A generalization of the Goldbach–Vinogradov theorem

by

T. Zhan (Jinan)

1. Introduction. The Goldbach–Vinogradov theorem states that every large odd integer is the sum of three primes. After Vinogradov proved this result in 1937, several generalizations of it were studied by a number of authors. One question is how small we may take $U = U(N)$ such that every large odd integer N can be written in the form

(1.1) $N = p_1 + p_2 + p_3, \quad p_1 \leq U = U(N) \quad (2 \nmid N, p_1, p_2, p_3 \text{ are primes}).$

The first non-trivial result concerning this problem was given by Pan [7], who proved in 1959 that U may be taken as small as $N^{2c/(2c+1)+\varepsilon}$ if $\zeta(\frac{1}{2} + it) \ll (|t|+1)^{c+\varepsilon}$ for any $\varepsilon > 0$. The classical result $c = 1/6$ then gives $U = N^{1/4+\varepsilon}$. In the present paper some improvements on Pan's result will be given. Let

$$R(N) = \sum_{\substack{p_1+p_2+p_3=N \\ p_1 \leq U, \, p_2 \leq y \\ N-y < p_3 \leq N}} 1.$$

In Section 2 we prove

THEOREM 1. *Suppose that $0 < \varepsilon < 1/2$, $U = y^{1/6+\varepsilon}$ and $y = N^{7/12+\varepsilon}$. Then for $2 \nmid N$,*

(1.2) $R(N) = \sigma(N)UyL^{-1}(\log U)^{-1}(\log y)^{-1}(1 + O(L^{-1})),$

where

$$\sigma(N) = \prod_{p|N}\left(1 - \frac{1}{(p-1)^2}\right)\prod_{p \nmid N}\left(1 + \frac{1}{(p-1)^3}\right) > \frac{1}{2} \quad \text{for } 2 \nmid N,$$

$L = \log N$ *and the O-constant depends on ε only.*

From Theorem 1 it follows that $U(N) = N^{7/72+\varepsilon}$ is permissible in (1.1).

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[95]

The proof of Theorem 1 depends on the Hardy–Littlewood circle method and Selberg's inequality

$$(1.3) \qquad \int_x^{2x} \Big| \sum_{t < n \le t+\Delta} \Lambda(n) - \Delta \Big|^2 \, dt \ll_{\varepsilon, A} \Delta^2 x (\log x)^{-A},$$

where $0 < \varepsilon < 5/6$, $A > 0$ are any constants and $x^{1/6+\varepsilon} \le \Delta \le x$ (see [9], for example). In fact, what we actually need in the proof is a generalization of (1.3) stated in Lemma 1 in Section 2. The main idea of the proof is to make use of Gallagher's lemma [1] and (1.3) in estimating integrals on major arcs.

As a consequence of (1.3) we deduce that if $g(n) = n^{1/6+\varepsilon}$ then for all integers $n \le x$ with at most $O(x(\log x)^{-A})$ exceptions the interval $(n, n + g(n))$ contains $\gg g(n)(\log n)^{-1}$ primes. A better result due to Harman [3] is known, namely, for $g(n) = n^{1/10+\varepsilon}$ the above statement is still true. Although it seems unlikely that one may get an improvement on Theorem 1 directly from Harman's result, we can prove the following Theorem 2 by combining the method of proof of Theorem 1 with Harman's sieve estimates used in proving his result. The idea of combining the circle method with the sieve method may be found in [5] or [6].

THEOREM 2. *Suppose that $0 < \varepsilon < 1/2$, $U = y^{1/10+\varepsilon}$ and $y = N^{7/12+\varepsilon}$. Then there exists an absolute constant $C_1 > 0$ such that*

$$(1.4) \qquad R(N) \ge C_1 U y L^{-3}.$$

Theorem 2 implies that one may take $U(N) = N^{7/120+\varepsilon}$ in (1.1). The proof of Theorem 2 is given in Sections 3 and 4.

In what follows we shall use the notations introduced above. $c > 0$ will denote a positive constant that may be different at each occurrence.

2. Proof of Theorem 1

LEMMA 1. *Suppose that $A > 0$, $B > 0$ and $0 < \varepsilon < 5/6$. Then for $(a, q) = 1$, $1 \le q \le (\log x)^B$ and $x^{1/6+\varepsilon} \le \Delta \le x$ we have*

$$(2.1) \qquad \int_x^{2x} \Big| \sum_{\substack{t < n \le t+\Delta \\ n \equiv a \,(q)}} \Lambda(n) - \frac{\Delta}{\phi(q)} \Big|^2 \, dt \ll_{\varepsilon, c, A} \Delta^2 x (\log x)^{-A}.$$

This is a generalization of (1.3) and can be shown in essentially the same way as in [9].

For the sake of simplicity instead of $R(N)$ we consider $\widetilde{R}(N)$ defined by

$$\widetilde{R}(N) = \sum_{\substack{N=n_1+n_2+n_3 \\ n_1 \leq U,\, n_2 \leq y \\ N-y < n_3 \leq N}} \Lambda(n_1)\Lambda(n_2)\Lambda(n_3)$$

and prove

$$(2.2) \qquad \widetilde{R}(N) = \sigma(N)Uy(1 + O(L^{-1})).$$

It is an easy matter to derive Theorem 1 from (2.2).

Applying the circle method, we obtain

$$\widetilde{R}(N) = \int\limits_0^1 \sum_{n_1 \leq U} \Lambda(n_1)e(n_1\alpha) \sum_{n_2 \leq y} \Lambda(n_2)e(n_2\alpha)$$

$$\times \sum_{N-y < n_3 \leq N} \Lambda(n_3)e(n_3\alpha)e(-N\alpha)\,d\alpha$$

$$= \left(\int\limits_m + \int\limits_E \right) \sum_{n_1 \leq U} \Lambda(n_1)e(n_1\alpha) \sum_{n_2 \leq y} \Lambda(n_2)e(n_2\alpha)$$

$$\times \sum_{N-y < n_3 \leq N} \Lambda(n_3)e(n_3\alpha)e(-N\alpha)\,d\alpha$$

$$= \int\limits_m + \int\limits_E, \quad \text{say,}$$

where m and E are defined by

$$m = \bigcup_{q \leq Q} \bigcup_{\substack{1 \leq a \leq q \\ (a,q)=1}} \left[\frac{a}{q} - \frac{1}{q\tau}, \frac{a}{q} + \frac{1}{q\tau} \right], \qquad Q = L^{c_1}, \ \tau = UL^{-c_2},$$

$$E = [-1/\tau, 1 - 1/\tau] \setminus m.$$

Denote by $I_{q,a}$ the interval $[a/q - 1/(q\tau), a/q + 1/(q\tau)]$. Taking $c_1 = 10$ and $c_2 = 12$, from the result of Vinogradov and the Siegel–Walfisz theorem (see [8], for example) we get

$$(2.3) \qquad \sum_{n \leq U} \Lambda(n)e(n\alpha) \ll UL^{-2} \quad \text{if } \alpha \in E,$$

$$(2.4) \quad \sum_{n \leq U} \Lambda(n)e(n\alpha) = \frac{\mu(q)}{\phi(q)} \sum_{n \leq U} e(n\lambda) + O(U\exp(-cL^{1/2}))$$

$$\text{if } \alpha = a/q + \lambda \in m.$$

Then it follows from (2.3) and the Cauchy–Schwarz inequality that

$$(2.5) \quad \int\limits_{E} \ll UL^{-2} \int\limits_{0}^{1} \Big| \sum_{n_2 \leq y} \varLambda(n_2) e(n_2 \alpha) \sum_{N-y < n_3 \leq N} \varLambda(n_3) e(n_3 \alpha) \Big| \, d\alpha$$

$$\ll yUL^{-1}.$$

By (2.4) we obtain

$$(2.6) \int\limits_{I_{q,a}} = \frac{\mu(q)}{\phi(q)} \int\limits_{-1/(q\tau)}^{1/(q\tau)} \sum_{n_1 \leq U} e(n_1 \lambda) \sum_{n_2 \leq y} \varLambda(n_2) e(n_2(a/q + \lambda))$$

$$\times \sum_{N-y < n_3 \leq N} \varLambda(n_3) e(n_3(a/q + \lambda)) e(-N(a/q + \lambda)) \, d\lambda$$

$$= \frac{\mu^2(q)}{\phi^2(q)} \int\limits_{-1/(q\tau)}^{1/(q\tau)} \sum_{n_1 \leq U} e(n_1 \lambda) \sum_{n_2 \leq y} e(n_2 \lambda)$$

$$\times \sum_{N-y < n_3 \leq N} \varLambda(n_3) e(n_3(a/q + \lambda)) e(-N(a/q + \lambda)) \, d\lambda$$

$$+ O\bigg(\frac{1}{\phi(q)} \int\limits_{-1/\tau}^{1/\tau} \Big| \sum_{n_1 \leq U} e(n_1 \lambda) \sum_{N-y < n_3 \leq N} \varLambda(n_3) e(n_3(a/q + \lambda)) \Big|$$

$$\times \Big| \sum_{n_2 \leq y} \varLambda(n_2) e(n_2(a/q + \lambda)) - \frac{\mu(q)}{\phi(q)} \sum_{m \leq y} e(m\lambda) \Big| \, d\lambda \bigg)$$

$$= I_1 + O(I_2), \quad \text{say.}$$

By the Cauchy–Schwarz inequality and Gallagher's lemma [1] we have

$$|I_2|^2 \ll \phi^{-2}(q) U^2 \int\limits_{0}^{1} \Big| \sum_{N-y < n \leq N} \varLambda(n) e(n(a/q + \lambda)) \Big|^2 \, d\lambda$$

$$\times \int\limits_{-1/\tau}^{1/\tau} \Big| \sum_{n \leq y} \varLambda(n) e(n(a/q + \lambda)) - \frac{\mu(q)}{\phi(q)} \sum_{m \leq y} e(m\lambda) \Big|^2 \, d\lambda$$

$$\ll \phi^{-2}(q) U^2 y \tau^{-2} L \int\limits_{1}^{y} \Big| \sum_{t < n \leq t+\tau} \varLambda(n) e\Big(n\frac{a}{q} \Big) - \frac{\mu(q)}{\phi(q)} \tau \Big|^2 \, dt$$

$$+ \phi^{-2}(q) U^2 y \tau L$$

$$\ll U^2 y \tau^{-2} L \max \int\limits_{1}^{y} \Big| \sum_{\substack{t < n \leq t+\tau \\ n \equiv l \, (q)}} \varLambda(n) - \frac{\tau}{\phi(q)} \Big|^2 \, dt + U^2 y^{2-\varepsilon}$$

where the "max" is taken over all $(l, q) = 1$ and $q \leq Q$. Since $Q = L^{c_1}$,

taking $A = 4c_1 + 2$ in Lemma 1, we obtain

$$(2.7) \qquad\qquad I_2 \ll Q^{-2}yUL^{-1}.$$

We now turn to I_1. It is easy to check that

$$(2.8) \quad I_1 = \frac{\mu^2(q)}{\phi^2(q)} \int_0^1 \sum_{n_1 \le U} e(n_1\lambda) \sum_{n_2 \le y} e(n_2\lambda)$$

$$\times \sum_{N-y<n_3\le N} \Lambda(n_3)e(n_3(a/q+\lambda))e(-N(a/q+\lambda))\, d\lambda$$

$$+ O\Big(\phi^{-2}(q) \int_{1/(q\tau)}^{1/2} \lambda^{-1} \Big| \sum_{n_2\le y} e(n_2\lambda) \sum_{N-y<n_3\le N} \Lambda(n_3)e(n_3(a/q+\lambda)) \Big|\, d\lambda \Big)$$

$$= \frac{\mu^2(q)}{\phi^2(q)} \sum_{\substack{n_1+n_2+n_3=N \\ n_1\le U,\, n_2\le y \\ N-y<n_3\le N}} \Lambda(n_3)e\Big(\frac{a}{q}(n_3-N)\Big) + O(\phi^{-1}(q)\tau yL)$$

$$= \frac{\mu^2(q)}{\phi^2(q)} \sum_{N-y<n_3\le N} \Lambda(n)e\Big(\frac{a}{q}(n_3-N)\Big) \sum_{\substack{n_1+n_2=N-n_3 \\ n_1\le U,\, n_2\le y}} 1 + O(\phi^{-1}(q)\tau yL)$$

$$= \frac{\mu^2(q)}{\phi^2(q)}U \sum_{N-y<n\le N} \Lambda(n)e\Big(\frac{a}{q}(n-N)\Big) + O(\phi^{-1}(q)\tau yL) + O(\phi^{-2}(q)U^2)$$

$$= \frac{\mu^2(q)}{\phi^2(q)}U \sum_{\substack{l=1 \\ (l,q)=1}}^q e\Big(\frac{al}{q}\Big)e\Big(-\frac{a}{q}N\Big) \sum_{\substack{N-y<n\le N \\ n\equiv l\,(q)}} \Lambda(n) + O(\phi^{-1}(q)\tau yL)$$

$$= \frac{\mu(q)}{\phi^3(q)}Uye\Big(-\frac{a}{q}N\Big)(1+O(L^{-1})) + O(\phi^{-1}(q)\tau yL).$$

In the last step we used the Siegel–Walfisz theorem in short intervals, namely,

$$(2.9) \qquad \sum_{\substack{N-y<n\le N \\ n\equiv l\,(q)}} \Lambda(n) = \frac{1}{\phi(q)}y + O(y\exp(-cL^{-1/2}))$$

holds for $(l,q) = 1$ and $q \le L^B$ ($B > 0$ is any constant).

From (2.6)–(2.8) it follows in a standard way that (see [8])

$$\int_m = Uy \sum_{q\le Q} \frac{\mu(q)}{\phi^3(q)}e\Big(-\frac{a}{q}N\Big) + O(UyL^{-1}) = \sigma(N)Uy(1+O(L^{-1})).$$

(2.2) is thus proved.

3. Preparation for the proof of Theorem 2. In this section we give an outline for the proof of Theorem 2 and some fundamental lemmas. Suppose that $\varepsilon > 0$ is a constant sufficiently small, N is an odd integer sufficiently large, $N^{7/12+\varepsilon} \leq y \leq N$, $Y \leq y \leq 2Y$ and $y^{\varepsilon} \leq U \leq y^{1-\varepsilon}$. Consider the set

$$\mathcal{A} = \mathcal{A}(N; y, U) = \{f(p_1, p_3) = N - p_1 - p_3 : p_1 \leq U, \ N - y < p_3 \leq N\}.$$

Since $-U \leq N - p_1 - p_3 \leq y$ for $N - p_1 - p_3 \in \mathcal{A}$, then

$$(3.1) \qquad\qquad R(N) = S(\mathcal{A}, y^{1/2}) + O(U^2).$$

From Buchstab's identity it follows that

$$S(\mathcal{A}, y^{1/2}) = S(\mathcal{A}, z_1) - \sum_{z_2 \leq p < y^{1/2}} S(\mathcal{A}_p, z_3(p))$$

$$- \sum_{z_1 \leq p < z_2} S(\mathcal{A}_p, z_4(p)) + \sum_{\substack{z_3(p) \leq q < p < y^{1/2} \\ p \geq z_2}} S(\mathcal{A}_{pq}, q)$$

$$+ \sum_{\substack{z_4(p) \leq q < p < z_2 \\ p \geq z_1}} S(\mathcal{A}_{pq}, q)$$

$$= \Sigma_1 - \Sigma_2 - \Sigma_3 + \Sigma_4 + \Sigma_5, \quad \text{say,}$$

where the parameters z_i $(1 \leq i \leq 4)$ are as in Harman [3], namely,

$$z_1 = Y^{26(1-3\varepsilon)/105}, \qquad z_2 = Y^{9/35},$$

$$z_3(p) = (Y^{1-3\varepsilon}p^{-1})^{1/3}, \qquad z_4(p) = (Y^{26/35-2\varepsilon}p^{-1})^{1/2}.$$

Obviously, $\Sigma_5 \geq 0$. If we replace $y^{1/2}$ by $(2Y)^{1/2}$ in Σ_2 and $y^{1/2}$ by $Y^{1/2}$ in Σ_4, it follows that

$$(3.2) \qquad\qquad S(\mathcal{A}, y^{1/2}) \geq \Sigma_1 - \Sigma_2 - \Sigma_3 + \Sigma_4.$$

We shall give a lower bound for Σ_1 and an upper bound for Σ_2 and Σ_3 by Iwaniec's linear sieve (Lemma 2). To estimate the error terms arising in Lemma 1 we shall use Lemma 3 below which can be shown by the Hardy–Littlewood circle method as in the proof of Theorem 1. However, instead of Selberg's inequality here we have to use some sieve estimates already proved in Harman [3]. (We actually need some generalized forms of Harman's results.) Similar to estimating these error terms, an asymptotic formula for a subsum of Σ_4 can be given. In this case Lemma 4 will be used instead of Lemma 3. Then the theorem will follow if the right side of (3.2) is $\gg UyL^{-3}$.

LEMMA 2 (Iwaniec [4]). *Let* $z \geq 2$, $D \geq z^2$ *and* $r(\mathcal{A}, d) = |\mathcal{A}_d| - W/d$. *Then*

$$S(\mathcal{A}, z) \leq WV(z)\{F(s) + E\} + R^+,$$

$$S(\mathcal{A}, z) \geq WV(z)\{f(s) - E\} - R^-$$

where $s = (\log D)/(\log z)$ and $E = c\varepsilon + O((\log D)^{-1/3})$. The remainder terms $R^{\pm}$ are of the form

$$R^{\pm} = \sum_{(D)} R^{\pm}_{(D)} = \sum_{(D)} \sum_{v \leq D^{\varepsilon}} C^{\pm}_{(D)}(v, \varepsilon) \sideset{}{'}\sum_{D_i \leq p_i \leq D_i^{1+\varepsilon^7}} r(\mathcal{A}, vp_1 \dots p_r),$$

where (D) runs over all subsequences $D_1 \geq \dots \geq D_r$, including the empty subsequence, of the sequence $D^{\varepsilon^2(1+\varepsilon^7)^n}$, $n \geq 0$, for which

$$D_1 D_2 \dots D_{2k} D_{2k+1}^3 \leq D \quad (0 \leq k \leq (r-1)/2)$$

in the case of R^{+}, and

$$D_1 D_2 \dots D_{2k-1} D_{2k}^3 \leq D \quad (0 \leq k \leq r/2)$$

in the case of R^{-}. Moreover, $\sum'$ indicates that v and p_i satisfy

$$v \,|\, P(D^{\varepsilon^2}), \quad p_i \,|\, P(z) \quad (1 \leq i \leq r).$$

Finally, we also have $|C^{\pm}_{(D)}(v, \varepsilon)| \leq 1$.

$F(s)$, $f(s)$, $P(z)$ and $V(z)$ are standard functions in sieve theory. For their properties see [2].

In our case we take $W = \widetilde{U}\widetilde{y}$,

$$\text{(3.3)} \qquad \begin{aligned} \widetilde{U} &= \sum_{p \leq U} 1 = U(\log U)^{-1}(1 + O(L^{-1})), \\ \widetilde{y} &= \sum_{N-y < p \leq N} 1 = yL^{-1}(1 + O(L^{-1})) \end{aligned}$$

and have

$$\text{(3.4)} \qquad V(z) = \prod_{p < z}(1 - p^{-1}) = e^{-\gamma}(\log z)^{-1}(1 + O((\log z)^{-1}))$$

where γ is the Euler constant.

In Lemmas 3 and 4 we suppose that $c(n) \ll \tau^c(n)$ for $n \leq D$, $c(n) = 0$ otherwise, and $T_0 = \sum_{n \leq D} n^{-1} c(n) \gg 1$. Obviously, $T_0 \ll L^c$ for $D \ll N$.

LEMMA 3. *Suppose that for any* $A > 0$, $B > 0$, $0 < \varepsilon \leq 9/10$ *and* $x^{1/10+\varepsilon} \leq \Delta \leq x$,

$$\text{(3.5)} \qquad \int_x^{2x} \left| \sum_{\substack{t < nk \leq t+\Delta \\ nk \equiv a \,(q)}} c(n) - \frac{\Delta}{\phi(q)} T_0 \right|^2 dt \ll_{\varepsilon,A,B} \Delta^2 x (\log x)^{-A}$$

holds for $(a, q) = 1$ *and* $q \leq (\log x)^B$. *Then*

$$R_1(N) = \sum_{\substack{nk+p_1+p_3=N \\ p_1 \leq U, \, nk \leq y \\ N-y < p_3 \leq N}} c(n) = \sigma(N) U y T_0 (\log U)^{-1} L^{-1}(1 + O(L^{-1})).$$

Proof. Similarly to the proof of Theorem 1 it suffices to show that

$$(3.6) \quad \widetilde{R}_1(N) = \sum_{\substack{nk+n_1+n_3=N \\ n_1 \leq U,\, nk \leq y \\ N-y < n_3 \leq N}} \Lambda(n_1)\Lambda(n_3)c(n) = \sigma(N)UyT_0(1 + O(L^{-1})).$$

Suppose that Q, τ, the major arcs m and the minor arcs E are defined as in Section 2. Take c_1 and c_2 sufficiently large. In the same way as in Section 2 we get from Vinogradov's result

$$\widetilde{R}_1(N) = \int_0^1 \sum_{n_1 \leq U} \Lambda(n_1)e(n_1\alpha) \sum_{nk \leq y} \Lambda(nk)e(nk\alpha)$$

$$\times \sum_{N-y < n_3 \leq N} \Lambda(n_3)e(n_3\alpha)e(-N\alpha)\, d\alpha$$

$$= \int_m + \int_E = \int_m + O(yUT_0L^{-A}),$$

and from (2.4) and (3.4),

$$\int_{I_{q,a}} = \int_{-1/(q\tau)}^{1/(q\tau)} \sum_{n_1 \leq U} \Lambda(n_1)e\left(n_1\left(\frac{a}{q}+\lambda\right)\right) \sum_{nk \leq y} c(n)e\left(nk\left(\frac{a}{q}+\lambda\right)\right)$$

$$\times \sum_{N-y < n_3 \leq N} \Lambda(n_3)e\left(n_3\left(\frac{a}{q}+\lambda\right)\right)e\left(-N\left(\frac{a}{q}+\lambda\right)\right) d\lambda$$

$$= \frac{\mu^2(q)}{\phi^2(q)} \sum_{\substack{n_1+nk+n_3=N \\ n_1 \leq U,\, nk \leq y \\ N-y < n_3 \leq N}} \Lambda(n_3)c(n)e\left(\frac{a}{q}(n_3-N)\right) + O(\phi^{-1}(q)\tau yL).$$

To evaluate the integral $\int_{I_{q,a}}$ we first rewrite it as

$$(3.7) \quad \int_{I_{q,a}} = \frac{\mu^2(q)}{\phi^2(q)} \sum_{\substack{l=1 \\ (l,q)=1}}^{q} e\left(\frac{a}{q}(l-N)\right) \sum_{\substack{N-y < n_3 \leq N \\ n_3 \equiv l\,(q)}} \Lambda(n_3)$$

$$\times \sum_{\substack{n_1+nk=N-n_3 \\ n_1 \leq U,\, nk \leq y}} c(n) + O(\phi^{-1}(q)\tau yL)$$

$$= \frac{\mu^2(q)}{\phi^2(q)} \sum_{\substack{l=1 \\ (l,q)=1}}^{q} e\left(\frac{a}{q}(l-N)\right) \sum_{\substack{N-y < n_3 \leq N-U \\ n_3 \equiv l\,(q)}} \Lambda(n_3)$$

$$\times \sum_{N-n_3-U \leq nk < N-n_3} c(n) + O(\phi^{-1}(q)\tau yL).$$

Making substitution $u = t + \Delta$ in (3.5) ($q = 1$), we obtain

$$(3.8) \quad \int_x^{2x} \Big| \sum_{u-\Delta < nk \leq u} c(n) - \Delta T_0 \Big|^2 du \ll_{\varepsilon,A,B} \Delta^2 x (\log x)^{-A} + \Delta^3 (\log x)^c.$$

Take $x = y$ and $\Delta = U$ in (3.7). It follows that

$$\sum_{U < m \leq y} \Big| \sum_{m-U < nk \leq m} c(n) - U T_0 \Big|^2 \ll U^2 y L^{-A}.$$

Hence by the Cauchy–Schwarz inequality we have

$$(3.9) \quad \sum_{U < m \leq y} \Big| \sum_{m-U < nk \leq m} c(n) - U T_0 \Big| \ll U y L^{-A/2}.$$

(3.7), (3.9) and the Siegel–Walfisz theorem in short intervals (2.9) then yield

$$\int_{I_{q,a}} = \frac{\mu^2(q)}{\phi^2(q)} U T_0 \sum_{\substack{l=1 \\ (l,q)=1}}^{q} e\Big(\frac{a}{q}(l-N)\Big) \sum_{\substack{N-y < n_3 \leq N-U \\ n_3 \equiv l \,(q)}} \Lambda(n_3)$$

$$+ O(\phi^{-1}(q)\tau y L) + O(Q^{-2}L^{-1}yU)$$

$$= \frac{\mu(q)}{\phi^3(q)} U y T_0 e\Big(-N\frac{a}{q}\Big) + O(Q^{-2}L^{-1}yU)$$

if c_1, c_2 are taken sufficiently large and $c_2 \geq c_1 + 2$. Hence

$$\int_m = U y T_0 \sum_{q \leq Q} \frac{\mu(q)}{\phi^3(q)} e\Big(-\frac{a}{q}N\Big) + O(U y L^{-1}) = \sigma(N) U y T_0 (1 + O(L^{-1})),$$

and (3.6) follows.

LEMMA 4. *Suppose that for any* $A > 0$, $B > 0$, $0 < \varepsilon \leq 9/10$ *and* $x^{1/10+\varepsilon} \leq \Delta \leq x$,

$$\int_x^{2x} \Big| \sum_{\substack{t < nk \leq t+\Delta \\ nk \equiv a \,(q)}} c(n)\Lambda(k) - \frac{\Delta}{\phi(q)} T_0 \Big|^2 dt \ll_{\varepsilon,A,B} \Delta^2 x (\log x)^{-A}$$

holds for $(a,q) = 1$ *and* $q \leq (\log x)^B$. *Then*

$$R_1(N) = \sum_{\substack{np+p_1+p_3=N \\ p_1 \leq U, nk \leq y \\ N-y < p_3 \leq N}} c(n) = \sigma(N) U y T_0 (\log U)^{-1} L^{-1} (1 + O(L^{-1})).$$

This lemma can be shown in the same way as Lemma 3.

4. Proof of Theorem 2. In this section we assume all the conditions of Theorem 2. For the sake of simplicity we only prove the theorem for $\varepsilon > 0$ sufficiently small. In fact, this is the most important case, the other case

can be treated in the same way. Furthermore, the constants c_i $(i = 1, 2, \ldots)$ in this section are independent of the c_i's in Section 3.

Estimation of Σ_1. Applying Lemma 2 to Σ_1 we obtain

$$\Sigma_1 \geq WV(z_1)(f(\log D/\log z_1) - c\varepsilon + O(L^{-1/3})) - R^-.$$

Take $D = y^{1-3\varepsilon}$ as in [3]. From (3.3), (3.4) and the result in [3, §5] it follows that

$$\Sigma_1 \geq c_1 \widetilde{U} \widetilde{y} (\log y)^{-1}(1 - c\varepsilon + O(L^{-1/3})) - R^-,$$

where $c_1 = 2\log(79/26)$. Writing R^- in the form

$$R^- = \sum_{(D)} R^-_{(D)} = \sum_{(D)} \sum_{n \leq D} c(n)r(\mathcal{A}, n)$$

it follows easily that $c(n) \ll \tau^c(n)$ and

$$R^-_{(D)} = \sum_{\substack{p_1 + nk + p_3 = N \\ N - y < p_3 \leq N \\ p_1 \leq U, \, nk \leq y, \, n \leq D}} c(n) - T_0 W.$$

Now,

$$\sum_{\substack{t < nl \leq t+k \\ n \leq D}} c(n) - T_0 U$$

is actually the error term arising from the sieve estimate in [3] and it was already proved there that

$$\int_{y/2}^{y} \left| \sum_{\substack{t < nk \leq t+U \\ n \leq D}} c(n) - T_0 U \right|^2 dt \ll U^2 y^{1-\delta}.$$

By essentially the same method we can show that

$$(4.1) \qquad \int_{y/2}^{y} \left| \sum_{\substack{t < nk \leq t+U \\ n \leq D, \, nk \equiv a \, (q)}} c(n) - \frac{U}{\phi(q)} T_0 \right|^2 dt \ll_{\varepsilon, A, B} U^2 y(\log x)^{-A}.$$

Hence from Lemma 3 and (4.1) we get $R^-_{(D)} \ll U y^{1-\delta}$, and so

$$\Sigma_1 \geq c_1 \widetilde{U} \widetilde{y} (\log y)^{-1}(1 - c\varepsilon + O(L^{-1/3})).$$

Estimation of Σ_2. Start from

$$\Sigma_2 = \sum_{z_2 \leq p < (2Y)^{1/2}} S(\mathcal{A}_p, z_3(p))$$

$$\leq \sum_{(P)} (\log P)^{-1} \sum_{P \leq n \leq 2P} \Lambda(n) S(\mathcal{A}_n, z_3(p)).$$

Applying Lemma 2 to $S(\mathcal{A}_n, z_3(p))$ with $D = Y^{1-3\varepsilon}P^{-1}$ we obtain

$$\Sigma_2 \leq \sum_{z_2 \leq p < (2Y)^{1/2}} S(\mathcal{A}_p, z_3(p)) \sum_{P \leq n \leq 2P} \Lambda(n)n^{-1}WV(z_3(P))$$

$$\times S(\mathcal{A}_n, z_3(P))(F(\log D / \log z_3(P)) + c\varepsilon + O(L^{-1/3}))$$

$$+ \sum_{(P)}(\log P)^{-1} \sum_{P \leq n \leq 2P} \Lambda(n)R_n^+$$

$$= \Sigma_2^{(1)} + \Sigma_2^{(2)}, \quad \text{say.}$$

From the discussion in [3, §5] we know that the main term

(4.2) $$\Sigma_2^{(1)} \leq c_2 \widetilde{U}\widetilde{y}(\log y)^{-1}(1 + c\varepsilon + O(L^{-1/3})),$$

where $c_2 = 2\log(26/9)$. The error term may be written as

$$\Sigma_2^{(2)} = \sum_{(P)}(\log P)^{-1}\sum_{(D)}R_{(D)}^+(P)$$

with

$$R_{(D)}^+(P) = \sum_{n \leq D} c(n)r(\mathcal{A}, n) = \sum_{\substack{p_1+nk+p_3=N \\ N-y<p_3\leq N \\ p_1 \leq U,\, nk \leq y,\, n \leq Y^{1-3\varepsilon}}} c(n) - T_0 W.$$

It is easy to see that

$$\sum_{\substack{t<nk\leq t+U \\ n \leq Y^{1-3\varepsilon}}} c(n) - T_0 U$$

is actually the error term of

$$\sum_{P \leq n \leq 2P} \Lambda(n)S(\mathcal{A}_n, z_3(p))$$

in [3]. Hence it follows in the same way as in the estimation of Σ_1 that

(4.3) $$\Sigma_2^{(2)} \ll Uy^{1-\delta}.$$

(4.2) and (4.3) yield

$$\Sigma_2 \leq c_2 \widetilde{U}\widetilde{y}(\log y)^{-1}(1 + c\varepsilon + O(L^{-1/3})).$$

Estimation of Σ_3. Similarly to the treatment of Σ_2, it follows from Lemma 1, Lemma 3 and the corresponding estimates in [3, §5] that

$$\Sigma_3 \leq c_3 \widetilde{U}\widetilde{y}(\log y)^{-1}(1 + c\varepsilon + O(L^{-1/3})),$$

where

$$c_3 = 2\log(2133/2028)(1 + 1/17 - 2\log(35/34)).$$

Estimation of Σ_4. Again, similarly to the estimation of Σ_4 in [3] we can give an asymptotic formula for a subsum Σ'_4 of Σ_4 (in the subsum p and q are taken over the same range as in [3]). We can apply Lemma 4 directly to Σ'_4 since from [3, §5] we know that the condition of Lemma 4 is fulfilled. Hence

$$\Sigma_4 \geq c_4 \widetilde{U} \widetilde{y}(\log y)^{-1}(1 - c\varepsilon + O(L^{-1})),$$

where $c_4 = 0.14$.

Since $c_1 - c_2 - c_3 + c_4 > 0$, Theorem 2 is proved.

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MATHEMATICAL INSTITUTE
SHANDONG UNIVERSITY
JINAN, SHANDONG 250100
P.R. CHINA