Zeros of Hecke $L$-functions associated with cusp forms

by

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1. Introduction. Let $f(z) = \sum_{n=1}^{\infty} a_n e(nz)$ be a holomorphic cusp form of even integral weight $k > 0$ with respect to the modular group $\Gamma = \text{SL}(2, \mathbb{Z})$, and define (for $\Re s > (k+1)/2$)

$$L_f(s) = \sum_{n=1}^{\infty} a_n n^{-s},$$

the associated Hecke $L$-function. We also assume that $f(z)$ is a Hecke eigenform [11] with $a_1 = 1$. Recall that we have the bound for the coefficients $|a_n| \leq d(n)n^{(k-1)/2}$ by Deligne’s proof of the Ramanujan–Petersson conjecture [2], [3], and the bound for the square mean [9], [18],

$$\sum_{n \leq N} |a_n|^2 \ll N^k.$$

It is well known [10] that $L_f(s)$ admits analytic continuation to $\mathbb{C}$ as an entire function and satisfies the functional equation

$$(2\pi)^{-s} \Gamma(s) L_f(s) = (-1)^{k/2}(2\pi)^{-k-s} \Gamma(k-s) L_f(k-s).$$

Moreover, $L_f(s)$ has Euler product representation ($\Re s > (k+1)/2$)

$$L_f(s) = \prod_p (1 - a_p p^{-s} + p^{k-1} p^{-2s})^{-1}.$$ 

The non-trivial zeros of $L_f(s)$ lie within the strip $(k-1)/2 < \Re s < (k+1)/2$, symmetrically to the real axis and the critical line $\sigma = k/2$. The Riemann Hypothesis for $L_f(s)$ asserts that all the non-trivial zeros of $L_f(s)$ lie on the critical line $\Re s = k/2$. Hafner [13], generalizing Selberg’s remarkable work [19] on $\zeta(s)$, has shown that a positive proportion of all non-trivial zeros are on the critical line.

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In this work, we establish the analogue of Selberg’s density theorem [20] for $L_f(s)$. Define, for $\sigma \geq k/2$ and $T \geq 1$,

$$N_f(\sigma, T) = |\{\beta + i\gamma : L_f(\beta + i\gamma) = 0, \beta \geq \sigma, 0 < \gamma \leq T\}|.$$

It was proved by Lekkerkerker [15] that

$$N_f\left(\frac{k-1}{2}, T\right) \sim \frac{1}{\pi} T \log T.$$

We will show that

**Theorem 1.1.** For some $a > 0$ we have

$$N_f(\sigma, T) \ll T^{1-a(\sigma-k/2)} \log T,$$

uniformly for $k/2 \leq \sigma \leq (k+1)/2$.

Our proof shows that one may take $a = 1/72$. However, we make no effort to obtain an optimal $a$ by our method.

Application of standard techniques of analytic number theory easily yields results of the type

$$N_f(\sigma, T) \ll T^{c(\sigma)} (\log T)^A,$$

where $c(\sigma) < 1$ for $\sigma > k/2$ and some $A > 0$. The significance of Theorem 1.1 lies in that $A$ can be taken to be 1. Selberg used the analogue of Theorem 1.1 for $\zeta(s)$ to prove his famous result on the moments of $\arg\zeta(1/2 + it)$. In view of recent work of Bombieri and Hejhal [1], there is a similar application to $\arg L_f(k/2 + it)$, which is the main motivation of the present paper.

**Corollary 1.2.** The functions

$$\frac{\log |L_f(k/2 + it)|}{\sqrt{\pi \log \log t}}, \quad \frac{\arg L_f(k/2 + it)}{\sqrt{\pi \log \log t}}$$

become distributed, in the limit of large $t$, like independent random variables, each having Gaussian density $\exp(-\pi u^2) du$.

To prove Theorem 1.1 by Selberg’s method, one considers not $L_f(s)$ itself, but $L_f(s)M_X(s)$, where the mollifier $M_X(s)$ is a Dirichlet polynomial of length $X = T^\theta$, $0 < \theta < 1/2$, and is chosen such that $L_f(s)M_X(s)$ is very close to 1 in the region $\sigma > k/2, 0 < t \leq T$, or more precisely such that the mean value

$$\frac{1}{T} \int_T^{2T} |L_f(\sigma + it)M_X(\sigma + it) - 1|^2 dt$$

is very small, i.e.

$$\int_T^{2T} |L_f(\sigma + it)M_X(\sigma + it) - 1|^2 dt \ll T^{1-a(\sigma-k/2)},$$
uniformly for \( k/2 \leq \sigma \leq (k + 1)/2 \). It is then possible to deduce, by a standard argument (see §3), that the zeros of \( L_f(s)M_X(s) \) and \textit{a fortiori} of \( L_f(s) \) in the region considered are comparatively few. The required mean value estimate is obtained as follows. If we prove
\[
\frac{2T}{T} \int \left| L_f(k/2 + it)M_X(k/2 + it) - 1 \right|^2 dt \ll T,
\]
and
\[
\frac{2T}{T} \int \left| L_f(k/2 + 1 + it)M_X(k/2 + 1 + it) - 1 \right|^2 dt \ll T^{1-a},
\]
then by a convexity theorem (see §3) we have
\[
\frac{2T}{T} \int \left| L_f(\sigma + it)M_X(\sigma + it) - 1 \right|^2 dt \ll T^{1-a(\sigma-k/2)},
\]
uniformly for \( k/2 \leq \sigma \leq k/2 + 1 \), which is all we need. The second inequality is easy to prove since \( L_f(s) \) is an absolutely convergent Dirichlet series for \( \Re s > (k + 1)/2 \) and \( M_X(s) \) is an approximate inverse to \( L_f(s) \) such that \( L_f(s)M_X(s) - 1 \) is given by a Dirichlet series of the type \( \sum_{n \geq y} b_n n^{-s} \), with \( y \) large. The first inequality represents the main difficulty, since it is not obtainable by a routine extension of Selberg’s work in the \( \zeta(s) \) case. We will replace \( L_f(s) \) by a Dirichlet polynomial of length \( \sim T \) using the approximate functional equation of \( L_f(s) \) in the form obtained by A. Good [7]. The resulting expression then becomes
\[
\frac{2T}{T} \int \left| P_{TX}(\sigma + it) \right|^2 dt,
\]
where \( P_{TX} \) is a Dirichlet polynomial of length \( TX \). However, in general no method succeeds in handling the above mean value once \( P(s) \) has length \( \gg T \). Therefore we have to make careful use of the special feature of \( M_X(s) \). In fact, the argument similar to Selberg [19] and Hafner [13], with some modification, is suitable here. For some technical reason we will prove the first inequality for \( \sigma = k/2 + 1/\log T \) rather than \( k/2 \) and then apply a convexity theorem to obtain Theorem 1.1.

We remark here that in our proof of Theorem 1.1 Deligne’s bound for the Fourier coefficients \( a_n \) is used but not crucial here. A weaker bound like \( a_n \ll n^{(k-1)/2+1/4+\varepsilon} \) which follows from Weil’s bound for the Kloosterman sums and the bound for the square mean mentioned before would suffice. Thus, our method should be applicable when \( f(z) \) is a Maass form, though the Ramanujan–Petersson conjecture remains unproved in this case.
We would like to mention that D. Farmer [6], using Hafner’s method and the spectral theory, establishes an asymptotic formula for the mean square of $L_f(s)$ weighted by a general mollifier of Levinson’s type. He mentions that this mean value theorem can be combined with Jutila’s method [14] to give a density result, but he does not give any details.

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2. Main lemma. Let $\psi_U(t)$ be a non-negative smooth function such that
\[
\psi_U(t) = \begin{cases} 
0 & \text{if } t \leq 1 - 1/U \text{ or } t \geq 2 + 1/U, \\
1 & \text{if } 1 + 1/U \leq t \leq 2 - 1/U;
\end{cases}
\]
and
\[
\psi^{(p)}_U(t) \ll U^p, \quad p \geq 0,
\]
where $U$ is a positive parameter and in our discussion it will be chosen as $O(1)$ later. The object of this section is to prove the following lemma, which is the analogue of Lemma 6 in [19].

**Lemma 2.1.** If $k/2 < \sigma \leq k/2 + 1/40$, $\varepsilon > 0$, and $\mu, \nu$ are positive coprime integers $\leq T$, then
\[
\int_{-\infty}^{\infty} \psi_U \left( \frac{t}{T} \right) |L_f(\sigma + it)|^2 \left( \frac{\mu}{\nu} \right)^{it} dt = \frac{1}{(\mu\nu)^\sigma} D_{\mu\nu}(2\sigma) \int_{-\infty}^{\infty} \psi_U \left( \frac{t}{T} \right) dt \]
\[
+ \frac{1}{(\mu\nu)^{k-\sigma}} D_{\mu\nu}(2k - 2\sigma) \int_{-\infty}^{\infty} \psi_U \left( \frac{t}{T} \right) \left( \frac{t}{2\pi} \right)^{2k-4\sigma} dt
\]
\[
+ O \left( \frac{(\mu\nu)^3 U^4 T^{4/5}}{2\sigma - k} \right),
\]
where
\[
D_{\mu\nu}(s) = \sum_{l=1}^{\infty} \frac{a_\mu a_{\nu l}}{l^s}.
\]
Proof. The proof is very similar to the treatment in [8] and [13], and so we give only a sketch. We have (denote $\sigma + it$ by $s$)
\[
\int_{-\infty}^{\infty} \psi_U \left( \frac{t}{T} \right) |L_f(\sigma + it)|^2 \left( \frac{\mu}{\nu} \right)^{it} dt
\]
\[= \int_{-\infty}^{\infty} \psi_U \left( \frac{t}{T} \right) L_f(\sigma + it)L_f(\sigma - it) \left( \frac{\mu}{\nu} \right)^{it} dt
\]
\[= \int_{-\infty}^{\infty} \left( \sum_{n=1}^{\infty} a_n n^{-\sigma}(n\nu)^{-it} \phi \left( \frac{2\pi n}{t} \sqrt{\frac{\nu}{\mu}} \right) \right) \psi_U \left( \frac{t}{T} \right) dt
\]
\[+ (2\pi)^{2\sigma-k} \frac{\Gamma(k-s)}{\Gamma(s)} \sum_{n,m=1}^{\infty} a_n a_m \frac{n}{\nu} \phi \left( \frac{2\pi n}{t} \sqrt{\frac{\mu}{\nu}} \right)
\]
\[\times \left( \sum_{m=1}^{\infty} a_m m^{-\sigma}(m\mu)^{it} \phi \left( \frac{2\pi m}{t} \sqrt{\frac{\mu}{\nu}} \right) \right)
\]
\[\times \frac{1}{\sqrt{\nu}} \left( \sum_{m=1}^{\infty} a_m m^{-\sigma}(m\mu)^{it} \phi \left( \frac{2\pi m}{t} \sqrt{\frac{\mu}{\nu}} \right) \right) dt
\]
\[+ O \left( \left( \frac{\mu}{\nu} + \frac{\nu}{\mu} \right) \log^2 T \right).
\]
Here we use the approximate functional equation for $L_f(\sigma \pm it)$ (see [7], Satz), and $\phi(\xi), \phi^*(\xi)$ are suitable smooth functions satisfying $\phi^*(\xi) = 1 - \phi(1/\xi)$, and $\phi(\xi) = 1, |\xi| \leq 2/3; \phi(\xi) = 0, |\xi| \geq 3/2$.

Multiplying out the expression in the above integrand and using the same argument and notation as in [8], §2 (see also [13], §3), we see that the above expression equals
\[
\sum_{n,m} \frac{a_n a_m}{(nm)^\sigma} \int_{-\infty}^{\infty} \psi_U \left( \frac{t}{T} \right) \left( \frac{m\mu}{n\nu} \right)^{it} \Phi \left( \frac{2\pi n}{t} \sqrt{\frac{\nu}{\mu}}, \frac{2\pi m}{t} \sqrt{\frac{\mu}{\nu}} \right) dt
\]
\[+ \sum_{n,m} \frac{a_n a_m}{(nm)^{k-\sigma}} (2\pi)^{-2k+4\sigma}
\]
\[\times \int_{-\infty}^{\infty} \psi_U \left( \frac{t}{T} \right) \left( \frac{n\mu}{m\nu} \right)^{it} \Phi^* \left( \frac{2\pi n}{t} \sqrt{\frac{\mu}{\nu}}, \frac{2\pi m}{t} \sqrt{\frac{\nu}{\mu}} \right) dt
\]
\[+ O \left( \frac{\log^2 T}{2\sigma - k} \left( \sqrt{\frac{\nu}{\mu}} + \sqrt{\frac{\mu}{\nu}} \right) \right) = S_1 + S_2 + S_3,
\]
say, where $\Phi, \Phi^*$ are certain smooth functions with compact supports, and $\Phi(\varrho, \varrho) = \phi(\varrho), \Phi^*(\varrho, \varrho) = \phi^*(\varrho)$. 
For $S_1$, the terms with $mn = n\nu$ give

\[
\sum_{l=1}^{\infty} \sum_{n=1}^{\infty} a_{\mu l} a_{\nu l} \int_{-\infty}^{\infty} \psi_U \left( \frac{t}{T} \right) \phi \left( \frac{2\pi \sqrt{\mu \nu l}}{t} \right) dt.
\]

For the terms with $mn \neq n\nu$, we let $mn - \nu l = l$. Without loss of generality we may assume $l > 0$. Then the non-diagonal terms with $l > 0$ give

\[
\sum_{l>0} \sum_{n=1}^{\infty} a_{n} a_{(n\nu+\frac{l}{2})/\mu} \times \int_{-\infty}^{\infty} \psi_U \left( \frac{t}{T} \right) \phi \left( \frac{2\pi n \nu + l/2}{t \sqrt{\mu \nu}} \right) e^{it(nn + l/2)} dt.
\]

For $S_2$, the terms $n\mu = m\nu$ give

\[
(2\pi)^{2k+4\sigma} \sum_{l=1}^{\infty} \sum_{n=1}^{\infty} a_{n} a_{\nu l} \int_{0}^{\infty} \psi_U \left( \frac{t}{T} \right) \phi^* \left( \frac{2\pi \sqrt{\mu \nu l}}{t} \right) t^{2k-4\sigma} dt.
\]

For the terms with $n\mu \neq m\nu$, we let $m\nu - n\mu = l$, and the non-diagonal terms with $l > 0$ give

\[
(\mu \nu)^{k-\sigma} (2\pi)^{-2k+4\sigma} \sum_{0<l\leq 2\pi UT} \sum_{n=1}^{\infty} a_{n} a_{\nu l} \int_{-\infty}^{\infty} \psi_U \left( \frac{t}{T} \right) \phi^* \left( \frac{2\pi \nu l}{t \sqrt{\mu \nu}} \right) e^{-it(l/2)} dt + O \left( (\mu \nu)^3 U^4 \log^2 T \right).
\]

Let

\[
G(s) = \int_{0}^{\infty} \phi(x) x^{-s-1} dx, \quad G^*(s) = \int_{0}^{\infty} \phi^*(x) x^{-s-1} dx.
\]

Then, by Mellin inversion,

\[
\phi(x) = \frac{1}{2\pi i} \int_{(2)} G(s) x^{-s} ds, \quad \phi^*(x) = \frac{1}{2\pi i} \int_{(2)} G^*(s) x^{-s} ds.
\]
Note that $G(s)$ and $G^\ast(s)$ are analytic except for a simple pole at $s = 0$ with residue 1, and $D_{\mu\nu}(s)$ has only a simple pole at $s = k$ for $\Re s \geq k - 1/2$,

$$D_{\mu\nu}(s) = P(\mu, s)P(\nu, s)D(s) \quad \text{with} \quad D(s) = \sum_{l=1}^\infty a_l^2 l^{-s};$$

$$P(a, s) = \prod_{p^j \parallel a} \left( \sum_{j=0}^\infty a_p^{r+j} a_p^{-j}s \right) \left( \sum_{j=0}^\infty a_p^2 p^{-j}s \right)^{-1};$$

$$D(s) \ll t^{1+\varepsilon}, \quad \Re s \geq k - 1/2;$$

$$P(a, s) \ll a^{(k-1)/2+\varepsilon}, \quad \Re s \geq k - 1/2;$$

$$|G(s)| + |G^\ast(s)| \ll t \frac{1}{|s(s+1)\ldots(s+l)|}.$$ 

Thus (1) and (3) equal respectively

(5) \begin{align*}
\frac{1}{(\mu\nu)^\sigma} \cdot \frac{1}{2\pi i} \int_{(2)} \left( \frac{1}{2\pi \sqrt{\mu\nu}} \right)^s G(s)D_{\mu\nu}(2\sigma + s) \int_{-\infty}^{\infty} \psi_U \left( \frac{t}{T} \right) t^s \, dt, 
\end{align*}

(6) \begin{align*}
\frac{1}{(\mu\nu)^{k-\sigma}} \cdot \frac{1}{2\pi i} \int_{(2)} \left( \frac{1}{2\pi \sqrt{\mu\nu}} \right)^s G^\ast(s)D_{\mu\nu}(2(k-\sigma) + s) 
\times \int_{-\infty}^{\infty} \psi_U \left( \frac{t}{T} \right) t^s \left( \frac{t}{2\pi} \right)^{2k-4\sigma} \, dt.
\end{align*}

Next we shift the lines of integration in (5), (6) to $\Re s = -1/2$ and $\Re s = -1/2 - 2k + 4\sigma$, respectively. The integrands have poles at $k - 2\sigma, 0$ and $2\sigma - k, 0$, respectively. The residues at $k - 2\sigma$ and $2\sigma - k$ cancel out and the residues at 0 give the main terms.

By the estimate given above, we deduce that

(5) + (6) = \begin{align*}
\frac{1}{(\mu\nu)^\sigma} D_{\mu\nu}(2\sigma) \int_{-\infty}^{\infty} \psi_U \left( \frac{t}{T} \right) dt 
+ \frac{1}{(\mu\nu)^{k-\sigma}} D_{\mu\nu}(2k-2\sigma) \int_{-\infty}^{\infty} \psi_U \left( \frac{t}{T} \right) \left( \frac{t}{2\pi} \right)^{2k-2\sigma} \, dt + O(T^{1/2+\varepsilon}).
\end{align*}

Define

$$H_l(s) = \int_0^\infty \phi(\xi) e^{2\pi i (l/\sqrt{\mu\nu}) \xi^{-1} - s} \, d\xi.$$ 

This is an entire function and by the Mellin inversion formula,

$$\phi(\xi) e^{2\pi i (l/\sqrt{\mu\nu}) \xi^{-1}} = \frac{1}{2\pi i} \int_{(2)} H_l(s) \xi^{-s} \, ds.$$
Thus the sum in (2) becomes
\[
(\mu\nu)^{\sigma} \sum_{0<l<\sqrt{\mu\nu}UT} \int_{\mu\nu(l+1/2)/2}^{\infty} a_n a_{\mu\nu(l+1)/\mu} (n\nu + l/2)^{2\sigma} \times \prod_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{2\pi i} \int_{(2)} \left( \frac{\sqrt{\mu\nu}}{2\pi} \right)^n H_i(s) D_{\mu\nu}(s+2\sigma, l) \int_{-\infty}^{\infty} \psi(t) t^s dt.
\]
Here
\[
D_{\mu\nu}(s, l) = \sum_{n=1}^{\infty} \frac{a_n a_{(n\nu+l)/\mu}}{(n\nu + l/2)^{s}}.
\]
We move the line of integration to \(\Re s = -1/5\). We have, on \(\Re s = -1/5\),
\[
\int_{-\infty}^{\infty} \psi(t) t^s dt \ll \frac{1}{(|s| + 1)^4} U^3 T^{4/5},
\]
and we also have Hafner’s result [12]
\[
D_{\mu\nu}(2\sigma + s, l) \ll \frac{l|s|^{1+\varepsilon}}{(\mu\nu)^{k/2 - 7/4}}.
\]
Thus the above expression is majorized by
\[
\frac{(\mu\nu)^{\sigma}}{(\mu\nu)^{k/2 - 7/4} U^1 T^{4/5}} \ll (\mu\nu)^{3} U^1 T^{4/5}.
\]
Similarly, we obtain the same bound for the sum occurring in (4). Thus the proof is complete.

In Section 4 we will give an alternative, and more elementary, treatment of \(D_{\mu\nu}(s, l)\) giving a slightly weaker analytic continuation result, which still suffices for the proof.

3. Proof of Theorem. Let
\[
L_f^{-1}(s) = \sum_{n=1}^{\infty} \frac{\mu_f(n)}{n^s}, \quad \Re s > \frac{k + 1}{2}.
\]
Thus
\[
\mu_f(p^r) = \begin{cases} 1 & \text{if } r = 0, \\ -a_p & \text{if } r = 1, \\ p^{k-1} & \text{if } r = 2, \\ 0 & \text{if } r \geq 3. \end{cases}
\]
Set \( \lambda_n = \mu_f(n)g_\xi(n) \), where
\[
g_\xi(n) = \begin{cases} 
1 & \text{if } 1 \leq n \leq \xi, \\
\frac{\log(\xi^2/n)}{\log \xi} & \text{if } \xi \leq n \leq \xi^2, \\
0 & \text{if } n \geq \xi^2,
\end{cases}
\]
and \( \xi = T^\theta, 0 < \theta < 1/4 \) will be specified later.

We define the mollifier
\[
M_{\xi^2}(s) = \sum_v \frac{\lambda_v}{v^s},
\]
where \( k/2 + A/\log \xi \leq \sigma \leq k/2 + \delta \), and \( A, \delta \) are sufficiently large positive numbers.

Using the multiplicativity of the Hecke eigenvalues \( a_n \) [11] and the definition of \( P(n, s) \),
\[
P(n, s) = \prod_{p^r \parallel n} P(p^r, s),
\]
we easily have
\[
P(p, s) = \frac{a_p}{1 + p^{k-1-s}}, \quad P(p^2, s) = \frac{a_p^2}{1 + p^{k-1-s}} - p^{k-1}.
\]

We have, using Lemma 2.1,
\[
\int_{-\infty}^{\infty} \psi_U \left( \frac{t}{T} \right) |L_f(\sigma + it)|^2 |M_{\xi^2}(\sigma + it)|^2 dt
\]
\[
= \sum_{v_1, v_2 \leq \xi^2} \frac{\lambda_{v_1} \lambda_{v_2}}{(v_1 v_2)^{2\sigma}} (v_1, v_2)^{2\sigma} D_{v_1/(v_1, v_2), v_2/(v_1, v_2)} (2\sigma) \int_{-\infty}^{\infty} \psi_U \left( \frac{t}{T} \right) dt
\]
\[
+ \sum_{v_1, v_2 \leq \xi^2} \frac{\lambda_{v_1} \lambda_{v_2}}{(v_1 v_2)^k} (v_1, v_2)^{2(k-\sigma)} D_{v_1/(v_1, v_2), v_2/(v_1, v_2)} (2k - 2\sigma)
\]
\[
\times \int_{-\infty}^{\infty} \psi_U \left( \frac{t}{T} \right) \left( \frac{t}{2\pi} \right)^{2k-4\sigma} dt + O(T^{4/5}U^3 \xi^{14} \log^3 \xi)
\]
\[
= S_1 + S_2 + S_3,
\]
say. Since we have
\[
D_{v_1/(v_1, v_2), v_2/(v_1, v_2)} (2\sigma) = D(2\sigma) P\left( \frac{v_1}{(v_1, v_2)}, 2\sigma \right) P\left( \frac{v_2}{(v_1, v_2)}, 2\sigma \right),
\]
where 

\[ D(s) = D_{11}(s) = \sum_{l=1}^{\infty} \frac{d_{l}^{2}}{l^{s}}, \]

it follows by Möbius inversion that

\[ S_{1} = D(2\sigma) \int_{-\infty}^{\infty} \psi_{U} \left( \frac{t}{T} \right) dt \]

\[
\times \sum_{v_{1}, v_{2} \leq \xi^{2}} \frac{\lambda_{v_{1}} \lambda_{v_{2}}}{(v_{1} v_{2})^{2\sigma}} \left( v_{1}, v_{2} \right)^{2\sigma} P \left( \frac{v_{1}}{(v_{1}, v_{2})}, 2\sigma \right) P \left( \frac{v_{2}}{(v_{1}, v_{2})}, 2\sigma \right) 
\]

\[ = D(2\sigma) \int_{-\infty}^{\infty} \psi_{U} \left( \frac{t}{T} \right) dt \]

\[
\times \sum_{v_{1}, v_{2} \leq \xi^{2}} \frac{\lambda_{v_{1}} \lambda_{v_{2}}}{(v_{1} v_{2})^{2\sigma}} \sum_{r | v_{1}, r | v_{2}} \mu(l) \left( \frac{r}{l} \right)^{2\sigma} P \left( \frac{v_{1}}{r/l}, 2\sigma \right) P \left( \frac{v_{2}}{r/l}, 2\sigma \right) 
\]

\[ = D(2\sigma) \int_{-\infty}^{\infty} \psi_{U} \left( \frac{t}{T} \right) dt \]

\[
\times \sum_{r \leq \xi^{2}, r \text{ cubefree}} \sum_{l | r} \mu(l) \left( \frac{r}{l} \right)^{2\sigma} P \left( \frac{v}{r/l}, 2\sigma \right)^{2}. 
\]

Similarly

\[ S_{2} = D(2k - 2\sigma) \int_{-\infty}^{\infty} \psi_{U} \left( \frac{t}{T} \right) \left( \frac{t}{2\pi} \right)^{2k-4\sigma} dt \]

\[
\times \sum_{r \leq \xi^{2}, r \text{ cubefree}} \sum_{l | r} \mu(l) \left( \frac{r}{l} \right)^{2(k-\sigma)} \left( \sum_{r | v} \frac{\lambda_{v}}{v^{2\sigma}} P \left( \frac{v}{r/l}, 2(k-\sigma) \right) \right)^{2}. 
\]

We distinguish two cases: (a) \( r \leq \xi \), and (b) \( \xi < r \leq \xi^{2} \).

First consider the case (a) \( r \leq \xi \). We deduce that, since

\[
\frac{1}{2\pi i} \int_{(2)} \frac{y^{s}}{s^{2}} ds = \begin{cases} \log y, & y \geq 1, \\ 0, & 0 < y \leq 1, \end{cases}
\]

we have

\[
\sum_{v} \frac{\lambda_{rv}}{(rv)^{2\sigma}} P(lv, 2\sigma) 
\]

\[ = \sum_{v} \frac{\mu_{f}(rv) g_{e}(rv)}{(rv)^{2\sigma}} P(lv, 2\sigma) \]
\[
= \frac{1}{2\pi i} \int_\Gamma \frac{\xi^s(\xi^s - 1)}{s^2} \left( \sum_v \frac{\mu_f(rv)}{(rv)^{2\sigma}} P(lv, 2\sigma) \right) \frac{ds}{\log \xi}
\]

\[
= \sum_v \frac{\mu_f(rv)}{(rv)^{2\sigma}} P(lv, 2\sigma)
+ \frac{1}{2\pi i} \int_\Gamma \frac{(\xi/r)^s(\xi^s - 1)}{s^2} \left( \sum_v \frac{\mu_f(rv)}{(rv)^{2\sigma}} \frac{1}{vs} P(lv, 2\sigma) \right) \frac{ds}{\log \xi},
\]

where \( \Gamma \) denotes the path \([ix, |x| \geq \delta] \cup \{\delta e^{i\theta}, \pi/2 \leq \theta \leq 3\pi/2\} \), and \( \delta \) is sufficiently small.

We observe that \((p^{e(r)} \parallel r, p^{e(l)} \parallel l)\)
\[
\sum_v \frac{\mu_f(rv)}{(rv)^{2\sigma}} P(lv, 2\sigma)
= \prod_p \left( 1 + \frac{\mu_f(p)}{p^{2\sigma}} P(p, 2\sigma) + \frac{\mu_f(p^2)}{p^{4\sigma}} P(p^2, 2\sigma) \right)
\times \prod_{p|r} \frac{\mu_f(p^{e(r)})}{(p^{e(r)})^{2\sigma}} P(p^{e(r)}, 2\sigma) + \frac{\mu_f(p^{e(r)+1})}{(p^{e(r)+1})^{2\sigma} p^s} P(p^{e(r)+1}, 2\sigma)
\times \prod_{p|l} \frac{\mu_f(p^{e(l)})}{(p^{e(l)})^{2\sigma}} P(p^{e(l)}, 2\sigma) + \frac{\mu_f(p^{e(l)+1})}{(p^{e(l)+1})^{2\sigma} p^s} P(p^{e(l)+1}, 2\sigma)
= \frac{1}{D^{(2\sigma)}} u(r, l, 2\sigma),
\]
say. Similarly
\[
\sum_v \frac{\mu_f(rv)}{(rv)^{2\sigma}} \cdot \frac{1}{vs} P(lv, 2\sigma)
= \prod_p \left( 1 + \frac{\mu_f(p)}{p^{2\sigma+s}} P(p, 2\sigma) + \frac{\mu_f(p^2)}{p^{4\sigma+2s}} P(p^2, 2\sigma) \right)
\times \prod_{p|r} \frac{\mu_f(p^{e(r)})}{(p^{e(r)})^{2\sigma}} P(p^{e(r)}, 2\sigma) + \frac{\mu_f(p^{e(r)+1})}{(p^{e(r)+1})^{2\sigma} p^s} P(p^{e(r)+1}, 2\sigma)
\times \prod_{p|l} \frac{\mu_f(p^{e(l)})}{(p^{e(l)})^{2\sigma}} P(p^{e(l)}, 2\sigma) + \frac{\mu_f(p^{e(l)+1})}{(p^{e(l)+1})^{2\sigma} p^s} P(p^{e(l)+1}, 2\sigma)
= G(s) v(r, l, 2\sigma, s),
\]
say.

It is easily verified that, for \( \Re s > -1/2 \),
\[
G(s) = \frac{1}{D^{(2\sigma+s)}} \prod_p \left( 1 + O\left( \frac{1}{p^{2(1+\Re s)}} \right) \right).
\]
We have, by Cauchy’s inequality,
\[
\left( \sum_{v} \frac{\lambda_{v}}{(rv)^{2\sigma}} P(lv, 2\sigma) \right)^{2} \ll \left| \sum_{v} \frac{\mu_{r}(rv)}{(rv)^{2\sigma}} P(lv, 2\sigma) \right|^{2} + \int_{\frac{1}{2}} \left| \left( \frac{x^{s}}{r} \frac{x^{s} - 1}{s^{2}} \right) \right| \times \left| \sum_{v} \frac{\mu_{r}(rv)}{(rv)^{2\sigma} v^{s}} P(lv, 2\sigma) \right|^{2} \left| ds \right| / \log^{2} x.
\]

For \( r \) cubefree, \( r = r_{1}r_{2}^{2}, \mu(r_{1}r_{2}) \neq 0 \), we infer that
\[
\sum_{l \mid r} |\mu(l)| \left( \frac{r}{l} \right)^{2\sigma} \left| \sum_{v} \frac{\mu_{r}(rv)}{(rv)^{2\sigma}} P(lv, 2\sigma) \right|^{2} \ll \prod_{p \mid r} \left( 1 + \frac{1}{p^{3/4}} \right) \frac{1}{D(2\sigma)^{2}} \left( \sum_{l \mid r_{1}} a_{l}^{2} \left( \frac{r_{1}}{l} \right)^{-3} \right) r_{2}^{2(k-1)-4\sigma},
\]
\[
\sum_{l \mid r} |\mu(l)| \left( \frac{r}{l} \right)^{2\sigma} \left| \sum_{v} \frac{\mu_{r}(rv)}{(rv)^{2\sigma} v^{s}} P(lv, 2\sigma) \right|^{2} \ll \prod_{p \mid r} \left( 1 + \frac{1}{p^{3/4}} \right) \frac{1}{D(2\sigma + s)^{2}} \left( \sum_{l \mid r_{1}} a_{l}^{2} \left( \frac{r_{1}}{l} \right)^{-3} \right) r_{2}^{2(k-1)-4\sigma}.
\]

From the zero-free region result for \( D(s) \) (see, for example, [17], Theorem 5.1) and a standard argument (due to Landau, see [21], §3.9 and §3.11), we have
\[
D(s) \neq 0, \quad 1 / D(s) \ll \log(|y| + 3),
\]
for \( s = x + iy, \ x \geq k - 2\delta / \log(|y| + 3) \). Note that
\[
\sum_{r \leq \xi} \frac{a_{r}^{2}}{r^{2\sigma}} \prod_{p \mid r} \left( 1 + \frac{1}{p^{3/4}} \right) = \sum_{r \leq \xi} \frac{a_{r}^{2}}{r^{2\sigma}} \sum_{u \mid r} \frac{|\mu(u)|}{u^{3/4}} = \sum_{u \leq \xi} \frac{|\mu(u)|}{u^{3/4+2\sigma}} \sum_{r \leq \xi/u} \frac{a_{ru}^{2}}{r^{2\sigma}} \ll \sum_{u} \frac{1}{u^{3/4+2\sigma}} \left( \sum_{(r, u) = 1} a_{r}^{2} \right) \left( \sum_{r \mid u} a_{ru}^{2} \right) \ll D(2\sigma) \sum_{u} \frac{d^{2}(u)}{u^{3/4+2\sigma}} \sum_{r \mid u} \frac{(ru)^{k-1} d^{2}(r)}{r^{2\sigma}} \ll D(2\sigma) \sum_{u} \frac{d^{2}(u)}{u^{7/4}} \sum_{r \mid u} \frac{d^{2}(r)}{r} \ll D(2\sigma).
\]

Here we have used Deligne’s bound for the Hecke eigenvalues \( a_{ru} \), but it is clear that the weaker and more elementary bound \( a_{r} \ll r^{(k-1)/2 + 1/4 + \varepsilon} \) suffices for the same purpose. Hence
\[
\sum_{r_1 r_2 \leq \xi, \mu(r_1 r_2) \neq 0} d(r_2) r_2^{2(k-1)-4\sigma} \sum_{t \mid r_1} \frac{a_t^2}{t^{2\sigma}} \prod_{p \mid t} \left(1 + \frac{1}{p^{3/4}}\right) \left(\frac{r_1}{t}\right)^{-3} \prod_{p \mid r_2} \left(1 + \frac{1}{p^{3/4}}\right)
\]
\[
\ll \sum_{t \leq \xi} \frac{a_t^2}{t^{2\sigma}} \prod_{p \mid t} \left(1 + \frac{1}{p^{3/4}}\right) \sum_{r \leq \xi/t} \frac{d(r)}{r^3} \sum_{r_2 \leq \sqrt{\xi}} \frac{d(r_2)}{r_2^2} \ll D(2\sigma).
\]

Thus,
\[
D(2\sigma) \sum_{r \leq \xi} \sum_{l \mid r} |\mu(l)| \left(\frac{r}{l}\right)^{2\sigma} \left| \sum_{v} \frac{\lambda_{rv}}{(rv)^{2\sigma}} P(lv, 2\sigma) \right|^2 \ll 1.
\]

(Note that \(2\sigma - k \gg 1/\log \xi\).

In case (b), \(\xi < r \leq \xi^2\), we have
\[
\sum_{v} \frac{\lambda_{rv}}{(rv)^{2\sigma}} P(lv, 2\sigma) = \frac{1}{2\pi i} \int_{(2)} \left(\frac{\xi^2}{r}\right)^s \frac{1}{\pi^2} \left(\sum_{v} \left|\frac{\mu_f(rv)}{(rv)^{2\sigma}} P(lv, 2\sigma)\right| \right) \frac{ds}{\log \xi}.
\]

The treatment is the same except that the above integrand has a double pole at \(s = 0\). Using \(\sum_{p \mid r} \log p/p \ll \log \log r\), we can establish that
\[
D(2\sigma) \sum_{\xi < r \leq \xi^2} \sum_{l \mid r} |\mu(l)| \left(\frac{r}{l}\right)^{2\sigma} \left| \sum_{v} \frac{\lambda_{rv}}{(rv)^{2\sigma}} P(lv, 2\sigma) \right|^2 \ll 1.
\]

Hence \(S_1 \ll T\). Similarly, \(S_2 \ll T\). If we choose \(\psi_U(t/T)\) to be the majorant of the characteristic function of \([T, 2T]\) (here \(U \ll 1\)), then we have, with \(\xi = T^{1/72}\),
\[
\int_{T}^{2T} |L_f(\sigma + it)|^2 |M_{\xi^2}(\sigma + it)|^2 \, dt \ll T.
\]

In particular, we have

**Lemma 3.1.** Let \(\sigma = k/2 + A/\log T\). Then
\[
\int_{T}^{2T} |L_f(\sigma + it) M_{\xi^2}(\sigma + it) - 1|^2 \, dt \ll T.
\]

We also have

**Lemma 3.2.**
\[
\int_{T}^{2T} |L_f(k/2 + 1 + it) M_{\xi^2}(k/2 + 1 + it) - 1|^2 \, dt \ll T^{1-1/72}.
\]

**Proof.** Lemma 3.2 follows immediately from the equality (see [16])
\[
\int_{0}^{T} \left| \sum_{n=1}^{\infty} a_n n^i t \right|^2 \, dt = \sum_{n=1}^{\infty} |a_n|^2 (T + O(n)).
\]
From Lemma 3.1, Lemma 3.2 and an easy modification of the classical convexity theorem (see [21], §7.8), we deduce that

**Theorem 3.3.** We have
\[ \int_T^{2T} |L_f(\sigma + it)M_\xi(\sigma + it) - 1|^2 \, dt \ll T^{1-\frac{1}{2}(\sigma-k/2)}, \]
uniformly for \( k/2 + A/\log T \leq \sigma \leq k/2 + 1 \).

Now we are in a position to prove our main theorem.

**Theorem 3.4.** We have
\[ N_f(\sigma, T) \ll T^{1-\frac{1}{2}(\sigma-k/2)} \log T, \]
uniformly for \( k/2 \leq \sigma \leq (k+1)/2 \).

First, we show the following proposition.

**Proposition 3.5.** We have
\[ \int_\sigma^{(k+1)/2} (N_f(\sigma', 2T) - N_f(\sigma', T)) \, d\sigma' \ll T^{1-\frac{1}{2}(\sigma-k/2)}, \]
uniformly for \( k/2 \leq \sigma \leq (k+1)/2 \).

**Proof.** It suffices to prove that
\[ \int_\sigma^{(k+1)/2} (N_f(\sigma', 2T) - N_f(\sigma', T)) \, d\sigma' \ll T^{1-\frac{1}{2}(\sigma-k/2)}, \]
for \( k/2 + A/\log \xi \leq \sigma \leq (k+1)/2 \).

Let \( \Phi(s) = 1 - (L_f(s)M_\xi(s) - 1)^2 \). The zeros of \( L_f(s) \) occur among those of \( \Phi(s) \), with at least the same multiplicities. By Littlewood’s lemma concerning the number of zeros of an analytic function in a rectangle [22], we have
\[ \int_\sigma^{(k+1)/2} (N_f(\sigma', 2T) - N_f(\sigma', T)) \, d\sigma' \leq \frac{1}{2\pi} \int_T^{2T} \log |\Phi(\sigma + it)| \, dt + \frac{1}{2\pi} \int_\sigma^{\infty} \arg \Phi(\sigma' + 2iT) \, d\sigma' \]
\[ - \frac{1}{2\pi} \int_\sigma^{\infty} \arg \Phi(\sigma' + iT) \, d\sigma'. \]
In the range \(((k+1)/2 + 4, \infty)\), we see that
\[ \arg \Phi(\sigma' + it) = O(2^{-\sigma'}). \]
Hence this part of the integral is $O(1)$. In the range $(k/2, (k + 1)/2 + 4)$, it follows from Jensen’s theorem [22] and a standard argument (see [19]) that
\[
\arg \Phi(\sigma' + iT) = O(\log T).
\]
We deduce that
\[
\int_{\sigma}^{\infty} \arg \Phi(\sigma' + iT) \, d\sigma' \ll \log T.
\]
Finally, since $\log(1 + |x|) \leq |x|$,
\[
\int_{T}^{2T} \log |\Phi(\sigma + iT)| \, dt \leq \int_{T}^{2T} |L_f(\sigma + iT)M_2(\sigma + iT) - 1|^2 \, dt = O(T^{1-\frac{1}{12}(\sigma-k/2)}).
\]
This proves the proposition.

**Proof of Theorem 3.4.** It suffices to assume that $\sigma - k/2 \geq 1/\log T$. Thus,
\[
N_f(\sigma, T) \leq \log T \int_{\sigma-1/\log T}^{\sigma} N_f(\sigma', T) \, d\sigma' \\
\leq \log T \int_{\sigma-1/\log T}^{(k+1)/2} N_f(\sigma', T) \, d\sigma' \\
\ll T^{1-\frac{1}{72}(\sigma-k/2)} \log T.
\]
Our proof is now complete.

**4. Appendix.** In this section, we will give a simple proof of Hafner’s result which is used in Section 2 without appealing to the spectral theory of the Laplacian acting on $L^2(\Gamma_0(a,b)\backslash \mathbb{H})$. Our approach is based upon the delta-symbol method introduced by Duke–Friedlander–Iwaniec [5] and does not require a discussion of exceptional eigenvalues for the congruence subgroups. Instead, we only need Weil’s bound for the Kloosterman sums. Our result is quantitatively a little weaker than Hafner’s but is sufficient for our application. Furthermore, our method can as well be applied to the case when $a_n$ is the Fourier coefficient of a Maass form so it appears to be of independent interest.

Let $a_n$ be the $n$th Fourier coefficient of a (holomorphic) Hecke eigenform of weight $k$. We consider the sum
\[
\sum_{m \mu - n \nu = l, \, x \leq n \leq 2x} a_n a_m, \quad \mu, \nu, l > 0, \, x \geq 10.
\]
Let \( g(\xi) \) be a smooth function on \( \mathbb{R} \) with compact support such that 
\[ 0 \leq g(\xi) \leq 1; \quad g(\xi) = 1 \text{ if } x \leq \xi \leq 2x; \quad \text{supp}(g(\xi)) \subset [x - x^{1-\theta}, 2x + x^{1-\theta}] \] for some \( 0 < \theta < 1; \) and \( g^{(p)}(\xi) \ll_p (x^{1-\theta})^{-p}, \ p \geq 0. \) Then we have
\[
\sum_{m\mu - n\nu = l, x \leq n \leq 2x} a_n a_m g(n) + O\left( \left( \frac{x\nu + l}{\mu} \right)^{(k-1)/2+\varepsilon} x^{(k-1)/2+\varepsilon} x^{1-\theta} \right).
\]

Let \( h(\xi) \) be another smooth function with compact support such that 
\[ 0 \leq h(\xi) \leq 1; \quad h(\xi) = 1, \text{ if } (\frac{3}{8} x\nu + l)/\mu \leq \xi \leq (\frac{5}{8} x\nu + l)/\nu; \quad \text{supp}(h(\xi)) \subset \left[ (\frac{1}{2} x\nu + l)/\mu, (\frac{9}{4} x\nu + l)/\nu \right]; \] and \( h^{(p)}(\xi) \ll_p (x\nu/\mu)^{-p}, \ p \geq 0. \) Clearly we have
\[
\sum_{m\mu - n\nu = l} a_n a_m g(n) = \sum_{m\mu - n\nu = l} a_n a_m g(n) h(m).
\]

Next we will recall the delta-symbol method introduced in [5].

Define
\[
\delta(n) = \begin{cases} 
1 & \text{if } n = 0, \\
0 & \text{if } n \neq 0.
\end{cases}
\]

Let \( \omega(t) \) be an even function on \( \mathbb{R} \) with \( \omega(0) = 0 \) and compactly supported such that \( \sum_{k=1}^{\infty} \omega(k) = 1. \) Let
\[
\delta_k(n) = \omega(k) - \omega\left( \frac{n}{k} \right).
\]

Then clearly \( \delta(n) = \sum_{k|n} \delta_k(n). \) Thus
\[
\delta(n) = \sum_k k^{-1} \sum_{h \mod k} e\left( \frac{hn}{k} \right) \delta_k(n).
\]

Put
\[
\Delta_c(n) = \sum_r r^{-1} \delta_{cr}(n).
\]

Writing \( r = (h, k), \ a = h/r, \ c = k/r, \) we have
\[
\delta(n) = \sum_c c^{-1} \sum_{a \mod c}^* e\left( \frac{an}{c} \right) \Delta_c(n).
\]

We will apply the above identity to integers \( |n| < N/2, \) say, with \( \omega(t) \) supported on \( K/2 < |t| < K, \) and whose derivatives satisfy \( \omega^{(j)}(t) \ll K^{-j-1}. \) Now, \( \delta_k(n) \) vanishes except for \( 1 \leq k < \max(K, N/K) = K \) by choosing \( K = N^{1/2}. \) Hence \( \Delta_c(n) \) vanishes except for \( 1 \leq c < K \) and \( \Delta_c(n) \ll K^{-1} \log K. \) Let \( \Delta_1 = x\nu/\mu, \ \Delta_2 = x^{1-\theta}, \ \Delta = \min(\Delta_1, \Delta_2). \) We infer that,
by (10) and (11),
\[ S = \sum_{m\mu - n\nu = l} a_n a_m g(n) h(m) \]
\[ = \sum_{m,n} a_n a_m g(n) h(m) \delta(m\mu - n\nu - l) = \sum_c c^{-1} S_c, \]
where
\[ (12) \quad S_c = \sum_{a \mod c}^{\ast} \sum_{m,n} a_m a_n g(n) h(m) e\left(\frac{a(m\mu - n\nu - l)}{c}\right) \Delta_c(m\mu - n\nu - l) \]
\[ = \sum_{a \mod c}^{\ast} e\left(-\frac{al}{c}\right) \sum_{m,n} b_m b_n e\left(\frac{a}{c}(m\mu - n\nu)\right) F(m,n), \]
with \( b_m = a_m m^{-(k-1)/2} \) and
\[ F(m,n) = (mn)^{(k-1)/2} h(m) g(n) \Delta_c(\mu m - \nu n - l). \]
Define \( \gamma = \max(\mu, \nu), K^2 = N = 8x\gamma l. \) It is easy to see that
\[ (13) \quad \frac{d}{dn} \Delta_c(n) \left\{ \begin{array}{ll}
\ll 1/K|n| & \text{if } |n| \gg Kc, \\
0 & \text{otherwise}.
\end{array} \right. \]
We have, for \( i + j \geq 1, \)
\[ (14) \quad \frac{\partial^{i+j}}{\partial \xi^i \partial \eta^j} F(\xi, \eta) \ll \left(\frac{\nu + l}{\mu}\right)^{(k-1)/2} x^{k-1/2} K^{-1} \left(\frac{\Delta_c}{K}\right)^{-i-j+1} \left(\frac{\mu + \nu}{|\mu \xi - \nu \eta - l|} + \frac{1}{\Delta}\right) \]
if \( |\mu \xi - \nu \eta - l| \gg Kc, \) and without the term \( (\mu + \nu)/|\mu \xi - \nu \eta - l| \) if otherwise.
We need the following Poisson-type formula [4]:

**Lemma 4.1.** Let \( F \) be a smooth and compactly supported function on \( \mathbb{R}^+ \).
For any integers \( c \geq 1 \) and \( (a,c) = 1 \) we have
\[ \sum_m b_m e\left(\frac{am}{c}\right) F(m) = \sum_r b_r e\left(\frac{-ar}{c}\right) \tilde{F}(r), \]
where \( a\tilde{a} \equiv 1 \mod c \) and \( \tilde{F}(r) \) is the Hankel-type transform
\[ \tilde{F}(y) = 2\pi i^{k-1} \int_0^\infty F(x) J_{k-1}\left(\frac{4\pi}{c} \sqrt{xy}\right) dx, \]
where \( J_\nu(z) \) is the usual Bessel function.

Applying Lemma 4.1 in each variable \( m, n \) in (12), we deduce that
\[ (15) \quad S_c = \sum_{a \mod c}^{\ast} e\left(\frac{al}{c}\right) \sum_{r_1, r_2} b_{r_1} b_{r_2} e\left(\frac{\pi r_1}{c_1} r_1 - \frac{\pi r_2}{c_2} r_2\right) \tilde{F}(r_1, r_2), \]
where
\[ \mu_1 = \frac{\mu}{(\mu, c)}, \quad c_1 = \frac{c}{(\mu, c)}, \quad \nu_1 = \frac{\nu}{(\nu, c)}, \quad c_2 = \frac{c}{(\nu, c)}, \]
and
\[ (16) \quad \tilde{F}(r_1, r_2) = 4\pi^2 \int_0^\infty \int_0^\infty F(x_1, x_2) J_{k-1}\left( \frac{4\pi}{c_1} \sqrt{x_1 r_1} \right) \left( \frac{4\pi}{c_2} \sqrt{x_2 r_2} \right) dx_1 dx_2. \]

By the recurrence formula
\[ \frac{d}{dz} (z^\nu J_\nu(z)) = z^\nu J_{\nu-1}(z) \]
and the bound \( J_\nu(z) \ll (1 + z)^{-1/2} \), we obtain, by partial integration twice in each variable in (16) and using (14),
\[ \sum_{r_1, r_2} b_{r_1} b_{r_2} |\tilde{F}(r_1, r_2)| \ll K(l\gamma x^{\theta+\epsilon} x^{3/4} / \mu)^{(k-1)/2} x^{(k-1)/2}. \]

The sum over \( a \) in (15) is a Kloosterman sum \( S(l, *, c) \) to which we apply Weil’s bound. Thus, we infer that
\[ S_c \ll (l, c)^{1/2} c^{1/2} \tau(c)(xl\gamma)^{1/2} (\gamma l x^{\theta+\epsilon})^{9/4} \left( \frac{x^\nu + l}{\mu} \right)^{(k-1)/2} x^{(k-1)/2}. \]
Hence
\[ S \ll (xl\gamma)^{3/4} (l\gamma x^{\theta+\epsilon})^{9/4} \left( \frac{x^\nu + l}{\mu} \right)^{(k-1)/2} x^{(k-1)/2} \]
\[ \ll \left( \frac{x^\nu + l}{\mu} \right)^{(k-1)/2} x^{(k-1)/2} \gamma^3 l^{3/4} x^{13/14}. \]

We conclude that
\[ \sum_{n \leq x} a_n a_{(n\nu+l)/\mu} \ll \left( \frac{x^\nu + l}{\mu} \right)^{(k-1)/2} x^{(k-1)/2} \gamma^3 l^{3/4} x^{13/14}, \]
on taking \( \theta = 1/13 \). Finally, since for \( \Re s > k \),
\[ D_{\mu, \nu}(s, l) := \sum_{n=1}^\infty \frac{a_n a_{(n\nu+l)/\mu}}{(n\nu + l/2)^s} = \int_1^{\infty} \frac{1}{(x^\nu + l/2)^s} d\left( \sum_{n \leq x} a_n a_{(n\nu+l)/\mu} \right) \]
\[ = s\nu \int_{1/2}^{\infty} \frac{\sum_{n \leq x} a_n a_{(n\nu+l)/\mu}}{(x^\nu + l/2)^{s+1}} dx, \]
we obtain
Theorem 4.2. \( D_{\mu,\nu}(s,l) \) can be analytically continued to \( \Re s > k - 1/14 \), and for \( \Re s > k - 1/14 \), \( s = \sigma + it \), we have
\[
D_{\mu,\nu}(s,l) \ll \frac{l^3|s|}{(\mu\nu)^{(k-1)/2}(\max(\mu,\nu))^3} \frac{1}{\sigma - (k - 1/14)}.
\]

References


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