pmatrix},
$$

$$
W_j := \begin{pmatrix} E_{j-1} & & & & & 0 \\ & 1 & & & & \\ & -k_1 & 1 & & 0 & \\ & \vdots & & 0 & & \ddots \\ 0 & & & & & \\ & -k_{s-j+1} & & & & 1 \end{pmatrix} \in \mathrm{SL}(s+1; \mathbb{Z}),
$$

for $1 \le j \le s+1$ with $W_{s+1} = E_{s+1}$, and

$$
G_k := (g_j^{(i)})_{1 \le i \le k,\, 1 \le j \le k} \qquad (1 \le k \le s+1).
$$

Then we can prove the equality

$$
(4.7) \qquad W_m W_{m-1} \dots W_1 A^m = \left(\begin{array}{c|c} 0 & G_m \\ \hline E_{s-m+1} & 0 \end{array} \right) \qquad (1 \le m \le s+1)
$$

by induction on m. In fact, (4.7) holds for $m = 1$. Suppose that (4.7) is valid for $m = k$ $(1 \le k \le s)$. Then we have, by (4.6),

$$
W_{k+1} W_k \dots W_1 A^{k+1} = W_{k+1} \left(\begin{array}{c|c} 0 & G_k \\ \hline E_{s-k+1} & 0 \end{array} \right) A = \left(\begin{array}{c|c} 0 & G_{k+1} \\ \hline E_{s-k} & 0 \end{array} \right).
$$

Hence, (4.7) holds. In particular,

$$
(4.8) \qquad G_{s+1} = W_{s+1} W_s W_{s-1} \dots W_1 A^{s+1} = W_s W_{s-1} \dots W_1 A^{s+1}.
$$

On the other hand, we can prove by induction on m that

$$
W^{-1} W_s W_{s-1} \dots W_{s-m+1} = \left(\begin{array}{c|c} W^{(s-m)} & 0 \\ & E_{m+1} \end{array} \right) \qquad (1 \le m \le s),
$$

where W is the matrix on the right-hand side of (4.3), and $W^{(l)}$ denotes the $(s+1) \times l$ matrix defined by

$$
W^{(l)} := (w_{ij})_{1 \le i \le s+1,\, 1 \le j \le l} \quad \text{with } (w_{ij})_{1 \le i \le s+1,\, 1 \le j \le s+1} := W^{-1}.
$$

In particular, setting $m = s$, we have

$$
W^{-1} W_s W_{s-1} \dots W_1 = E_{s+1}.
$$

Therefore, in view of (4.8), we get

$$
(g_j^{(i)})_{1 \le i \le s+1,\, n-s \le j \le n} = W A^n \qquad (n \ge s+1),
$$

which implies the lemma. ∎

In what follows, ${}^tS$ denotes the transpose of a matrix S. The cross-transposed matrix ${}^\times S$ of S is defined by ${}^\times S := (s_{m-i+1,n-j+1})_{1\le i\le m, 1\le j\le n}$ for a matrix $S = (s_{i,j})_{1\le i\le m, 1\le j\le n}$. We remark that ${}^\times(ST) = {}^\times S {}^\times T$, ${}^\times({}^tS) = {}^t({}^\times S)$, ${}^\times E = E$, and so ${}^\times({}^t(S^{-1})) = ({}^\times({}^tS))^{-1}$ for $S \in \mathrm{GL}(n, \mathbb{K})$.

LEMMA 3. $|\sigma^{n-1}(a_1)| = f_n \ (n \ge 1)$.

P r o o f. Taking the transpose and then cross-transpose of both sides of (4.3), and taking the product with ${}^t(1, 1, \ldots, 1) \in \mathbb{Z}^{s+1}$ from the right, we obtain the lemma by (3.1), noting that $f_n = \sum_{i=1}^{s+1} f_n^{(i)}$. ∎

Throughout this paper, $\chi(w; x)$ denotes the characteristic set of a given word $w = w_1 w_2 \ldots w_n \ldots \in K^* \cup K^\infty$ $(w_n \in K)$ with respect to $x \in K$, i.e.,

$$\chi(w; x) := \{n \in \mathbb{N} : w_n = x\}.$$

We define the polynomials $p_n^{(i)} = p_n^{(i)}(z) \in \mathbb{Z}[z]$ by

$$(4.9) \qquad p_n^{(i)} := \sum_{m \in \chi(\sigma^{n-1}(a_1); a_i)} z^m \qquad (1 \le i \le s+1, \ n \ge 1).$$

We put

$$(4.10) \qquad q_n^{(i)} = q_n^{(i)}(z) := z^{f_n} p_n^{(i)}(z^{-1}) \qquad (1 \le i \le s+1, \ n \ge 1).$$

$q_n^{(i)}$ is clearly a polynomial in z by Lemma 3. The mirror image ${}^R u$ of a given word $u = u_1 u_2 \ldots u_j \ldots u_k \in K^*$ $(u_j \in K)$ is defined by

$$ {}^R u := u_k u_{k-1} \ldots u_j \ldots u_1, \qquad {}^R\lambda := \lambda.$$

For brevity, we write ${}^R u^k := ({}^R u)^k = {}^R(u^k)$ $(u \in K^*, \ k \ge 0)$. We denote by ${}^k(h)$ the rational function defined by

$${}^k(h) := 1 + h + \ldots + h^{k-1} \qquad (k \ge 1)$$

for a function h with $0 \ne h \in \mathbb{Q}(z)$.

In view of (4.9) and (4.10), we have

$$(4.11) \qquad q_n^{(i)} := z^{-1} \sum_{m \in \chi({}^R(\sigma^{n-1}(a_1)); a_i)} z^m \qquad (1 \le i \le s+1, \ n \ge 1).$$

Hence, noting that

$${}^R u_n = {}^R u_{n-s-1} {}^R u_{n-s}^{k_1} \ldots {}^R u_{n-2}^{k_{s-1}} {}^R u_{n-1}^{k_s} \qquad (n \ge s+2),$$

which follows from (4.5), we obtain by Lemma 3 the following

LEMMA 4.

$$q_n^{(i)} = q_{n-s-1}^{(i)} + \sum_{j=1}^{s} b_n^{(j)} q_{n-s+j-1}^{(i)} \qquad (n \ge s+2),$$

where $b_n^{(i)} = b_n^{(i)}(z)$ $(1 \le i \le s, \ n \ge s+2)$ *are polynomials of z defined by*

$$b_n^{(1)} := {}^{k_1}\!\big(z^{f_{n-s}}\big) \cdot z^{f_{n-s-1}},$$

$$b_n^{(2)} := {}^{k_2}\!\big(z^{f_{n-s+1}}\big) \cdot z^{k_1 f_{n-s}+f_{n-s-1}},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$b_n^{(s-1)} := {}^{k_{s-1}}\!\big(z^{f_{n-2}}\big) \cdot z^{k_{s-2} f_{n-3}+\dots+k_1 f_{n-s}+f_{n-s-1}},$$

$$b_n^{(s)} := {}^{k_s}\!\big(z^{f_{n-1}}\big) \cdot z^{k_{s-1} f_{n-2}+k_{s-2} f_{n-3}+\dots+k_1 f_{n-s}+f_{n-s-1}}.$$

R e m a r k 3. $b_n^{(i)}$ is a polynomial in z when $n \geq s-i+2$, while, in general, it is not a polynomial in z when $1 \leq n \leq s-i+1$.

LEMMA 5. *Let Q_n be the matrix defined by*

$$(4.12) \qquad Q_n := (q_{n-s+j-1}^{(i)})_{1 \leq i \leq s+1,\, 1 \leq j \leq s+1} \qquad (n \geq s+1).$$

Then we have

$$Q_{s+1} = C_1 C_2 \dots C_{s+1},$$

where $C_j \in \mathrm{SL}(s+1; \mathbb{Z}[z])$ $(1 \leq j \leq s+1)$ are defined by

$$C_j := \left(\begin{array}{c|c|c} & b_j^{(s-j+2)} & \\ & b_j^{(s-j+3)} & \\ E_{j-1} & \vdots & 0 \\ & b_j^{(s)} & \\ \cline{1-2} & 1 & \\ & 0 & \\ & \vdots & \cline{2-3} \\ 0 & \vdots & E_{s-j+1} \\ & 0 & \end{array}\right) \qquad (2 \leq j \leq s),$$

$$C_1 = C_{s+1} := E_{s+1}.$$

P r o o f. In view of (3.1), we have

$$b_2^{(s)} = {}^{k_s}\!\big(z^{f_1}\big) \cdot z^{k_{s-1} f_0 + k_{s-2} f_{-1} + \dots + k_1 f_{-s+2} + f_{-s+1}} = {}^{k_s}\!\big(z^{f_1}\big) \cdot z;$$

similarly,

$$b_3^{(s-1)} = {}^{k_{s-1}}\!\big(z^{f_1}\big) \cdot z,$$

$$b_3^{(s)} = {}^{k_s}\!\big(z^{f_2}\big) \cdot z^{k_{s-1} f_1 + 1};$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$b_s^{(2)} = {}^{k_2}\!\big(z^{f_1}\big) \cdot z,$$

$$b_s^{(3)} = {}^{k_3}\!\big(z^{f_2}\big) \cdot z^{k_2 f_1 + 1},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$b_s^{(s)} = {}^{k_s}\!\big(z^{f_{s-1}}\big) \cdot z^{k_{s-1} f_{s-2} + k_{s-2} f_{s-3} + \dots + k_2 f_1 + 1};$$

$$b_{s+1}^{(1)} = {}^{k_1}(z^{f_1}) \cdot z,$$
$$b_{s+1}^{(2)} = {}^{k_2}(z^{f_2}) \cdot z^{k_1 f_1 + 1},$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$b_{s+1}^{(s)} = {}^{k_s}(z^{f_s}) \cdot z^{k_{s-1}f_{s-1} + k_{s-2}f_{s-2} + \dots + k_1 f_1 + 1}.$$

Hence, we get, by (4.10) and the mirror images of (4.6),

$$\mathbf{q}_1 = \mathbf{e}_1,$$
$$\mathbf{q}_2 = \mathbf{e}_2 + b_2^{(s)}\mathbf{q}_1,$$
$$\mathbf{q}_3 = \mathbf{e}_3 + b_3^{(s-1)}\mathbf{q}_1 + b_3^{(s)}\mathbf{q}_2,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\mathbf{q}_s = \mathbf{e}_s + b_s^{(2)}\mathbf{q}_1 + b_s^{(3)}\mathbf{q}_2 + \dots + b_s^{(s)}\mathbf{q}_{s-1},$$
$$\mathbf{q}_{s+1} = \mathbf{e}_{s+1} + b_{s+1}^{(1)}\mathbf{q}_1 + b_{s+1}^{(2)}\mathbf{q}_2 + \dots + b_{s+1}^{(s)}\mathbf{q}_s,$$

where $\mathbf{q}_j$ and $\mathbf{e}_j$ are the vectors defined by

$$\mathbf{q}_j := {}^t(q_j^{(1)}, q_j^{(2)}, \dots, q_j^{(s+1)}),$$
$$\mathbf{e}_j := {}^t(0, \dots, 0, \underset{\underset{j\text{th component}}{\uparrow}}{1}, 0, \dots, 0) \in (\mathbb{Z}[z])^{s+1} \quad (1 \le j \le s+1).$$

Therefore, we obtain Lemma 5. ∎

LEMMA 6. *Let $D \in \mathrm{SL}(s+1; \mathbb{Q}(z))$ be the matrix defined by*

$$D := \begin{pmatrix} 1 & & & & & \\ b_1^{(1)} & 1 & & & & \\ b_1^{(2)} & b_2^{(1)} & & & 0 & \\ \cdot & b_2^{(2)} & 1 & & & \\ \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & 1 & \\ b_1^{(s)} & b_2^{(s-1)} & \cdot & \cdot & b_s^{(1)} & 1 \end{pmatrix},$$

and let B_n ($n \ge 1$) be the matrices defined by (4.1) with $\mathbf{b}_n = {}^t(b_n^{(1)}, \dots, b_n^{(s)})$, where $b_n^{(i)}$ ($1 \le i \le s+1$, $n \ge 1$) are the rational functions of z given in Lemma 4. Then the identity $DQ_{s+1} = B_1 B_2 \dots B_{s+1}$ holds, where Q_{s+1} is the matrix given by (4.12).

Proof. Let C_j ($1 \le j \le s+1$) be as in Lemma 5, and let D_j ($1 \le j \le s+1$) be given by

$$D_j := B_{s-j+2} B_{s-j+3} \dots B_{s+1} C_{s+1}^{-1} C_s^{-1} \dots C_{s-j+2}^{-1}.$$

Denote by F_j $(1 \le j \le s+1)$ the matrix

$$F_j := \begin{pmatrix} 1 & & & & \\ b_{s-j+2}^{(1)} & 1 & & & \\ b_{s-j+2}^{(2)} & b_{s-j+3}^{(1)} & & 0 & \\ . & b_{s-j+3}^{(2)} & . & & \\ . & . & . & & \\ . & . & . & 1 & \\ b_{s-j+2}^{(j-1)} & b_{s-j+3}^{(j-2)} & . & b_s^{(1)} & 1 \end{pmatrix} \in \mathrm{SL}(j; \mathbb{Q}[z]),$$

with $F_1 = E_1$. Then we can prove

(4.13) $$D_j = \left(\begin{array}{c|c} 0 & F_j \\ \hline E_{s-j+1} & 0 \end{array} \right) \quad (1 \le j \le s+1)$$

by induction on j. In fact, (4.13) holds for $j = 1$. We suppose that (4.13) is valid when $j = k$ with $1 \le k \le s$. Then we have

$$B_{s-k+1} D_k C_{s-k+1} = \left(\begin{array}{c|c} & \begin{matrix} 1 \\ b_{s-k+1}^{(1)} \\ b_{s-k+1}^{(2)} \end{matrix} & F_k \\ 0 & & \\ \hline E_{s-k} & \begin{matrix} . \\ . \\ b_{s-k+1}^{(s)} \end{matrix} & 0 \end{array} \right) C_{s-k+1}^{-1} = D_{k+1}.$$

Hence, (4.13) is valid. In particular, we have $D = D_{s+1} = B_1 B_2 \ldots$ $\ldots B_{s+1} C_{s+1}^{-1} C_s^{-1} \ldots C_1^{-1}$. In view of Lemma 5, we get Lemma 6. ∎

LEMMA 7. *Let* D, B_n, *and* Q_n *be as above, and let* $R_n \in \mathrm{SL}(s+1; \mathbb{Q}(z))$ $(n \ge 0)$ *be the matrices defined by*

(4.14) $$R_n := B_1 B_2 \ldots B_n \quad (n \ge 1), \qquad R_0 := E_{s+1}.$$

Then $Q_n = D^{-1} R_n$ $(n \ge s+1)$.

P r o o f. It follows from (4.14) and Lemma 6 that $D^{-1} R_n = D^{-1} B_1 B_2 \ldots$ $\ldots B_{s+1} \ldots B_n = Q_{s+1} B_{s+2} \ldots B_n$ $(n \ge s+1)$, which equals Q_n, since Lemma 4 together with (4.12) implies $Q_m B_{m+1} = Q_{m+1}$ $(m \ge s+1)$. ∎

LEMMA 8. *Let* $\Delta \in \mathrm{SL}(s+1; \mathbb{Z})$ *and* $R_n^* \in \mathrm{SL}(s+1; \mathbb{Z}[z])$ $(n \ge 0)$ *be defined by*

$$\Delta := \begin{pmatrix} 1 & & & & \\ 1 & 1 & & 0 & \\ . & . & . & & \\ . & . & . & . & \\ 1 & 1 & . & . & 1 \end{pmatrix},$$

$$R_n^* := B_1^* B_2^* \ldots B_n^* \quad (n \ge 1), \qquad R_0^* := E_{s+1}.$$

Then we have

$$R_n^* = \Delta Q_n \qquad (\forall n \geq s+1),$$

where Q_n is the matrix (4.12), and $B_n^* \in \mathrm{SL}(s+1; \mathbb{Z}[z])$ $(n \geq 1)$ denotes the matrix defined by

$$B_n^* := \begin{pmatrix} 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 1 \\ 1 & & & & & & & \vdots \\ & 1 & & & 0 & & & \\ & & \cdot & & & & & 1 \\ & & & \cdot & & & & b_n^{(s-n+2)} \\ & & & & \cdot & & & b_n^{(s-n+3)} \\ & & 0 & & & 1 & & \cdot \\ & & & & & & 1 & b_n^{(s)} \end{pmatrix} \qquad (2 \leq n \leq s)$$

with

$$B_1^* := \left(\begin{array}{c|c} 0 \dots 0 & 1 \\ \hline & 1 \\ E_s & \vdots \\ & 1 \end{array} \right),$$

and $B_n^* := B_n$ $(n \geq s+1)$.

Proof. In view of Lemma 7, it is sufficient to show that the identity is valid when $n = s+1$ for the proof of this lemma. Let $Y_n \in \mathrm{SL}(s+1; \mathbb{Z}[z])$ $(0 \leq n \leq s)$ be the matrix defined by

$$Y_n := \left(\begin{array}{c|c} X_n & 0 \\ \hline 0 & E_n \end{array} \right)$$

with

$$X_n := (x_{ij})_{n+1 \leq i \leq s+1, \, n+1 \leq j \leq s+1},$$

where $X := X_0 = (x_{ij})_{1 \leq i \leq s+1, 1 \leq j \leq s+1} \in \mathrm{SL}(s+1; \mathbb{Z}[z])$ denotes the matrix

$$\begin{pmatrix} 1 & & & & & & \\ b_1^{(1)} - 1 & 1 & & & & 0 & \\ b_1^{(2)} - b_2^{(1)} & b_2^{(1)} - 1 & 1 & & & & \\ b_1^{(3)} - b_2^{(2)} & b_2^{(2)} - b_3^{(1)} & b_3^{(1)} - 1 & & \cdot & & \\ \cdot & \cdot & \cdot & & & \cdot & \\ \cdot & \cdot & \cdot & & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & & & 1 \\ b_1^{(s)} - b_2^{(s-1)} & b_2^{(s-1)} - b_3^{(s-2)} & b_3^{(s-2)} - b_4^{(s-3)} & \cdot & \cdot & b_s^{(1)} - 1 & 1 \end{pmatrix}.$$

In particular, $Y_0 = X$ and $Y_s = E_{s+1}$. Then we have

(4.15) $$Y_n B_{n+1}^* = B_{n+1} Y_{n+1} \qquad (0 \leq n \leq s)$$

by direct calculation. Hence we get

$$D\Delta^{-1}B_1^*B_2^*\dots B_s^* = Y_0 B_1^* B_2^* \dots B_s^* = B_1 Y_1 B_2^* \dots B_s^*$$
$$= B_1 B_2 Y_2 B_3^* \dots B_s^* = \dots = B_1 B_2 \dots B_s Y_s$$
$$= B_1 B_2 \dots B_s,$$

so that we obtain, by Lemma 6,

$$R_{s+1}^* = \Delta D^{-1} D\Delta^{-1} B_1^* B_2^* \dots B_s^* B_{s+1}^*$$
$$= \Delta D^{-1} B_1 B_2 \dots B_s B_{s+1} = \Delta Q_{s+1}. \quad \blacksquare$$

We can write

$$R_n^* = (r_{n-s+j-1}^{*(i)})_{1\le i\le s+1,\, 1\le j\le s+1} \quad (n \ge 0)$$

by the definition of R_n^*. Then we have the following

LEMMA 9.

$$(4.16) \qquad r_n^{*(i)} = \sum_{k=1}^{i} q_n^{(k)}, \quad \forall n \ge 1 \ (1 \le i \le s+1).$$

Proof. It follows from Lemma 8 that (4.16) holds for $n \ge s+1$. We assume $1 \le n \le s$. Then we have by (4.15)

$$R_n^* = B_1^* B_2^* \dots B_n^* = \Delta D^{-1} D\Delta^{-1} B_1^* B_2^* \dots B_n^*$$
$$= \Delta D^{-1} Y_0 B_1^* B_2^* \dots B_n^* = \Delta D^{-1} B_1 B_2 \dots B_n Y_n.$$

Hence, we get, by Lemma 6,

$$R_n^* = \Delta Q_{s+1} B_{s+1}^{-1} B_s^{-1} \dots B_1^{-1} B_1 B_2 \dots B_n Y_n$$
$$= \Delta Q_{s+1} B_{s+1}^{-1} B_s^{-1} \dots B_{n+1}^{-1} Y_n.$$

Let $\mathbf{e}_j$ be the jth fundamental vector as in the proof of Lemma 5. Multiplying by the vector $\mathbf{e}_{s+1}$, we obtain

$$R_n^* \mathbf{e}_{s+1} = \Delta Q_{s+1} B_{s+1}^{-1} B_s^{-1} \dots B_{n+3}^{-1} B_{n+2}^{-1} B_{n+1}^{-1} Y_n \mathbf{e}_{s+1}$$
$$= \Delta Q_{s+1} B_{s+1}^{-1} B_s^{-1} \dots B_{n+3}^{-1} B_{n+2}^{-1} B_{n+1}^{-1} \mathbf{e}_{s+1}$$
$$= \Delta Q_{s+1} B_{s+1}^{-1} B_s^{-1} \dots B_{n+3}^{-1} B_{n+2}^{-1} \mathbf{e}_s$$
$$= \Delta Q_{s+1} B_{s+1}^{-1} B_s^{-1} \dots B_{n+3}^{-1} \mathbf{e}_{s-1}$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$= \Delta Q_{s+1} B_{s+1}^{-1} \mathbf{e}_{n+1} = \Delta Q_{s+1} \mathbf{e}_n,$$

noting that

$$B_j^{-1} = \left(\begin{array}{c|c} -\mathbf{b}_j & E_s \\ \hline 1 & {}^t\mathbf{0}_s \end{array} \right).$$

This implies that the identity (4.16) is valid when $1 \le n \le s$. $\blacksquare$

Proof of Theorem 1(i). In view of (4.10) with (4.9), we obtain, by Lemma 9,

$$(4.17) \qquad r_n^{*(i)} = r_n^{*(i)}(z) = z^{f_n} \cdot \sum_{m \in \chi(\sigma^{n-1}(a_1); a_1, \ldots, a_i)} z^{-m} \in \mathbb{Z}[z]$$

for all $n \geq 1$, where $\chi(w; x_1, \ldots, x_i)$ stands for the set defined by

$$\chi(w; x_1, \ldots, x_i) := \bigcup_{j=1}^{i} \chi(w; x_j)$$

for a given word $w \in K^* \cup K^\infty$ with $x_j \in K$. In particular, we get by Lemma 3

$$r_n^{*(s+1)}(z) = z^{f_n} \cdot \sum_{m=1}^{f_n} z^{-m} = {}^{f_n}(z).$$

Therefore, in view of Lemma 1, we obtain

$$(4.18) \qquad \mathbf{\Psi}_n = \mathbf{\Psi}_n(z) = [\mathbf{0}; \mathbf{b}_1^*, \mathbf{b}_2^*, \ldots, \mathbf{b}_n^*] \qquad (n \geq 1),$$

where

$$(4.19) \qquad \mathbf{b}_n^* := \begin{cases} {}^t(\underbrace{1, \ldots, 1}_{s-n+1}, b_n^{(s-n+2)}, b_n^{(s-n+3)}, \ldots, b_n^{(s)}) \in (\mathbb{Z}[z])^s \\ \hspace{6cm} (1 \leq n \leq s), \\ {}^t(b_n^{(1)}, b_n^{(2)}, \ldots, b_n^{(s)}) \in (\mathbb{Z}[z])^s \qquad (n \geq s+1) \end{cases}$$

and

$$(4.20) \qquad \mathbf{\Psi}_n := \frac{z-1}{1-z^{-f_n}} \cdot \begin{pmatrix} \displaystyle\sum_{m \in \chi(\sigma^{n-1}(a_1); a_1)} z^{-m} \\ \displaystyle\sum_{m \in \chi(\sigma^{n-1}(a_1); a_1, a_2)} z^{-m} \\ \vdots \\ \displaystyle\sum_{m \in \chi(\sigma^{n-1}(a_1); a_1, \ldots, a_s)} z^{-m} \end{pmatrix}.$$

Choosing $z = g \in \mathbb{N}$ with $g \geq 2$ and letting $n \to \infty$, we get Theorem 1(i). ∎

We put $\mathbf{\Psi} = \mathbf{\Psi}(g) := [\mathbf{0}; \mathbf{b}_1^*, \ldots, \mathbf{b}_n^*, \ldots]$, which equals (3.2) that converges not only for $g \geq 2$ but also for $g = 1$ (see Remark 4).

The expression (3.4) in Theorem 2(iii) is obtained by applying the transformation T to (4.18), and subtracting $\mathbf{b}_1^* = {}^t(1, 1, \ldots, 1)$ from both sides. It follows from the convergence of (4.20) as $n \to \infty$ that (3.4) converges in $\mathbb{K}^s$ with respect to the metric induced by the norm $\| \ \|$.

We shall prove the transcendence results in Theorems 1–3. We need the following lemmas (Lemmas 10–14).

LEMMA 10. *The characteristic polynomial $f(x)$ of $\{f_n\}_{n=-s,-s+1,-s+2,\ldots}$ is the minimal polynomial of a Pisot number α $(k_s < \alpha < k_s + 1)$, i.e.,*

(i) $f(x)$ *is an irreducible polynomial over $\mathbb{Q}$,*

(ii) *there exists only one real number α satisfying $f(\alpha) = 0$ with $\alpha > 1$,*

(iii) $|\beta| < 1$ *for every algebraic conjugate β $(\neq \alpha)$ of α.*

P r o o f. We have $f(k_s + 1) > 0$, since

$$f(k_s + 1) \geq (k_s + 1)^{s+1}\left(1 - k_s \sum_{j=1}^{s+1}(k_s + 1)^{-j}\right) > 0.$$

It is clear that $f(k_s) < 0$. Hence, $f(x)$ has at least one real root in the interval $(k_s, k_s + 1)$. Let α be one of them, and let

$$f(x)/(x - \alpha) =: \sum_{j=1}^{s+1} c_j x^{j-1} \in \mathbb{R}[x].$$

Then we have

(4.21) $c_n = \alpha^{s-n+1} - \displaystyle\sum_{j=n}^{s} k_j \alpha^{j-n} = \sum_{j=1}^{n} k_{n-j}\alpha^{-j}, \quad 1 \leq n \leq s+1.$

Hence we get

$$1 = c_{s+1} > c_s > \ldots > c_1 > 0,$$

by applying the fact that $1 = \sum_{j=1}^{s+1} k_{s-j+1}\alpha^{-j}$, $\alpha > 0$, and the conditions $k_s \geq \ldots \geq k_1 \geq k_0 = 1$. Therefore, Kakeya's theorem implies that the equation $f(x)/(x - \alpha) = 0$ has its solutions only inside the unit circle $|x| < 1$ (cf. Kakeya [10], Hurwitz [9]). Hence, α is a simple root of $f(x)$.

Now, we show the irreducibility of $f(x)$. Let us consider the expansion in the JP algorithm for the vector $\mathbf{c} := {}^t(c_1, \ldots, c_s) \in \mathbb{R}^s$. Put $\mathbf{k} := {}^t(k_1, \ldots, k_s)$. Then, in view of (4.21), we get $T^{-1}(\mathbf{c}) = \mathbf{k} + \mathbf{c}$. Hence, we obtain a periodic expansion

(4.22) $\mathbf{c} = [\mathbf{0}; \overset{*}{\mathbf{k}}] := [\mathbf{0}; \mathbf{k}, \mathbf{k}, \ldots].$

Denote by $V_n \in \mathrm{SL}(s+1; \mathbb{Z})$ the matrix

$$V_n := \begin{pmatrix} {}^t\mathbf{0}_s & 1 \\ E_s & \mathbf{k} \end{pmatrix}^n \quad (n \geq 1), \quad V_0 := E.$$

Write $V_n = (v^{(i)}_{n-s+j-1})_{1 \leq i \leq s+1, 1 \leq j \leq s+1}$. The sequences

$$\{v^{(i)}_n\}_{n=-s,-s+1,-s+2,\ldots} \quad (1 \leq i \leq s+1)$$

satisfy the linear recurrence relation

$$(4.23) \qquad v_n^{(i)} = \sum_{j=0}^{s} k_j v_{n-s+j-1}^{(i)}, \qquad n \geq 1,$$

with the initial conditions given by $V_0 = E$. Since α is a simple root of $f(x)$, we can write, for suitable $\theta_j^{(i)} \in \mathbb{R}$ ($1 \leq i \leq s+1$, $0 \leq j \leq s$),

$$(4.24) \qquad v_n^{(i)} = \theta_0^{(i)} \alpha^n + \sum_{j=1}^{s} \theta_j^{(i)} t_j^{(i)}(n), \qquad \theta_0^{(i)} > 0,$$

where $t_j^{(i)}(n)$ ($1 \leq i \leq s+1$, $0 \leq j \leq s$) are functions of n of the form

$$n^k \alpha_l^n \quad \text{with } 1 > |\alpha_l| > 0 \ (0 \neq l = l(i,j) \leq s, \ k = k(i,j) \in \mathbb{Z}^+),$$

and α_l ($0 \leq l \leq s$, $\alpha_0 = \alpha$) are the roots of $f(x)$ counted with multiplicity. In view of (4.22), we have $\mathbf{c} = T((V_{n-1})_\pi (\mathbf{k} + \mathbf{c}))$ ($n \geq 1$). Hence, by Lemma 1 and (4.23) we get

$$c_i = \sum_{j=0}^{s} c_{j+1} v_{n-s+j}^{(i)} \Big/ \sum_{j=0}^{s} c_{j+1} v_{n-s+j}^{(s+1)},$$

so that

$$v_n^{(s+1)} c_i - v_n^{(i)} = \frac{\sum_{j=1}^{s+1} c_j (v_n^{(s+1)} v_{n-s+j-1}^{(i)} - v_n^{(i)} v_{n-s+j-1}^{(s+1)})}{\sum_{j=1}^{s+1} c_j v_{n-s+j-1}^{(s+1)}}.$$

Here, by (4.24) we get

$$v_n^{(s+1)} v_{n-s+j-1}^{(i)} - v_n^{(i)} v_{n-s+j-1}^{(s+1)} = o(\alpha^n).$$

Hence, we obtain

$$(4.25) \qquad v_n^{(s+1)} c_i - v_n^{(i)} = o(1),$$

where the o-constant is independent of n. It follows from (4.25) that the right-hand side of (4.22) converges to $\mathbf{c}$. Now, we suppose that $1, c_1, \ldots, c_s$ are linearly dependent over $\mathbb{Q}$. Then we have

$$\sum_{i=1}^{s+1} x_i c_i = 0, \qquad \mathbf{0} \neq {}^t(x_1, \ldots, x_{s+1}) \in \mathbb{Z}^{s+1}.$$

This relation together with (4.25) implies that there exists $m \in \mathbb{N}$, independent of n, such that

$$\sum_{i=1}^{s+1} x_i v_n^{(i)} = 0 \qquad (\forall n \geq m),$$

which contradicts the fact that $\det V_n \neq 0$ ($V_n \in \mathrm{SL}(s+1; \mathbb{Z})$). Therefore, $1, c_1, \ldots, c_s$ are linearly independent. In view of (4.21), the numbers

$1/\alpha^n$ $(1 \leq n \leq s+1)$ are linearly independent over $\mathbb{Q}$, from which the irreducibility of the polynomial $f(x)$ follows. This completes the proof of Lemma 10. ∎

LEMMA 11. *Let* $\boldsymbol{\alpha} \in (\mathbb{Q}(\alpha))^s$ *be the vector defined by*

$$\begin{pmatrix} \alpha^s/(\alpha^s + \alpha^{s-1} + \ldots + 1) \\ (\alpha^s + \alpha^{s-1})/(\alpha^s + \alpha^{s-1} + \ldots + 1) \\ \vdots \\ (\alpha^s + \alpha^{s-1} + \ldots + \alpha)/(\alpha^s + \alpha^{s-1} + \ldots + 1) \end{pmatrix}.$$

Then $\boldsymbol{\alpha}$ *has the expression*

(4.26) $$\boldsymbol{\alpha} = [\mathbf{0}; \mathbf{b}_1^*(1), \ldots, \mathbf{b}_n^*(1), \ldots] \quad (= \boldsymbol{\Psi}(1)),$$

which is admissible in the JP algorithm.

Proof. It follows from the convergence of (4.22) that the JP expression in (4.26) converges. By direct calculation, we can check that the identity (4.26) holds. From (4.21),

(4.27) $$0 < [\mathbf{0}; \mathbf{b}_n^*(1), \mathbf{b}_{n+1}^*(1), \mathbf{b}_{n+2}^*(1), \ldots] < 1 \quad (\forall n \geq 1)$$

follows, which implies the admissibility of (4.26). ∎

Proof of Theorem 2(i). By Lemma 11 together with (4.17) and (4.18), we get, for each $1 \leq i \leq s+1$,

$$\lim_{n\to\infty} |\sigma^n(a_1)|_{a_i}/|\sigma^n(a_1)| = \alpha^{s-i+1}/(\alpha^s + \alpha^{s-1} + \ldots + \alpha + 1),$$

which is irrational by Lemma 10. Hence, the infinite words $\varepsilon_i(\omega)$ $(1 \leq i \leq s+1)$ are not ultimately periodic, where ε_i is the coding defined by $\varepsilon_i(a_j) = \delta_{ij}$ $(1 \leq j \leq s+1)$ with $\delta_{ii} = 1$, $\delta_{ij} = 0$ $(i \neq j)$. On the other hand, $\varepsilon_i(\omega)$ is the sequence of coefficients of the Laurent series of the function $\Phi^{(i)}(z)$, which is bounded. Therefore, the functions $\Phi^{(i)}(z)$ $(1 \leq i \leq s+1)$ are not rational. Hence, the assertion (i) is a consequence of the following lemma. ∎

LEMMA 12 (Fatou). *If a power series with integral coefficients represents an algebraic function that is not a rational function, then its radius of convergence is smaller than one* (see pp. 368–371 in Fatou [7], or p. 139, #167 in Pólya and Szegö [15]).

Let K be an alphabet. For given words $u = u_1 u_2 \ldots \in K^*$ and $v = v_1 v_2 \ldots \in K^* \cup K^\infty$ $(u_i, v_j \in K)$, we denote by uv the word $u_1 u_2 \ldots v_1 v_2 \ldots \in K^* \cup K^\infty$. If $w = uv$ $(u \in K^*, v \in K^* \cup K^\infty)$, then u and v will be called a *prefix* and a *suffix* of w, respectively. A word u satisfying $w = tuv$ $(t \in K^*, uv \in K^* \cup K^\infty)$ will be referred to as a *subword* of w, where $uv := u$ when $u \in K^\infty$ and $v = \lambda$. We denote by $e(u, v)$ $(u, v \in K^* \cup K^\infty)$ the number

defined by

$$e(u,v) := \inf\{n \in \mathbb{N} : u_n \neq v_n\} - 1 \quad (u \neq v), \qquad e(u,u) := \infty$$

for $u = u_1 u_2 \ldots$, $v = v_1 v_2 \ldots$ $(u_i, v_j \in K)$. ($K^* \cup K^\infty$ becomes a complete metric space with the metric $d(u,v) := 1/(e(u,v)+1)$.)

Let σ be the substitution (1.2) over K, ω be its fixed point as in Section 3, and let J be the triple $J := (K, \sigma, a)$. (J is a D0L system, cf. [16, 17].) u^* indicates the infinite periodic word $uuu\ldots$ for a given word $u \in K^+ := K^* - \{\lambda\}$. We denote by $\varrho = \varrho(J)$ the positive number defined by

$$(4.28) \qquad \varrho := \limsup_{n \to \infty} \frac{e(\omega, (\sigma^n(a))^*)}{|\sigma^n(a)|}.$$

LEMMA 13. *Let α be the number as in Lemma* 10. *Then we have*

$$\varrho(J) \geq \begin{cases} k_s + \alpha^{-1} & \text{if } k_s \geq 2, \\ 2 + 1/(\alpha^{s+1} - 1) & \text{if } k_s = 1. \end{cases}$$

P r o o f. We write $v \neg w$ $(v, w \in K^* \cup K^\infty)$ if w is a prefix of v. The binary relation $\neg$ is transitive. In view of (4.5), we have

$$\omega \neg u_{n+1} \neg u_n^{k_s} u_{n-1} \quad (n \geq s+1), \qquad u_n \neg u_m \quad (n \geq m \geq 1).$$

Hence, we get

$$\varrho \geq k_s + \lim_{n \to \infty} |u_{n-1}|/|u_n| = k_s + \alpha^{-1},$$

by Lemmas 3 and 10. If $k_s = 1$, then $k_j = 1$ $(1 \leq j \leq s+1)$ and

$$u_n = u_{n-1} u_{n-2} \ldots u_{n-s-1} \quad (n \geq s+2)$$

by (4.5). Then, we have

$\omega \neg u_{n+2}$

$\quad = u_{n+1} u_n u_{n-1} \ldots u_{n-s+1}$

$\quad = u_n u_{n-1} \ldots u_{n-s} u_n u_{n-1} \ldots u_{n-s+1}$

$\quad \neg u_n u_{n-1} \ldots u_{n-s} u_{n-s} u_{n-s-1}$

$\quad = u_n u_{n-1} \ldots u_{n-s} u_{n-s-1} u_{n-s-2} \ldots u_{n-2s-1} u_{n-s-1}$

$\quad = u_n u_n u_{n-s-2} \ldots u_{n-2s-1} u_{n-s-1}$

$\quad \neg u_n u_n u_{n-s-2} \ldots u_{n-2s-1} u_{n-2s-1} u_{n-2s-2}$

$\quad = u_n u_n u_{n-s-2} \ldots u_{n-2s-1} u_{n-2s-2} u_{n-2s-3} \ldots u_{n-3s-2} u_{n-2s-2}$

$\quad = u_n u_n u_{n-s-1} u_{n-2s-3} \ldots u_{n-3s-2} u_{n-2s-2}$

$\quad \neg u_n u_n u_{n-s-1} u_{n-2s-3} \ldots u_{n-3s-2} u_{n-3s-2} u_{n-3s-3}$

$\quad = u_n u_n u_{n-s-1} u_{n-2s-3} \ldots u_{n-3s-2} u_{n-3s-3} u_{n-3s-4} \ldots u_{n-4s-3} u_{n-3s-3}$

$\quad = u_n u_n u_{n-s-1} u_{n-2s-2} u_{n-3s-4} \ldots u_{n-4s-3} u_{n-3s-3}$

$\quad \neg u_n u_n u_{n-s-1} u_{n-2s-2} u_{n-3s-4} \ldots u_{n-4s-3} u_{n-4s-3} u_{n-4s-4}$

$$= u_n u_n u_{n-s-1} u_{n-2s-2} u_{n-3s-4} \ldots u_{n-4s-3} u_{n-4s-4} u_{n-4s-5} \ldots$$

$$\ldots u_{n-5s-4} u_{n-4s-4}$$

$$= u_n u_n u_{n-s-1} u_{n-2s-2} u_{n-3s-3} u_{n-4s-5} \ldots u_{n-5s-4} u_{n-4s-4}$$

$$\neg\ u_n u_n u_{n-s-1} u_{n-2s-2} u_{n-3s-3} u_{n-4s-5} \ldots u_{n-5s-4} u_{n-5s-4} u_{n-5s-5}.$$

Repeating the above process, we see that $u_n u_n u_{n-(s+1)} u_{n-2(s+1)} \cdots$ $\ldots u_{n-j(s+1)}$ is a prefix of ω when $n \geq (j+2)(s+1)$. On the other hand, we have

$$u_n \neg\ u_{n-s-1} u_{n-s-2} \neg\ u_{n-s-1} u_{n-2s-2} u_{n-2s-3}$$

$$\neg\ u_{n-s-1} u_{n-2s-2} u_{n-3s-3} u_{n-3s-4}$$

$$\neg\ \ldots \neg\ u_{n-(s+1)} u_{n-2(s+1)} \cdots u_{n-j(s+1)}$$

for $n \geq j(s+1) + 1$. Therefore, we obtain

$$\varrho \geq 2 + \lim_{n\to\infty} |u_{n-(s+1)} u_{n-2(s+1)} \cdots u_{n-j(s+1)}|/|u_n|$$

$$= 2 + (\alpha^{-(s+1)} + \alpha^{-2(s+1)} + \ldots + \alpha^{-j(s+1)}).$$

Since j can be taken arbitrarily large, we get the lemma. ■

Proof of Theorem 1(ii). We assume that $2 \leq g \in \mathbb{N}$, and put

$$\Psi_n^{(i)} = \Psi_n^{(i)}(g) = s_n^{(i)}/t_n^{(i)} \quad \text{with } s_n^{(i)}, t_n^{(i)} \in \mathbb{N}, \ \gcd(s_n^{(i)}, t_n^{(i)}) = 1,$$

where $\Psi_n^{(i)}$ is the ith component of the $\boldsymbol{\Psi}_n$. Then $t_n^{(i)}$ is a divisor of the number $g^{f_n} - 1$, by (4.20). Hence, we have $1 \leq t_n^{(i)} < g^{f_n}$. We also get by (4.20) the following expression in the g-adic expansion:

$$\Psi_n^{(i)} = 0.\tau_i(u_n)^*$$

with a period $\tau_i(u_n)$, where τ_i is the coding as in Theorem 1. Thus, in view of the expression of $\Psi^{(i)}$ and $\Psi_n^{(i)}$ in the g-adic expansion, we see that for any $\varepsilon > 0$ the inequality

$$(4.29) \qquad |\Psi^{(i)} - s_n^{(i)}/t_n^{(i)}| < g^{-(\varrho-\varepsilon)f_n} \quad (1 \leq i \leq s)$$

holds for infinitely many n, where $\varrho = \varrho(J)$ denotes the number (4.28). Hence, the same is true for the inequality

$$(4.30) \qquad |\Psi^{(i)} - s_n^{(i)}/t_n^{(i)}| < t_n^{(i)^{-(\varrho-\varepsilon)}} \quad (1 \leq i \leq s).$$

On the other hand, $\Psi^{(i)} = 0.\tau_i(\omega)$ in base g, and Lemma 11 implies

$$\lim_{n\to\infty} \frac{|\tau_i(\sigma^n(a_1))|_{g-1}}{|\tau(\sigma^n(a_1))|} = \frac{\alpha^s + \alpha^{s-1} + \ldots + \alpha^{s-i+1}}{\alpha^s + \alpha^{s-1} + \ldots + \alpha + 1}$$

which is irrational for $1 \leq i \leq s$ by Lemma 10, so that the number $\Psi^{(i)}$ ($1 \leq i \leq s$) is irrational. Thus, the inequality (4.30) is valid for infinitely many rational numbers $s_n^{(i)}/t_n^{(i)}$, because of the fact that $\varrho > 0$ and $s_n^{(i)}/t_n^{(i)}$ tends

to $\Psi^{(i)}$ as $n \to \infty$ by (4.29). Therefore, by Lemma 13 we obtain

$$\mu(\Psi^{(i)}) \geq \varrho > 2 \quad (1 \leq i \leq s),$$

where $\mu(\Psi)$ denotes the irrationality measure of a number Ψ. Hence, Roth's theorem leads to the assertion (ii) of Theorem 1. ■

Proof of Theorem 3. Let τ be as in Theorem 3. Using Lemma 13, we can show that $\mu(0.\tau(\omega)) > 2$ in the same manner as in the proof of Theorem 1(ii), noting that $0.\tau(\omega)$ is irrational by the following lemma. ■

LEMMA 14. *Under the same notations as in Theorem 3, $\tau(\omega) \in \{0, 1, \ldots$* *$\ldots, g-1\}^\infty$ is not an ultimately periodic word when* $\operatorname{rank} U(\tau) \geq 2$.

Proof. Put

$$\tau(\omega) = t_1 t_2 \ldots t_l \ldots, \quad t_l \in \Lambda := \{0, 1, \ldots, g-1\},$$

and

$$N = N(n) := \sum_{i=1}^{s+1} f_n^{(i)} |u_i|, \quad u_i := \tau(a_i),$$

where $\{f_n^{(i)}\}_{-s,-s+1,-s+2,\ldots}$ denotes the sequence defined in Lemma 2. Then we have

$$|t_1 t_2 \ldots t_N|_j = \sum_{i=1}^{s+1} f_n^{(i)} |u_i|_j \quad (j \in \Lambda)$$

by Lemma 2. Hence by (4.27) we get

$$\lim_{n \to \infty} \frac{|t_1 t_2 \ldots t_{N(n)}|_j}{N(n)} = \sum_{i=1}^{s+1} \alpha^{s-i+1} |u_i|_j \Big/ \sum_{i=1}^{s+1} \alpha^{s-i+1} |u_i| = \xi_j,$$

say. It follows from Lemma 10 that the quantity ξ_j is rational if and only if the vector ${}^t(|u_1|_j, |u_2|_j, \ldots, |u_{s+1}|_j)$ is linearly dependent on the vector ${}^t(|u_1|, |u_2|, \ldots, |u_{s+1}|)$. Hence, it is easily seen that ξ_j is irrational for some j under the condition $\operatorname{rank} U \geq 2$, since $\sum_{j=0}^{g-1} |u_i|_j = |u_i|$. Thus, the lemma follows. ■

Proof of Theorem 1(iii). The expression (4.26) shows that Theorem 1(iii) is valid when $g = 1$, since the quantities α^i ($0 \leq i \leq s$) are linearly independent by Lemma 10. We assume that $g \geq 2$. It follows from (4.17) and (4.29) that for any $\varepsilon > 0$,

$$|\Psi^{(i)} - r_n^{*(i)}/r_n^{*(s+1)}| < r_n^{*(s+1)^{-(\varrho-\varepsilon)}} \quad (1 \leq i \leq s)$$

holds for infinitely many n, since $1 + (g-1)r_n^{*(s+1)} = g^{f_n}$. Hence, by Lemma 13, we can find a positive number γ such that

$$(4.31) \qquad |\Psi^{(i)} r_n^{*(s+1)} - r_n^{*(i)}| < r_n^{*(s+1)^{-\gamma}} \quad (1 \leq i \leq s)$$

for infinitely many n. Now, we suppose that a numbers $\Psi^{(1)}, \Psi^{(2)}, \ldots$
$\ldots, \Psi^{(s)}, \Psi^{(s+1)} (= 1)$ are linearly dependent over $\mathbb{Q}$. Then by (4.31) we
have

$$-\sum_{i=1}^{s+1} x_i r_n^{*(i)} = r_n^{*(s+1)} \sum_{i=1}^{s+1} x_i \Psi^{(i)} - \sum_{i=1}^{s+1} x_i r_n^{*(i)}$$

$$= \sum_{i=1}^{s+1} x_i (\Psi^{(i)} r_n^{*(s+1)} - r_n^{*(i)}) = o(1)$$

for suitable integers x_i with $(x_1, x_2, \ldots, x_{s+1}) \neq (0, 0, \ldots, 0) \in \mathbb{Z}^{s+1}$. Hence
$\sum_{i=1}^{s+1} x_i r_n^{*(i)} = 0$ for all sufficiently large n, since $r_n^{*(i)} \in \mathbb{Z}$, and so $\det R_n^* = 0$ ($\forall n \gg 1$), which contradicts the fact $R_n^* \in \mathrm{SL}(s+1; \mathbb{Z})$. This completes
the proof of Theorem 1(iii). ■

R e m a r k 4. In the inequality (4.31), we can take $\gamma = \varrho - 1 - \varepsilon$ with
$\varrho > 2$ when $g \geq 2$, and $\varrho = 1 - \log(\max_{1 \leq i \leq s} |\alpha_i|)/\log \alpha > 1$ when $g = 1$,
where $\varepsilon > 0$ and the α_i are the algebraic conjugates of α with $\alpha_0 = \alpha$.

P r o o f   o f   T h e o r e m 2(ii). The linear independence result in Theo-
rem 2 can be proved in the parallel manner to the proof of Theorem 1. From
the inequality

$$\|\Psi^{(i)}(z) - r_n^{*(i)}(z)/r_n^{*(s+1)}(z)\| < e^{-(\varrho - \varepsilon)f_n}$$

with $\varepsilon > 0$, $n > n_0(\varepsilon)$, $1 \leq i \leq s$, we have

$$\left\| \sum_{i=1}^{s+1} x_i(z) r_n^{*(i)}(z) \right\| = o(1)$$

for a suitable fixed vector ${}^t(x_1, x_2, \ldots, x_{s+1}) \in (\mathbb{C}[z])^{s+1} \setminus \{\mathbf{0}_{s+1}\}$ under
the hypothesis that the functions $\Psi^{(1)}(z), \Psi^{(2)}(z), \ldots, \Psi^{(s)}(z), \Psi^{(s+1)}(z) = 1$
are linearly dependent over $\mathbb{C}(z)$. This leads to a contradiction. Therefore,
noting the identity $\Psi^{(i)}(z) = (z-1) \sum_{j=1}^{i} \Phi^{(j)}(z)$, we get Theorem 2(ii). ■

It remains to prove the admissibility results. We have already shown that
the expression

$$\Xi = \Xi(z) = [\mathbf{0}; \mathbf{b}_2^*(z), \mathbf{b}_3^*(z), \ldots, \mathbf{b}_n^*(z), \ldots]$$

converges in the JPP algorithm with $\Xi := {}^t(\Xi^{(1)}, \Xi^{(2)}, \ldots, \Xi^{(s)})$, where $\Xi^{(i)}$
is the function given in Theorem 2, and so does the expression

$$\Psi = \Psi(g) = [\mathbf{0}; \mathbf{b}_1^*(g), \mathbf{b}_2^*(g), \ldots, \mathbf{b}_n^*(g), \ldots]$$

in the JP algorithm with $g \in \mathbb{N}$. Recall that $\mathbf{b}_n^*(z) \in (\mathbb{Z}[z])^s$ and $\mathbf{b}_n^*(g) \in \mathbb{N}^s$. Thus, for the proof of the admissibility results, it is enough to show
$\|[\mathbf{0}; \mathbf{b}_n^*(z), \mathbf{b}_n^*(z), \ldots]\| < 1$ ($n \geq 2$), and $0 < [\mathbf{0}; \mathbf{b}_n^*(g), \mathbf{b}_n^*(g), \ldots] < 1$
($n \geq 1$).

P r o o f o f T h e o r e m 2(iii). We set

$$_m\boldsymbol{\Psi} = {}_m\boldsymbol{\Psi}(z) := [\mathbf{0}; \mathbf{b}_m^*, \mathbf{b}_{m+1}^*, \ldots] \quad (m \geq 1),$$
$$_m\boldsymbol{\Psi}_n = {}_m\boldsymbol{\Psi}(z)_n := [\mathbf{0}; \mathbf{b}_m^*, \mathbf{b}_{m+1}^*, \ldots, \mathbf{b}_n^*] \quad (1 \leq m \leq n)$$

with

$$^t({}_m\varPsi^{(1)}, {}_m\varPsi^{(2)}, \ldots, {}_m\varPsi^{(s)}) := {}_m\boldsymbol{\Psi}, \qquad {}_m\varPsi^{(0)} := 0,$$
$$^t({}_m\varPsi_n^{(1)}, {}_m\varPsi_n^{(2)}, \ldots, {}_m\varPsi_n^{(s)}) := {}_m\boldsymbol{\Psi}_n.$$

We use the following lemma due to Parusnikov [14].

LEMMA 15 (Parusnikov). *Let* $\mathbf{b}_j := {}^t(b_j^{(1)}, b_j^{(2)}, \ldots, b_j^{(s)}) \in (\mathbb{C}[z])^s$ $(1 \leq j \leq k)$ *be vectors satisfying*

$$\deg b_j^{(i)} < \deg b_j^{(s)} \neq 0 \quad \text{for } i \neq s \text{ and } 1 \leq j \leq k \ (\deg 0 := -\infty),$$

and let $P_n = (p_j^{(i)})_{1 \leq i \leq s+1, n-s \leq j \leq n} \in \mathrm{SL}(s+1; \mathbb{C}[z])$ $(0 \leq n \leq k)$ *be the matrices defined by* (4.1). *Then*

$$\deg p_n^{(1)} = \sum_{j=2}^n \deg b_j^{(s)} \quad (1 \leq n \leq k), \quad \deg p_1^{(1)} = 0,$$

$$\deg p_n^{(i)} = \deg b_1^{(i-1)} + \sum_{j=2}^n \deg b_j^{(s)} \quad (2 \leq i \leq s+1, \ 1 \leq n \leq k).$$

P r o o f. By induction on n. ∎

It follows from (3.1) and (4.19) with the definition (3.3) of $b_n^{(i)}$ that the inequalities $0 \leq \deg b_n^{*(i)} < \deg b_n^{*(s)}$ hold for all $n \geq 2$ and $1 \leq i \leq s-1$. Therefore, we obtain

$$\|_m\varPsi_n^{(i)}(z)\| < e^{-1} \quad (2 \leq m \leq n, \ 1 \leq i \leq s)$$

by Lemmas 1 and 15. Letting $n \to \infty$, we get

$$\|_m\varPsi^{(i)}(z)\| < 1 \quad (2 \leq m, \ 1 \leq i \leq s),$$

from which the admissibility of the expression (3.4) in the JPP algorithm follows, noting that $_m\varPsi_n^{(i)}(z)$ converges as $n \to \infty$ in $\mathbb{K}$. ∎

For the proof of Theorem 1(iv) we need the following lemma, which corresponds to Lemma 15.

LEMMA 16. *Let* $\mathbf{b}_j := {}^t(b_j^{(1)}, b_j^{(2)}, \ldots, b_j^{(s)}) \in \mathbb{Z}^s$ $(1 \leq j \leq h)$ *be vectors satisfying*

$$1 \leq b_j^{(i)} \leq b_j^{(h)} \quad \text{for } 1 \leq i \leq h \leq s \text{ and } 1 \leq j \leq h,$$

and let $P_n = (p_j^{(i)})_{1 \leq i \leq s+1, n-s \leq j \leq n} \in \mathrm{SL}(s+1; \mathbb{Z})$ $(1 \leq n \leq h)$ *be the matrices defined by* (4.1). *Then*

$$(4.32) \qquad \begin{aligned} p_n^{(i)} &\leq p_n^{(l)} \quad \textit{for } 1 \leq i \leq l \leq s \textit{ and } 1 \leq n \leq h, \\ p_m^{(i)} &\leq p_n^{(i)} \quad \textit{for } 1 \leq m \leq n \leq h \textit{ and } 1 \leq i \leq s. \end{aligned}$$

P r o o f (by induction on n). The lemma holds when $n = 1$. Suppose that it is valid for all $n \leq k$ $(k < h)$ with $1 \leq k \leq s$. Then we have

$$p_{k+1} = \left(\begin{array}{c|c} 0 & \\ \hline E_{s-k+1} & {}_kP \end{array} \right) \left(\begin{array}{c|c} {}^t\mathbf{0} & 1 \\ \hline E_s & \mathbf{b}_{k+1} \end{array} \right),$$

where ${}_kP$ denotes the $(s+1) \times k$ matrix defined by ${}_kP := (p_j^{(i)})_{1 \leq i \leq s+1, 1 \leq j \leq k}$. Hence, setting $b_{k+1}^{(0)} := 1$, we get

$$p_{k+1}^{(j)} = \sum_{m=1}^{k} b_{k+1}^{(s-k+m)} p_m^{(j)} \quad (1 \leq j \leq k),$$

$$p_{k+1}^{(j)} = b_{k+1}^{(j-k-1)} + \sum_{m=1}^{k} b_{k+1}^{(s-k+m)} p_m^{(j)} \quad (k+1 \leq j \leq s+1).$$

Therefore, we obtain (4.32) with $n = k+1$ by induction hypothesis. Now, we suppose that the lemma is valid for all $n \leq k$ $(k < h)$ with $k \geq s+1$. Then it is easy to check that (4.32) holds when $n = k+1$ by using the identity $P_{k+1} = P_k B_{k+1}$. ∎

P r o o f   o f   T h e o r e m 1(iv). In view of the expression (3.2) with (3.3), we get $0 < {}_m\Psi_n^{(i)}(g) \leq 1$ by Lemmas 1 and 16. Thus, we obtain $0 \leq {}_m\Psi^{(i)}(g) \leq 1$ for all $m \geq 1$ and $1 \leq i \leq s$. Let us suppose that ${}_m\Psi^{(i)}(g) = 0$ or 1 for $m = m_0$, $i = i_0$. Then the admissible expression of $\boldsymbol{\Psi}(g)$ in the JP algorithm terminates with the length $m_0 + i_0 - 1$, which contradicts the statement (iii) of Theorem 1. ∎

The proof of Theorems 1–3 is now completed.

5. Characteristic sets of ω, special cases. Let ω be the fixed point of σ over K defined by (1.2) as in the previous sections. We denote by $\int_i C$ and ∂C the sequences defined by

$$\int_i C := \left\{ i + \sum_{m=1}^{n-1} c_m \right\}_{n=1,2,\dots}, \qquad \partial C := \{c_{n+1} - c_n\}_{n=1,2,\dots},$$

for a given sequence $C = \{c_n\}_{n=1,2,\dots}$, respectively.

THEOREM 4. *Let L_j $(1 \le j \le s+1)$ be the set $\{h_j^{(1)}, h_j^{(1)}+h_j^{(2)}, \ldots, h_j^{(1)} + h_j^{(s+1)}\}$ with $h_j^{(i)} := |\sigma^{j-1}(a_i)|$ $(1 \le i \le s+1)$, and let $v_j : K^* \to L_j^*$ be the morphism defined by*

$$v_j(a_i) := (h_j^{(1)})^{k_{s-i+1}-1}(h_j^{(1)} + h_j^{(i+1)}) \quad (1 \le i \le s), \qquad v_j(a_{s+1}) := h_j^{(1)}.$$

Then

$$\chi(\omega; a_j) = \int_{h_j^{(1)}} v_j(\omega) \quad (1 \le j \le s+1).$$

Proof. In view of (4.4), we have, for each $1 \le j \le s+1$,

$$\sigma^j(a_i) = \sigma^{j-1}(a_1)^{k_{s-i+1}}\sigma^{j-1}(a_{i+1}) \quad (1 \le i \le s),$$
$$\sigma^j(a_{s+1}) = \sigma^{j-1}(a_1).$$

Here, the symbol a_j appears exactly k_{s-i+1} times in the words $\sigma^j(a_i)$ as its prefix for all $1 \le i \le s+1$ when $j = 1$, the symbol a_j $(2 \le j \le s+1)$ appears exactly once in the word $\sigma^{j-1}(a_1)$ as its suffix, so that the symbol a_j appears once in the same positions of the words $\sigma^j(a_i)$ from left for all $1 \le i \le s+1$ when $2 \le j \le s+1$. Therefore, noting the fact that $\omega = \sigma^j(\omega_1)\sigma^j(\omega_2)\ldots\sigma^j(\omega_m)\ldots$, we get $\partial\chi(\omega; a_j) = v_j(\omega)$, where $\chi(\omega; a_j)$ is supposed to be a sequence in the increasing order of its terms, which leads to the theorem. ∎

EXAMPLE. Taking $s = 3$, $k_s = 1$, we have the following

$$\int_1 \omega(2,2,2,1) = \{1,3,5,7,9,11,13,15,16,18,20,22,24,26,28,30,\ldots\},$$
$$\int_2 \omega(4,4,3,2) = \{2,6,10,14,17,21,25,29,31,35,39,43,46,50,54,\ldots\},$$
$$\int_4 \omega(8,7,6,4) = \{4,12,19,27,33,41,48,56,60,68,75,83,89,97,\ldots\},$$
$$\int_8 \omega(15,14,12,8) = \{8,23,37,52,64,79,93,108,116,131,145,160,\ldots\},$$

which gives a partition of the set $\mathbb{N}$ into 4 parts, where $\omega(w_1, w_2, w_3, w_4)$ denotes the sequence $v(\omega)$ with $v(a_i) = w_i$.

Finally, we consider the case $s = 1$ in Theorem 1. In this case, the JP algorithm turns out to be that of the simple continued fraction expansion. We put $k := k_1$, and $a_1 = a$, $a_2 = b$. Then we have $f(x) = x^2 - kx - 1$, and so $\alpha = (k + (k^2 + 4)^{1/2})/2$ in Lemma 10. We put

$$\chi(\omega; a) = \{s_1 < s_2 < \ldots < s_n < \ldots\}, \quad \chi(\omega; b) = \{t_1 < t_2 < \ldots < t_n < \ldots\}.$$

Then Theorem 4 implies

(5.1) $$t_n = kn + s_n \quad (n \ge 1).$$

It is clear that

(5.2) $$\chi(\omega; a) \cup \chi(\omega; b) = \mathbb{N}, \quad \chi(\omega; a) \cap \chi(\omega; b) = \emptyset.$$

It can be seen that the sequences $\chi(\omega; a)$ and $\chi(\omega; b)$ are uniquely determined by the properties (5.1) and (5.2). On the other hand, if we set

$$\theta := 1 + 1/\alpha = 1 + (-k + (k^2+4)^{1/2})/2, \quad \eta := 1 + \alpha = 1 + (k + (k^2+4)^{1/2})/2,$$

then we get the equality $\eta = k + \theta$. Thus, we have $\lfloor \eta n \rfloor = kn + \lfloor \theta n \rfloor$ $(n \in \mathbb{N})$. In addition, we also have $1/\theta + 1/\eta = 1$. Hence, the two sets $\{\lfloor \theta n \rfloor : n \in \mathbb{N}\}$ and $\{\lfloor \eta n \rfloor : n \in \mathbb{N}\}$ form a partition of $\mathbb{N}$, since θ and η are irrational, which is a well-known property of Beatty sequences (cf. [8], or [18], p. 29). Therefore we obtain

$$\chi(\omega; a) = \{\lfloor \theta n \rfloor : n \in \mathbb{N}\}, \quad \chi(\omega; b) = \{\lfloor \eta n \rfloor : n \in \mathbb{N}\}.$$

From this fact, in view of Theorem 1(i), we get the identity

$$(z-1) \sum_{n=1}^{\infty} z^{-\lfloor \theta n \rfloor} = [0; b_1, b_2, \ldots] \quad (|z| > 1, \ z \in \mathbb{C})$$

where b_n denotes the polynomial of z defined by

$$b_n := z^{f_{n-2}} \sum_{m=0}^{k-1} z^{m f_{n-1}} \quad (n \geq 1).$$

If we take $k = 1$ and $z = 2$, then we again have the identity (1.1).

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