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Zeros of Hecke L-functions associated with cusp forms

by

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1. Introduction. Let $f(z) = \sum_{n=1}^{\infty} a_n e(nz)$ be a holomorphic cusp form of even integral weight $k > 0$ with respect to the modular group $\Gamma = \mathrm{SL}(2,\mathbb{Z})$, and define (for $\Re s > (k+1)/2$)

$$L_f(s) = \sum_{n=1}^{\infty} a_n n^{-s},$$

the associated Hecke L-function. We also assume that $f(z)$ is a Hecke eigenform [11] with $a_1 = 1$. Recall that we have the bound for the coefficients

$$|a_n| \le d(n) n^{(k-1)/2}$$

by Deligne's proof of the Ramanujan–Petersson conjecture [2], [3], and the bound for the square mean [9], [18],

$$\sum_{n\le N} |a_n|^2 \ll N^k.$$

It is well known [10] that $L_f(s)$ admits analytic continuation to $\mathbb{C}$ as an entire function and satisfies the functional equation

$$(2\pi)^{-s}\Gamma(s)L_f(s) = (-1)^{k/2}(2\pi)^{-(k-s)}\Gamma(k-s)L_f(k-s).$$

Moreover, $L_f(s)$ has Euler product representation ($\Re s > (k+1)/2$)

$$L_f(s) = \prod_p (1 - a_p p^{-s} + p^{k-1}p^{-2s})^{-1}.$$

The non-trivial zeros of $L_f(s)$ lie within the strip $(k-1)/2 < \Re s < (k+1)/2$, symmetrically to the real axis and the critical line $\sigma = k/2$. The Riemann Hypothesis for $L_f(s)$ asserts that all the non-trivial zeros of $L_f(s)$ lie on the critical line $\Re s = k/2$. Hafner [13], generalizing Selberg's remarkable work [19] on $\zeta(s)$, has shown that a positive proportion of all non-trivial zeros are on the critical line.

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In this work, we establish the analogue of Selberg's density theorem [20] for $L_f(s)$. Define, for $\sigma \ge k/2$ and $T \ge 1$,

$$N_f(\sigma, T) = |\{\beta + i\gamma : L_f(\beta + i\gamma) = 0,\ \beta \ge \sigma,\ 0 < \gamma \le T\}|.$$

It was proved by Lekkerkerker [15] that

$$N_f\left(\frac{k-1}{2}, T\right) \sim \frac{1}{\pi} T \log T.$$

We will show that

Theorem 1.1. *For some $a > 0$ we have*

$$N_f(\sigma, T) \ll T^{1-a(\sigma-k/2)} \log T,$$

uniformly for $k/2 \le \sigma \le (k+1)/2$.

Our proof shows that one may take $a = 1/72$. However, we make no effort to obtain an optimal a by our method.

Application of standard techniques of analytic number theory easily yields results of the type

$$N_f(\sigma, T) \ll T^{c(\sigma)} (\log T)^A,$$

where $c(\sigma) < 1$ for $\sigma > k/2$ and some $A > 0$. The significance of Theorem 1.1 lies in that A can be taken to be 1. Selberg used the analogue of Theorem 1.1 for $\zeta(s)$ to prove his famous result on the moments of $\arg\zeta(1/2 + it)$. In view of recent work of Bombieri and Hejhal [1], there is a similar application to $\arg L_f(k/2 + it)$, which is the main motivation of the present paper.

Corollary 1.2. *The functions*

$$\frac{\log |L_f(k/2 + it)|}{\sqrt{\pi \log\log t}}, \qquad \frac{\arg L_f(k/2 + it)}{\sqrt{\pi \log\log t}}$$

become distributed, in the limit of large t, like independent random variables, each having Gaussian density $\exp(-\pi u^2) du$.

To prove Theorem 1.1 by Selberg's method, one considers not $L_f(s)$ itself, but $L_f(s) M_X(s)$, where the mollifier $M_X(s)$ is a Dirichlet polynomial of length $X = T^\theta$, $0 < \theta < 1/2$, and is chosen such that $L_f(s) M_X(s)$ is very close to 1 in the region $\sigma > k/2$, $0 < t \le T$, or more precisely such that the mean value

$$\frac{1}{T} \int_T^{2T} |L_f(\sigma + it) M_X(\sigma + it) - 1|^2 \, dt$$

is very small, i.e.

$$\int_T^{2T} |L_f(\sigma + it) M_X(\sigma + it) - 1|^2 \, dt \ll T^{1-a(\sigma-k/2)},$$

uniformly for $k/2 \le \sigma \le (k+1)/2$. It is then possible to deduce, by a standard argument (see §3), that the zeros of $L_f(s)M_X(s)$ and *a fortiori* of $L_f(s)$ in the region considered are comparatively few. The required mean value estimate is obtained as follows. If we prove

$$\int_T^{2T} |L_f(k/2+it)M_X(k/2+it)-1|^2\,dt \ll T,$$

and

$$\int_T^{2T} |L_f(k/2+1+it)M_X(k/2+1+it)-1|^2\,dt \ll T^{1-a},$$

then by a convexity theorem (see §3) we have

$$\int_T^{2T} |L_f(\sigma+it)M_X(\sigma+it)-1|^2\,dt \ll T^{1-a(\sigma-k/2)},$$

uniformly for $k/2 \le \sigma \le k/2+1$, which is all we need. The second inequality is easy to prove since $L_f(s)$ is an absolutely convergent Dirichlet series for $\Re s > (k+1)/2$ and $M_X(s)$ is an approximate inverse to $L_f(s)$ such that $L_f(s)M_X(s)-1$ is given by a Dirichlet series of the type $\sum_{n\ge y} b_n n^{-s}$, with y large. The first inequality represents the main difficulty, since it is not obtainable by a routine extension of Selberg's work in the $\zeta(s)$ case. We will replace $L_f(s)$ by a Dirichlet polynomial of length $\sim T$ using the approximate functional equation of $L_f(s)$ in the form obtained by A. Good [7]. The resulting expression then becomes

$$\int_T^{2T} |P_{TX}(\sigma+it)|^2\,dt,$$

where P_{TX} is a Dirichlet polynomial of length TX. However, in general no method succeeds in handling the above mean value once $P(s)$ has length $\gg T$. Therefore we have to make careful use of the special feature of $M_X(s)$. In fact, the argument similar to Selberg [19] and Hafner [13], with some modification, is suitable here. For some technical reason we will prove the first inequality for $\sigma = k/2 + 1/\log T$ rather than $k/2$ and then apply a convexity theorem to obtain Theorem 1.1.

We remark here that in our proof of Theorem 1.1 Deligne's bound for the Fourier coefficients a_n is used but not crucial here. A weaker bound like $a_n \ll n^{(k-1)/2+1/4+\varepsilon}$ which follows from Weil's bound for the Kloosterman sums and the bound for the square mean mentioned before would suffice. Thus, our method should be applicable when $f(z)$ is a Maass form, though the Ramanujan–Petersson conjecture remains unproved in this case.

We would like to mention that D. Farmer [6], using Hafner's method and the spectral theory, establishes an asymptotic formula for the mean square of $L_f(s)$ weighted by a general mollifier of Levinson's type. He mentions that this mean value theorem can be combined with Jutila's method [14] to give a density result, but he does not give any details.

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2. Main lemma. Let $\psi_U(t)$ be a non-negative smooth function such that

$$\psi_U(t) = \begin{cases} 0 & \text{if } t \le 1 - 1/U \text{ or } t \ge 2 + 1/U, \\ 1 & \text{if } 1 + 1/U \le t \le 2 - 1/U; \end{cases}$$

and

$$\psi_U^{(p)}(t) \ll U^p, \qquad p \ge 0,$$

where U is a positive parameter and in our discussion it will be chosen as $O(1)$ later. The object of this section is to prove the following lemma, which is the analogue of Lemma 6 in [19].

LEMMA 2.1. *If $k/2 < \sigma \le k/2 + 1/40$, $\varepsilon > 0$, and μ, ν are positive coprime integers $\le T$, then*

$$\begin{aligned}\int_{-\infty}^{\infty} \psi_U\Big(\frac{t}{T}\Big) |L_f(\sigma+it)|^2 \Big(\frac{\mu}{\nu}\Big)^{it}\,dt &= \frac{1}{(\mu\nu)^{\sigma}} D_{\mu\nu}(2\sigma) \int_{-\infty}^{\infty} \psi_U\Big(\frac{t}{T}\Big)\,dt \\ &\quad + \frac{1}{(\mu\nu)^{k-\sigma}} D_{\mu\nu}(2k-2\sigma) \int_{-\infty}^{\infty} \psi_U\Big(\frac{t}{T}\Big)\Big(\frac{t}{2\pi}\Big)^{2k-4\sigma}\,dt \\ &\quad + O\Big(\frac{(\mu\nu)^3 U^4 T^{4/5}}{2\sigma - k}\Big),\end{aligned}$$

where

$$D_{\mu\nu}(s) = \sum_{l=1}^{\infty} \frac{a_{\mu l} a_{\nu l}}{l^s}.$$

P r o o f. The proof is very similar to the treatment in [8] and [13], and so we give only a sketch. We have (denote $\sigma + it$ by s)

$$\begin{aligned}
&\int_{-\infty}^{\infty} \psi_U\Big(\frac{t}{T}\Big)|L_f(\sigma+it)|^2\Big(\frac{\mu}{\nu}\Big)^{it}\,dt\\
&= \int_{-\infty}^{\infty} \psi_U\Big(\frac{t}{T}\Big)L_f(\sigma+it)L_f(\sigma-it)\Big(\frac{\mu}{\nu}\Big)^{it}\,dt\\
&= \int_{-\infty}^{\infty} \bigg(\sum_{n=1}^{\infty} a_n n^{-\sigma}(n\nu)^{-it}\phi\Big(\frac{2\pi n}{t}\sqrt{\frac{\nu}{\mu}}\Big)\\
&\quad + (2\pi)^{2s-k}\frac{\Gamma(k-s)}{\Gamma(s)}\sum_{n=1}^{\infty} a_n n^{\sigma-k}\Big(\frac{n}{\nu}\Big)^{it}\phi^*\Big(\frac{2\pi n}{t}\sqrt{\frac{\mu}{\nu}}\Big)\bigg)\\
&\quad \times \bigg(\sum_{m=1}^{\infty} a_m m^{-\sigma}(m\mu)^{it}\phi\Big(\frac{2\pi m}{t}\sqrt{\frac{\mu}{\nu}}\Big)\\
&\quad + (2\pi)^{2\bar{s}-k}\frac{\Gamma(k-\bar{s})}{\Gamma(\bar{s})}\sum_{m=1}^{\infty} a_m m^{\sigma-k}\Big(\frac{m}{\mu}\Big)^{-it}\phi^*\Big(\frac{2\pi m}{t}\sqrt{\frac{\nu}{\mu}}\Big)\bigg)\psi_U\Big(\frac{t}{T}\Big)\,dt\\
&\quad + O\bigg(\bigg(\sqrt{\frac{\mu}{\nu}}+\sqrt{\frac{\nu}{\mu}}\bigg)\log^2 T\bigg).
\end{aligned}$$

Here we use the approximate functional equation for $L_f(\sigma \pm it)$ (see [7], Satz), and $\phi(\xi)$, $\phi^*(\xi)$ are suitable smooth functions satisfying $\phi^*(\xi) = 1-\phi(1/\xi)$, and $\phi(\xi) = 1$, $|\xi| \le 2/3$; $\phi(\xi) = 0$, $|\xi| \ge 3/2$.

Multiplying out the expression in the above integrand and using the same argument and notation as in [8], §2 (see also [13], §3), we see that the above expression equals

$$\begin{aligned}
&\sum_{n,m}\frac{a_n a_m}{(nm)^{\sigma}}\int_{-\infty}^{\infty} \psi_U\Big(\frac{t}{T}\Big)\Big(\frac{m\mu}{n\nu}\Big)^{it}\Phi\Big(\frac{2\pi n}{t}\sqrt{\frac{\nu}{\mu}},\frac{2\pi m}{t}\sqrt{\frac{\mu}{\nu}}\Big)\,dt\\
&\qquad + \sum_{n,m}\frac{a_n a_m}{(nm)^{k-\sigma}}(2\pi)^{-2k+4\sigma}\\
&\qquad \times \int_{-\infty}^{\infty} t^{2k-4\sigma}\psi_U\Big(\frac{t}{T}\Big)\Big(\frac{n\mu}{m\nu}\Big)^{it}\Phi^*\Big(\frac{2\pi n}{t}\sqrt{\frac{\mu}{\nu}},\frac{2\pi m}{t}\sqrt{\frac{\nu}{\mu}}\Big)\,dt\\
&\qquad + O\bigg(\frac{\log^2 T}{2\sigma-k}\bigg(\sqrt{\frac{\nu}{\mu}}+\sqrt{\frac{\mu}{\nu}}\bigg)\bigg) = S_1+S_2+S_3,
\end{aligned}$$

say, where Φ, Φ^* are certain smooth functions with compact supports, and $\Phi(\varrho,\varrho) = \phi(\varrho)$, $\Phi^*(\varrho,\varrho) = \phi^*(\varrho)$.

For S_1, the terms with $m\mu = n\nu$ give

$$\sum_{l=1}^{\infty} \frac{a_{\mu l} a_{\nu l}}{(\mu\nu l^2)^\sigma} \int_{-\infty}^{\infty} \psi_U\Big(\frac{t}{T}\Big)\phi\Big(\frac{2\pi\sqrt{\mu\nu}l}{t}\Big)\,dt. \tag{1}$$

For the terms with $m\mu \neq n\nu$, we let $m\mu - n\nu = l$. Without loss of generality we may assume $l > 0$. Then the non-diagonal terms with $l > 0$ give

$$\begin{aligned}
&\sum_{l>0}\sum_{n=1}^{\infty} \frac{a_n a_{(n\nu+l)/\mu}}{\big(n\frac{n\nu+l}{\mu}\big)^\sigma} \\
&\quad\times \int_{-\infty}^{\infty} \psi_U\Big(\frac{t}{T}\Big)\Big(\frac{n\nu+l}{n\nu}\Big)^{it} \varPhi\Big(\frac{2\pi n}{t}\sqrt{\frac{\nu}{\mu}}, \frac{2\pi(n\nu+l)}{t\sqrt{\mu\nu}}\Big)\,dt \\
&= (\mu\nu)^\sigma \sum_{0<l\le\sqrt{\mu\nu}UT^\varepsilon}\sum_{n=1}^{\infty} \frac{a_n a_{(n\nu+l)/\mu}}{(n\nu+l/2)^{2\sigma}} \\
&\quad\times \int_{-\infty}^{\infty} \psi_U\Big(\frac{t}{T}\Big)\phi\Big(\frac{2\pi(n\nu+l/2)}{t\sqrt{\mu\nu}}\Big)e^{it\frac{l}{n\nu+l/2}}\,dt \\
&\quad + O\Big(\frac{(\mu\nu)^3U^4\log^2 T}{2\sigma-k}\Big).
\end{aligned} \tag{2}$$

For S_2, the terms $n\mu = m\nu$ give

$$(2\pi)^{-2k+4\sigma}\sum_{l=1}^{\infty} \frac{a_{\mu l} a_{\nu l}}{(\mu\nu l^2)^{k-\sigma}} \int_{0}^{\infty} \psi_U\Big(\frac{t}{T}\Big)\phi^*\Big(\frac{2\pi\sqrt{\mu\nu}l}{t}\Big)t^{2k-4\sigma}\,dt. \tag{3}$$

For the terms with $n\mu \neq m\nu$, we let $m\nu - n\mu = l$, and the non-diagonal terms with $l > 0$ give

$$\begin{aligned}
&(\mu\nu)^{k-\sigma}(2\pi)^{-2k+4\sigma} \sum_{0<l\le\sqrt{\mu\nu}UT^\varepsilon}\sum_{n=1}^{\infty} \frac{a_n a_{(n\mu+l)/\nu}}{(n\mu+l/2)^{2k-2\sigma}} \\
&\times \int_{-\infty}^{\infty} \psi_U\Big(\frac{t}{T}\Big)t^{2k-4\sigma}\phi^*\Big(\frac{2\pi(n\mu+l/2)}{t\sqrt{\mu\nu}}\Big)e^{-it\frac{l}{n\nu+l/2}}\,dt + O\Big(\frac{(\mu\nu)^3U^4\log^2 T}{2\sigma-k}\Big).
\end{aligned} \tag{4}$$

Let

$$G(s) = \int_0^{\infty} \phi(x)x^{s-1}\,dx, \qquad G^*(s) = \int_0^{\infty} \phi^*(x)x^{s-1}\,dx.$$

Then, by Mellin inversion,

$$\phi(x) = \frac{1}{2\pi i}\int_{(2)} G(s)x^{-s}\,ds, \qquad \phi^*(x) = \frac{1}{2\pi i}\int_{(2)} G^*(s)x^{-s}\,ds.$$

Note that $G(s)$ and $G^*(s)$ are analytic except for a simple pole at $s = 0$ with residue 1, and $D_{\mu\nu}(s)$ has only a simple pole at $s = k$ for $\Re s \ge k - 1/2$,

$$D_{\mu\nu}(s) = P(\mu, s)P(\nu, s)D(s) \quad \text{with } D(s) = \sum_{l=1}^{\infty} a_l^2 l^{-s};$$

$$P(a, s) = \prod_{p^r \| a} \Big(\sum_{j=0}^{\infty} a_{p^{r+j}} a_{p^j} p^{-js}\Big)\Big(\sum_{j=0}^{\infty} a_{p^j}^2 p^{-js}\Big)^{-1};$$

$$D(s) \ll t^{1+\varepsilon}, \quad \Re s \ge k - 1/2;$$

$$P(a, s) \ll a^{(k-1)/2+\varepsilon}, \quad \Re s \ge k - 1/2;$$

$$|G(s)| + |G^*(s)| \ll_l \frac{1}{|s(s+1)\ldots(s+l)|}.$$

Thus (1) and (3) equal respectively

$$\frac{1}{(\mu\nu)^\sigma} \cdot \frac{1}{2\pi i} \int_{(2)} \left(\frac{1}{2\pi\sqrt{\mu\nu}}\right)^s G(s) D_{\mu\nu}(2\sigma + s) \int_{-\infty}^{\infty} \psi_U\left(\frac{t}{T}\right) t^s \, dt, \tag{5}$$

$$\frac{1}{(\mu\nu)^{k-\sigma}} \cdot \frac{1}{2\pi i} \int_{(2)} \left(\frac{1}{2\pi\sqrt{\mu\nu}}\right)^s G^*(s) D_{\mu\nu}(2(k-\sigma) + s) \times \int_{-\infty}^{\infty} \psi_U\left(\frac{t}{T}\right) t^s \left(\frac{t}{2\pi}\right)^{2k-4\sigma} dt. \tag{6}$$

Next we shift the lines of integration in (5), (6) to $\Re s = -1/2$ and $\Re s = -1/2 - 2k + 4\sigma$, respectively. The integrands have poles at $k - 2\sigma, 0$ and $2\sigma - k, 0$, respectively. The residues at $k - 2\sigma$ and $2\sigma - k$ cancel out and the residues at 0 give the main terms.

By the estimate given above, we deduce that

$$\begin{aligned}(5) + (6) &= \frac{1}{(\mu\nu)^\sigma} D_{\mu\nu}(2\sigma) \int_{-\infty}^{\infty} \psi_U\left(\frac{t}{T}\right) dt \\ &+ \frac{1}{(\mu\nu)^{k-\sigma}} D_{\mu\nu}(2k - 2\sigma) \int_{-\infty}^{\infty} \psi_U\left(\frac{t}{T}\right)\left(\frac{t}{2\pi}\right)^{2k-2\sigma} dt + O(T^{1/2+\varepsilon}).\end{aligned}$$

Define

$$H_l(s) = \int_0^{\infty} \phi(\xi) e^{2\pi i (l/\sqrt{\mu\nu})\xi^{-1}} \xi^{s-1} \, d\xi.$$

This is an entire function and by the Mellin inversion formula,

$$\phi(\xi) e^{2\pi i (l/\sqrt{\mu\nu})\xi^{-1}} = \frac{1}{2\pi i} \int_{(2)} H_l(s) \xi^{-s} \, ds.$$

Thus the sum in (2) becomes

$$(\mu\nu)^{\sigma}\sum_{0<|l|\le\sqrt{\mu\nu}UT^{\varepsilon}}\sum_{n=1}^{\infty}\frac{a_n a_{(n\nu+l)/\mu}}{(n\nu+l/2)^{2\sigma}}$$

$$\times\int_{-\infty}^{\infty}\psi_U\left(\frac{t}{T}\right)\phi\left(\frac{2\pi(n\nu+l/2)}{t\sqrt{\mu\nu}}\right)e^{itl/(n\nu+l/2)}\,dt$$

$$=(\mu\nu)^{\sigma}\sum_{0<l\le\sqrt{\mu\nu}UT^{\varepsilon}}\frac{1}{2\pi i}\int_{(2)}\left(\frac{\sqrt{\mu\nu}}{2\pi}\right)^{s}H_l(s)D_{\mu\nu}(s+2\sigma,l)\int_{-\infty}^{\infty}\psi_U\left(\frac{t}{T}\right)t^s\,dt.$$

Here

$$D_{\mu\nu}(s,l)=\sum_{n=1}^{\infty}\frac{a_n a_{(n\nu+l)/\mu}}{(n\nu+l/2)^{s}}.$$

We move the line of integration to $\Re s=-1/5$. We have, on $\Re s=-1/5$,

$$\int_{-\infty}^{\infty}\psi_U\left(\frac{t}{T}\right)t^s\,dt\ll\frac{1}{(|s|+1)^4}U^3T^{4/5},$$

$$H_l(s)\ll\frac{\sqrt{\mu\nu}}{l}|s|,$$

and we also have Hafner's result [12]

$$D_{\mu\nu}(2\sigma+s,l)\ll\frac{l|s|^{1+\varepsilon}}{(\mu\nu)^{k/2-7/4}}.$$

Thus the above expression is majorized by

$$\frac{(\mu\nu)^{\sigma}}{(\mu\nu)^{k/2-7/4}}\mu\nu U^4T^{4/5}\ll(\mu\nu)^3U^4T^{4/5}.$$

Similarly, we obtain the same bound for the sum occurring in (4). Thus the proof is complete.

In Section 4 we will give an alternative, and more elementary, treatment of $D_{\mu\nu}(s,l)$ giving a slightly weaker analytic continuation result, which still suffices for the proof.

3. Proof of Theorem.

Let

$$L_f^{-1}(s)=\sum_{n=1}^{\infty}\frac{\mu_f(n)}{n^s},\qquad\Re s>\frac{k+1}{2}.$$

Thus

$$\mu_f(p^r)=\begin{cases}1 & \text{if } r=0,\\ -a_p & \text{if } r=1,\\ p^{k-1} & \text{if } r=2,\\ 0 & \text{if } r\ge 3.\end{cases}$$

Set $\lambda_n = \mu_f(n) g_\xi(n)$, where

$$g_\xi(n) = \begin{cases} 1 & \text{if } 1 \le n \le \xi, \\ \dfrac{\log(\xi^2/n)}{\log \xi} & \text{if } \xi \le n \le \xi^2, \\ 0 & \text{if } n \ge \xi^2, \end{cases}$$

and $\xi = T^\theta$, $0 < \theta < 1/4$ will be specified later.

We define the mollifier

$$M_{\xi^2}(s) = \sum_v \frac{\lambda_v}{v^s},$$

where $k/2 + A/\log \xi \le \sigma \le k/2 + \delta$, and A, δ^{-1} are sufficiently large positive numbers.

Using the multiplicativity of the Hecke eigenvalues a_n [11] and the definition of $P(n,s)$,

$$P(n,s) = \prod_{p^r \| n} P(p^r, s),$$

$$P(p^r, s) = \Big(\sum_{j=0}^{\infty} a_{p^{j+r}} a_{p^j} p^{-js} \Big) \Big(\sum_{j=0}^{\infty} a_{p^j}^2 p^{-js} \Big)^{-1},$$

we easily have

$$P(p,s) = \frac{a_p}{1 + p^{k-1-s}}, \qquad P(p^2, s) = \frac{a_p^2}{1 + p^{k-1-s}} - p^{k-1}.$$

We have, using Lemma 2.1,

$$\begin{aligned} &\int_{-\infty}^{\infty} \psi_U\Big(\frac{t}{T}\Big) |L_f(\sigma + it)|^2 |M_{\xi^2}(\sigma + it)|^2 \, dt \\ &= \sum_{v_1, v_2 \le \xi^2} \frac{\lambda_{v_1} \lambda_{v_2}}{(v_1 v_2)^{2\sigma}} (v_1, v_2)^{2\sigma} D_{v_1/(v_1,v_2), v_2/(v_1,v_2)}(2\sigma) \int_{-\infty}^{\infty} \psi_U\Big(\frac{t}{T}\Big) \, dt \\ &\quad + \sum_{v_1, v_2 \le \xi^2} \frac{\lambda_{v_1} \lambda_{v_2}}{(v_1 v_2)^{k}} (v_1, v_2)^{2(k-\sigma)} D_{v_1/(v_1,v_2), v_2/(v_1,v_2)}(2k - 2\sigma) \\ &\quad \times \int_{-\infty}^{\infty} \psi_U\Big(\frac{t}{T}\Big) \Big(\frac{t}{2\pi}\Big)^{2k - 4\sigma} \, dt + O(T^{4/5} U^3 \xi^{14} \log^3 \xi) \\ &= S_1 + S_2 + S_3, \end{aligned}$$

say. Since we have

$$D_{v_1/(v_1,v_2), v_2/(v_1,v_2)}(2\sigma) = D(2\sigma) P\Big(\frac{v_1}{(v_1, v_2)}, 2\sigma\Big) P\Big(\frac{v_2}{(v_1, v_2)}, 2\sigma\Big),$$

where

$$D(s) = D_{11}(s) = \sum_{l=1}^{\infty} \frac{a_l^2}{l^s},$$

it follows by Möbius inversion that

$$\begin{aligned} S_1 &= D(2\sigma) \int_{-\infty}^{\infty} \psi_U\left(\frac{t}{T}\right) dt \\ &\quad \times \sum_{v_1, v_2 \le \xi^2} \frac{\lambda_{v_1}\lambda_{v_2}}{(v_1 v_2)^{2\sigma}} (v_1, v_2)^{2\sigma} P\left(\frac{v_1}{(v_1, v_2)}, 2\sigma\right) P\left(\frac{v_2}{(v_1, v_2)}, 2\sigma\right) \\ &= D(2\sigma) \int_{-\infty}^{\infty} \psi_U\left(\frac{t}{T}\right) dt \\ &\quad \times \sum_{v_1, v_2 \le \xi^2} \frac{\lambda_{v_1}\lambda_{v_2}}{(v_1 v_2)^{2\sigma}} \sum_{r|v_1,\, r|v_2} \sum_{l|r} \mu(l) \left(\frac{r}{l}\right)^{2\sigma} P\left(\frac{v_1}{r/l}, 2\sigma\right) P\left(\frac{v_2}{r/l}, 2\sigma\right) \\ &= D(2\sigma) \int_{-\infty}^{\infty} \psi_U\left(\frac{t}{T}\right) dt \\ &\quad \times \sum_{r \le \xi^2,\, r \text{ cubefree}} \sum_{l|r} \mu(l) \left(\frac{r}{l}\right)^{2\sigma} \left(\sum_{r|v} \frac{\lambda_v}{v^{2\sigma}} P\left(\frac{v}{r/l}, 2\sigma\right)\right)^2. \end{aligned}$$

Similarly

$$\begin{aligned} S_2 &= D(2k - 2\sigma) \int_{-\infty}^{\infty} \psi_U\left(\frac{t}{T}\right) \left(\frac{t}{2\pi}\right)^{2k-4\sigma} dt \\ &\quad \times \sum_{r \le \xi^2,\, r \text{ cubefree}} \sum_{l|r} \mu(l) \left(\frac{r}{l}\right)^{2(k-\sigma)} \left(\sum_{r|v} \frac{\lambda_v}{v^k} P\left(\frac{v}{r/l}, 2(k-\sigma)\right)\right)^2. \end{aligned}$$

We distinguish two cases: (a) $r \le \xi$, and (b) $\xi < r \le \xi^2$.

First consider the case (a) $r \le \xi$. We deduce that, since

$$\frac{1}{2\pi i} \int_{(2)} \frac{y^s}{s^2}\, ds = \begin{cases} \log y, & y \ge 1\ , \\ 0, & 0 < y \le 1, \end{cases}$$

we have

$$\begin{aligned} &\sum_v \frac{\lambda_{rv}}{(rv)^{2\sigma}} P(lv, 2\sigma) \\ &\qquad = \sum_v \frac{\mu_f(rv) g_\xi(rv)}{(rv)^{2\sigma}} P(lv, 2\sigma) \end{aligned}$$

$$= \frac{1}{2\pi i}\int\limits_{(2)} \frac{\xi^s(\xi^s-1)}{s^2}\Big(\sum_v \frac{\mu_f(rv)}{(rv)^{2\sigma+s}}P(lv,2\sigma)\Big)\frac{ds}{\log\xi}$$

$$= \sum_v \frac{\mu_f(rv)}{(rv)^{2\sigma}}P(lv,2\sigma)$$

$$+ \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{(\xi/r)^s(\xi^s-1)}{s^2}\Big(\sum_v \frac{\mu_f(rv)}{(rv)^{2\sigma}}\cdot\frac{1}{v^s}P(lv,2\sigma)\Big)\frac{ds}{\log\xi},$$

where Γ denotes the path $\{ix,|x|\ge\delta\}\cup\{\delta e^{i\theta},\pi/2\le\theta\le3\pi/2\}$, and δ is sufficiently small.

We observe that ($p^{e_p(r)}\parallel r$, $p^{e_p(l)}\parallel l$)

$$\sum_v \frac{\mu_f(rv)}{(rv)^{2\sigma}}P(lv,2\sigma)$$

$$= \prod_p\Big(1+\frac{\mu_f(p)}{p^{2\sigma}}P(p,2\sigma)+\frac{\mu_f(p^2)}{p^{4\sigma}}P(p^2,2\sigma)\Big)$$

$$\times\prod_{p|r}\frac{\frac{\mu_f(p^{e_p(r)})}{(p^{e_p(r)})^{2\sigma}}P(p^{e_p(l)},2\sigma)+\frac{\mu_f(p^{e_p(r)+1})}{(p^{e_p(r)+1})^{2\sigma}}P(p^{e_p(l)+1},2\sigma)}{1+\frac{\mu_f(p)}{p^{2\sigma}}P(p,2\sigma)+\frac{\mu_f(p^2)}{p^{4\sigma}}P(p^2,2\sigma)}$$

$$= \frac{1}{D(2\sigma)}u(r,l,2\sigma),$$

say. Similarly

$$\sum_v \frac{\mu_f(rv)}{(rv)^{2\sigma}}\cdot\frac{1}{v^s}P(lv,2\sigma)$$

$$= \prod_p\Big(1+\frac{\mu_f(p)}{p^{2\sigma+s}}P(p,2\sigma)+\frac{\mu_f(p^2)}{p^{4\sigma+2s}}P(p^2,2\sigma)\Big)$$

$$\times\prod_{p|r}\frac{\frac{\mu_f(p^{e_p(r)})}{(p^{e_p(r)})^{2\sigma}}P(p^{e_p(l)},2\sigma)+\frac{\mu_f(p^{e_p(r)+1})}{(p^{e_p(r)+1})^{2\sigma}p^s}P(p^{e_p(l)+1},2\sigma)}{1+\frac{\mu_f(p)}{p^{2\sigma+s}}P(p,2\sigma)+\frac{\mu_f(p^2)}{p^{4\sigma+2s}}P(p^2,2\sigma)}$$

$$= G(s)v(r,l,2\sigma,s),$$

say.

It is easily verified that, for $\Re s>-1/2$,

$$G(s)=\frac{1}{D(2\sigma+s)}\prod_p\Big(1+O\Big(\frac{1}{p^{2(1+\Re s)}}\Big)\Big).$$

We have, by Cauchy's inequality,

$$\Big(\sum_v \frac{\lambda_{rv}}{(rv)^{2\sigma}} P(lv,2\sigma)\Big)^2 \ll \Big|\sum_v \frac{\mu_f(rv)}{(rv)^{2\sigma}} P(lv,2\sigma)\Big|^2 + \int_\Gamma \Big|\Big(\frac{\xi}{r}\Big)^s \frac{\xi^s-1}{s^2}\Big| \times \Big|\sum_v \frac{\mu_f(rv)}{(rv)^{2\sigma} v^s} P(lv,2\sigma)\Big|^2 \frac{|ds|}{\log^2 \xi}.$$

For r cubefree, $r = r_1 r_2^2$, $\mu(r_1 r_2) \neq 0$, we infer that

$$\sum_{l|r} |\mu(l)| \Big(\frac{r}{l}\Big)^{2\sigma} \Big|\sum_v \frac{\mu_f(rv)}{(rv)^{2\sigma}} P(lv,2\sigma)\Big|^2 \ll \prod_{p|r}\Big(1+\frac{1}{p^{3/4}}\Big) \frac{1}{D(2\sigma)^2} \Big(\sum_{t|r_1} \frac{a_t^2}{t^{2\sigma}} \Big(\frac{r_1}{t}\Big)^{-3}\Big) r_2^{2(k-1)-4\sigma},$$

$$\sum_{l|r} |\mu(l)| \Big(\frac{r}{l}\Big)^{2\sigma} \Big|\sum_v \frac{\mu_f(rv)}{(rv)^{2\sigma} v^s} P(lv,2\sigma)\Big|^2 \ll \prod_{p|r}\Big(1+\frac{1}{p^{3/4}}\Big) \frac{1}{|D(2\sigma+s)|^2} \Big(\sum_{t|r_1} \frac{a_t^2}{t^{2\sigma}} \Big(\frac{r_1}{t}\Big)^{-3}\Big) r_2^{2(k-1)-4\sigma}.$$

From the zero-free region result for $D(s)$ (see, for example, [17], Theorem 5.1) and a standard argument (due to Landau, see [21], §3.9 and §3.11), we have

$$D(s) \neq 0, \quad 1/D(s) \ll \log(|y|+3),$$

for $s = x+iy$, $x \geq k - 2\delta/\log(|y|+3)$. Note that

$$\begin{aligned}\sum_{r\leq\xi} \frac{a_r^2}{r^{2\sigma}} \prod_{p|r}\Big(1+\frac{1}{p^{3/4}}\Big) &= \sum_{r\leq\xi} \frac{a_r^2}{r^{2\sigma}} \sum_{u|r} \frac{|\mu(u)|}{u^{3/4}} = \sum_{u\leq\xi} \frac{|\mu(u)|}{u^{3/4+2\sigma}} \sum_{r\leq\xi/u} \frac{a_{ru}^2}{r^{2\sigma}} \\ &\ll \sum_u \frac{1}{u^{3/4+2\sigma}} \Big(\sum_{(r,u)=1} \frac{a_r^2}{r^{2\sigma}}\Big)\Big(\sum_{r|u^\infty} \frac{a_{ru}^2}{r^{2\sigma}}\Big) \\ &\ll D(2\sigma) \sum_u \frac{d^2(u)}{u^{3/4+2\sigma}} \sum_{r|u^\infty} \frac{(ru)^{k-1} d^2(r)}{r^{2\sigma}} \\ &\ll D(2\sigma) \sum_u \frac{d^2(u)}{u^{7/4}} \sum_{r|u^\infty} \frac{d^2(r)}{r} \ll D(2\sigma).\end{aligned}$$

Here we have used Deligne's bound for the Hecke eigenvalues a_{ru}, but it is clear that the weaker and more elementary bound $a_r \ll r^{(k-1)/2+1/4+\varepsilon}$ suffices for the same purpose. Hence

$$\sum_{r_1 r_2^2\le\xi,\,\mu(r_1r_2)\ne0} d(r_2)r_2^{2(k-1)-4\sigma}\sum_{t\mid r_1}\frac{a_t^2}{t^{2\sigma}}\prod_{p\mid t}\left(1+\frac{1}{p^{3/4}}\right)\left(\frac{r_1}{t}\right)^{-3}\prod_{p\mid \frac{r_1}{t}}\left(1+\frac{1}{p^{3/4}}\right)$$
$$\ll\sum_{t\le\xi}\frac{a_t^2}{t^{2\sigma}}\prod_{p\mid t}\left(1+\frac{1}{p^{3/4}}\right)\sum_{r\le\xi/t}\frac{d(r)}{r^3}\sum_{r_2\le\sqrt{\xi}}\frac{d(r_2)}{r_2^2}\ll D(2\sigma).$$

Thus,

$$D(2\sigma)\sum_{r\le\xi}\sum_{l\mid r}|\mu(l)|\left(\frac{r}{l}\right)^{2\sigma}\left|\sum_v\frac{\lambda_{rv}}{(rv)^{2\sigma}}P(lv,2\sigma)\right|^2\ll 1.$$

(Note that $2\sigma-k\gg 1/\log\xi$.)

In case (b), $\xi<r\le\xi^2$, we have

$$\sum_v\frac{\lambda_{rv}}{(rv)^{2\sigma}}P(lv,2\sigma)=\frac{1}{2\pi i}\int_{(2)}\left(\frac{\xi^2}{r}\right)^s\frac{1}{s^2}\left(\sum_v\frac{\mu_f(rv)}{(rv)^{2\sigma}v^s}P(lv,2\sigma)\right)\frac{ds}{\log\xi}.$$

The treatment is the same except that the above integrand has a double pole at $s=0$. Using $\sum_{p\mid r}\log p/p\ll\log\log r$, we can establish that

$$D(2\sigma)\sum_{\xi<r\le\xi^2}\sum_{l\mid r}|\mu(l)|\left(\frac{r}{l}\right)^{2\sigma}\left|\sum_v\frac{\lambda_{rv}}{(rv)^{2\sigma}}P(lv,2\sigma)\right|^2\ll 1.$$

Hence $S_1\ll T$. Similarly, $S_2\ll T$. If we choose $\psi_U(t/T)$ to be the majorant of the characteristic function of $[T,2T]$ (here $U\ll 1$), then we have, with $\xi=T^{1/72}$,

$$\int_T^{2T}|L_f(\sigma+it)|^2|M_{\xi^2}(\sigma+it)|^2\,dt\ll T.$$

In particular, we have

Lemma 3.1. *Let* $\sigma=k/2+A/\log T$. *Then*

$$\int_T^{2T}|L_f(\sigma+it)M_{\xi^2}(\sigma+it)-1|^2\,dt\ll T.$$

We also have

Lemma 3.2.

$$\int_T^{2T}|L_f(k/2+1+it)M_{\xi^2}(k/2+1+it)-1|^2\,dt\ll T^{1-1/72}.$$

Proof. Lemma 3.2 follows immediately from the equality (see [16])

$$\int_0^T\left|\sum_{n=1}^\infty a_n n^{it}\right|^2dt=\sum_{n=1}^\infty|a_n|^2(T+O(n)).$$

From Lemma 3.1, Lemma 3.2 and an easy modification of the classical convexity theorem (see [21], §7.8), we deduce that

THEOREM 3.3. *We have*

$$\int_T^{2T} |L_f(\sigma+it)M_{\xi^2}(\sigma+it)-1|^2\,dt \ll T^{1-\frac{1}{72}(\sigma-k/2)},$$

uniformly for $k/2 + A/\log T \le \sigma \le k/2+1$.

Now we are in a position to prove our main theorem.

THEOREM 3.4. *We have*

$$N_f(\sigma,T) \ll T^{1-\frac{1}{72}(\sigma-k/2)}\log T,$$

uniformly for $k/2 \le \sigma \le (k+1)/2$.

First, we show the following proposition.

PROPOSITION 3.5. *We have*

$$\int_\sigma^{(k+1)/2} N_f(\sigma',T)\,d\sigma' \ll T^{1-\frac{1}{72}(\sigma-k/2)},$$

uniformly for $k/2 \le \sigma \le (k+1)/2$.

Proof. It suffices to prove that

$$\int_\sigma^{(k+1)/2} (N_f(\sigma',2T)-N_f(\sigma',T))\,d\sigma' \ll T^{1-\frac{1}{72}(\sigma-k/2)},$$

for $k/2 + A/\log\xi \le \sigma \le (k+1)/2$.

Let $\Phi(s) = 1-(L_f(s)M_{\xi^2}(s)-1)^2$. The zeros of $L_f(s)$ occur among those of $\Phi(s)$, with at least the same multiplicities. By Littlewood's lemma concerning the number of zeros of an analytic function in a rectangle [22], we have

$$\begin{aligned}\int_\sigma^{(k+1)/2} &(N_f(\sigma',2T)-N_f(\sigma',T))\,d\sigma' \\ &\le \frac{1}{2\pi}\int_T^{2T}\log|\Phi(\sigma+it)|\,dt + \frac{1}{2\pi}\int_\sigma^{\infty}\arg\Phi(\sigma'+2iT)\,d\sigma' \\ &\quad - \frac{1}{2\pi}\int_\sigma^{\infty}\arg\Phi(\sigma'+iT)\,d\sigma'.\end{aligned}$$

In the range $((k+1)/2+4,\infty)$, we see that

$$\arg\Phi(\sigma'+it) = O(2^{-\sigma'}).$$

Hence this part of the integral is $O(1)$. In the range $(k/2,(k+1)/2+4)$, it follows from Jensen's theorem [22] and a standard argument (see [19]) that

$$\arg \Phi(\sigma'+iT)=O(\log T).$$

We deduce that

$$\int_{\sigma}^{\infty} \arg \Phi(\sigma'+iT)\,d\sigma' \ll \log T.$$

Finally, since $\log(1+|x|)\le |x|$,

$$\begin{aligned}\int_{T}^{2T} \log|\Phi(\sigma+it)|\,dt &\le \int_{T}^{2T} |L_f(\sigma+it)M_{\xi^2}(\sigma+it)-1|^2\,dt\\ &= O(T^{1-\frac{1}{72}(\sigma-k/2)}).\end{aligned}$$

This proves the proposition.

Proof of Theorem 3.4. It suffices to assume that $\sigma-k/2\ge 1/\log T$. Thus,

$$\begin{aligned}N_f(\sigma,T) &\le \log T \int_{\sigma-1/\log T}^{\sigma} N_f(\sigma',T)\,d\sigma'\\ &\le \log T \int_{\sigma-1/\log T}^{(k+1)/2} N_f(\sigma',T)\,d\sigma'\\ &\ll T^{1-\frac{1}{72}(\sigma-k/2)}\log T.\end{aligned}$$

Our proof is now complete.

4. Appendix. In this section, we will give a simple proof of Hafner's result which is used in Section 2 without appealing to the spectral theory of the Laplacian acting on $L^2(\mathbf{\Gamma}_0(a,b)\backslash\mathbf{H})$. Our approach is based upon the delta-symbol method introduced by Duke–Friedlander–Iwaniec [5] and does not require a discussion of exceptional eigenvalues for the congruence subgroups. Instead we only need Weil's bound for the Kloosterman sums. Our result is quantitatively a little weaker than Hafner's but is sufficient for our application. Furthermore, our method can as well be applied to the case when a_n is the Fourier coefficient of a Maass form so it appears to be of independent interest.

Let a_n be the nth Fourier coefficient of a (holomorphic) Hecke eigenform of weight k. We consider the sum

$$\sum_{m\mu-n\nu=l,\ x\le n\le 2x} a_n a_m, \qquad \mu,\nu,l>0,\ x\ge 10. \tag{7}$$

Let $g(\xi)$ be a smooth function on $\mathbb{R}^1$ with compact support such that $0 \le g(\xi) \le 1$; $g(\xi) = 1$ if $x \le \xi \le 2x$; $\mathrm{supp}(g(\xi)) \subset [x - x^{1-\theta}, 2x + x^{1-\theta}]$ for some $0 < \theta < 1$; and $g^{(p)}(\xi) \ll_p (x^{1-\theta})^{-p}$, $p \ge 0$. Then we have

$$\tag{8} \sum_{m\mu - n\nu = l,\, x \le n \le 2x} a_n a_m = \sum_{m\mu - n\nu = l} a_n a_m g(n) + O\biggl(\biggl(\frac{x\nu + l}{\mu}\biggr)^{(k-1)/2+\varepsilon} x^{(k-1)/2+\varepsilon} x^{1-\theta}\biggr).$$

Let $h(\xi)$ be another smooth function with compact support such that $0 \le h(\xi) \le 1$; $h(\xi) = 1$, if $\bigl(\frac{3}{4}x\nu + l\bigr)/\mu \le \xi \le \bigl(\frac{9}{4}x\nu + l\bigr)/\mu$; $\mathrm{supp}(h(\xi)) \subset \bigl[\bigl(\frac{1}{2}x\nu + l\bigr)/\mu, \bigl(\frac{5}{2}x\mu + l\bigr)/\nu\bigr]$; and $h^{(p)}(\xi) \ll_p (x\nu/\mu)^{-p}$, $p \ge 0$. Clearly we have

$$\tag{9} \sum_{m\mu - n\nu = l} a_n a_m g(n) = \sum_{m\mu - n\nu = l} a_n a_m g(n) h(m).$$

Next we will recall the delta-symbol method introduced in [5].

Define

$$\tag{10} \delta(n) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n \ne 0. \end{cases}$$

Let $\omega(t)$ be an even function on $\mathbb{R}^1$ with $\omega(0) = 0$ and compactly supported such that $\sum_{k=1}^{\infty} \omega(k) = 1$. Let

$$\delta_k(n) = \omega(k) - \omega\biggl(\frac{n}{k}\biggr).$$

Then clearly $\delta(n) = \sum_{k \mid n} \delta_k(n)$. Thus

$$\delta(n) = \sum_k k^{-1} \sum_{h \bmod k} e\biggl(\frac{hn}{k}\biggr) \delta_k(n).$$

Put

$$\Delta_c(n) = \sum_r r^{-1} \delta_{cr}(n).$$

Writing $r = (h, k)$, $a = h/r$, $c = k/r$, we have

$$\tag{11} \delta(n) = \sum_c c^{-1} \sideset{}{^*}\sum_{a \bmod c} e\biggl(\frac{an}{c}\biggr) \Delta_c(n).$$

We will apply the above identity to integers $|n| < N/2$, say, with $\omega(t)$ supported on $K/2 < |t| < K$, and whose derivatives satisfy $\omega^{(j)}(t) \ll K^{-j-1}$. Now, $\delta_k(n)$ vanishes except for $1 \le k < \max(K, N/K) = K$ by choosing $K = N^{1/2}$. Hence $\Delta_c(n)$ vanishes except for $1 \le c < K$ and $\Delta_c(n) \ll K^{-1} \log K$. Let $\Delta_1 = x\nu/\mu$, $\Delta_2 = x^{1-\theta}$, $\Delta = \min(\Delta_1, \Delta_2)$. We infer that, by (10) and (11),

$$S = \sum_{m\mu - n\nu = l} a_n a_m g(n) h(m)$$
$$= \sum_{m,n} a_n a_m g(n) h(m) \delta(m\mu - n\nu - l) = \sum_c c^{-1} S_c,$$

where

$$S_c = \sideset{}{^*}\sum_{a \bmod c} \sum_{m,n} a_m a_n g(n) h(m) e\left(\frac{a(m\mu - n\nu - l)}{c}\right) \Delta_c(m\mu - n\nu - l) \tag{12}$$
$$= \sideset{}{^*}\sum_{a \bmod c} e\left(\frac{-al}{c}\right) \sum_{m,n} b_m b_n e\left(\frac{a}{c}(m\mu - n\nu)\right) F(m,n),$$

with $b_m = a_m m^{-(k-1)/2}$ and

$$F(m,n) = (mn)^{(k-1)/2} h(m) g(n) \Delta_c(\mu m - \nu n - l).$$

Define $\gamma = \max(\mu, \nu)$, $K^2 = N = 8x\gamma l$. It is easy to see that

$$\frac{d}{dn} \Delta_c(n) \begin{cases} \ll 1/K|n| & \text{if } |n| \gg Kc, \\ = 0 & \text{otherwise.} \end{cases} \tag{13}$$

We have, for $i + j \geq 1$,

$$\frac{\partial^{i+j}}{\partial \xi^i \partial \eta^j} F(\xi, \eta) \tag{14}$$
$$\ll \left(\frac{x\nu + l}{\mu}\right)^{(k-1)/2} x^{(k-1)/2} K^{-1} \left(\frac{\Delta c}{K}\right)^{-i-j+1} \left(\frac{\mu + \nu}{|\mu\xi - \nu\eta - l|} + \frac{1}{\Delta}\right)$$

if $|\mu\xi - \nu\eta - l| \gg Kc$, and without the term $(\mu + \nu)/|\mu\xi - \nu\eta - l|$ if otherwise. We need the following Poisson-type formula [4]:

LEMMA 4.1. *Let F be a smooth and compactly supported function on $\mathbb{R}^+$. For any integers $c \geq 1$ and $(a, c) = 1$ we have*

$$\sum_m b_m e\left(\frac{am}{c}\right) F(m) = \sum_r b_r e\left(\frac{-\overline{a}r}{c}\right) \breve{F}(r),$$

where $a\overline{a} \equiv 1 \pmod{c}$ and $\breve{F}(r)$ is the Hankel-type transform

$$\breve{F}(y) = 2\pi i^k c^{-1} \int_0^\infty F(x) J_{k-1}\left(\frac{4\pi}{c}\sqrt{xy}\right) dx,$$

where $J_\nu(z)$ is the usual Bessel function.

Applying Lemma 4.1 in each variable m, n in (12), we deduce that

$$S_c = \sideset{}{^*}\sum_{a \bmod c} e\left(\frac{al}{c}\right) \sum_{r_1, r_2} b_{r_1} b_{r_2} e\left(\frac{\overline{a\mu}_1}{c_1} r_1 - \frac{\overline{a\nu}_1}{c_2} r_2\right) \breve{F}(r_1, r_2), \tag{15}$$

where
$$\mu_1=\frac{\mu}{(\mu,c)},\quad c_1=\frac{c}{(\mu,c)},\quad \nu_1=\frac{\nu}{(\nu,c)},\quad c_2=\frac{c}{(\nu,c)},$$
and

(16) $$\breve{F}(r_1,r_2)=\frac{4\pi^2}{c_1c_2}\int_0^\infty\int_0^\infty F(x_1,x_2)J_{k-1}\Big(\frac{4\pi}{c_1}\sqrt{x_1r_1}\Big)J_{k-1}\Big(\frac{4\pi}{c_2}\sqrt{x_2r_2}\Big)\,dx_1\,dx_2.$$

By the recurrence formula
$$\frac{d}{dz}(z^\nu J_\nu(z))=z^\nu J_{\nu-1}(z)$$
and the bound $J_\nu(z)\ll(1+z)^{-1/2}$, we obtain, by partial integration twice in each variable in (16) and using (14),
$$\sum_{r_1,r_2}b_{r_1}b_{r_2}|\breve{F}(r_1,r_2)|\ll K(l\gamma x^{\theta+\varepsilon})^{9/4}\Big(\frac{x\nu+l}{\mu}\Big)^{(k-1)/2}x^{(k-1)/2}.$$
The sum over a in (15) is a Kloosterman sum $S(l,*,c)$ to which we apply Weil's bound. Thus, we infer that
$$S_c\ll(l,c)^{1/2}c^{1/2}\tau(c)(xl\gamma)^{1/2}(\gamma lx^{\theta+\varepsilon})^{9/4}\Big(\frac{x\nu+l}{\mu}\Big)^{(k-1)/2}x^{(k-1)/2}.$$
Hence
$$\begin{aligned}S&\ll(xl\gamma)^{3/4}(l\gamma x^{\theta+\varepsilon})^{9/4}\Big(\frac{x\nu+l}{\mu}\Big)^{(k-1)/2}x^{(k-1)/2}\\&\ll\Big(\frac{x\nu+l}{\mu}\Big)^{(k-1)/2}x^{(k-1)/2}x^{3/4+9\theta/4+\varepsilon}\gamma^3l^3.\end{aligned}$$
We conclude that
$$\begin{aligned}\sum_{n\le x}a_na_{(n\nu+l)/\mu}&\ll\Big(\frac{x\nu+l}{\mu}\Big)^{(k-1)/2}x^{(k-1)/2}(l^3x^{3/4+9\theta/4+\varepsilon}\gamma^3+x^{1-\theta+\varepsilon})\\&\ll\Big(\frac{x\nu+l}{\mu}\Big)^{(k-1)/2}x^{(k-1)/2}\gamma^3l^3x^{13/14},\end{aligned}$$
on taking $\theta=1/13$. Finally, since for $\Re s>k$,
$$\begin{aligned}D_{\mu,\nu}(s,l):=\sum_{n=1}^\infty\frac{a_na_{(n\nu+l)/\mu}}{(n\nu+l/2)^s}&=\int_{1/2}^\infty\frac{1}{(x\nu+l/2)^s}\,d\Big(\sum_{n\le x}a_na_{(n\nu+l)/\mu}\Big)\\&=s\nu\int_{1/2}^\infty\frac{\sum_{n\le x}a_na_{(n\nu+l)/\mu}}{(x\nu+l/2)^{s+1}}\,dx,\end{aligned}$$
we obtain

THEOREM 4.2. *$D_{\mu,\nu}(s,l)$ can be analytically continued to $\Re s > k-1/14$, and for $\Re s > k-1/14$, $s=\sigma+it$, we have*

$$D_{\mu,\nu}(s,l) \ll \frac{l^3|s|}{(\mu\nu)^{(k-1)/2}}(\max(\mu,\nu))^3\frac{1}{\sigma-(k-1/14)}.$$

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