ACTA ARITHMETICA
LXXI.1 (1995)

Large deviations of Montgomery type and its application to the theory of zeta-functions

by

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1. Introduction. Let $s = \sigma + it$ be a complex variable, and $\zeta(s)$ the Riemann zeta-function. For any $\sigma > 1/2$ and any $T > 0$, put

$$V(T, R, \sigma; \zeta) = \mu_1(\{t \in [-T, T] \mid \log \zeta(\sigma + it) \in R\}),$$

where μ_1 is the one-dimensional Lebesgue measure, and R is a closed rectangle in the complex plane $\mathbb{C}$ with the edges parallel to the axes. Bohr–Jessen [1] proved the existence of the limit

$$W(R, \sigma; \zeta) = \lim_{T\to\infty} (2T)^{-1} V(T, R, \sigma; \zeta).$$

Consider a special case

$$R = R(\ell) = \{z \in \mathbb{C} \mid -\ell \le \operatorname{Re} z \le \ell,\ -\ell \le \operatorname{Im} z \le \ell\},$$

where $\ell > 0$, and put

$$\widetilde{W}(\ell, \sigma; \zeta) = 1 - W(R(\ell), \sigma; \zeta).$$

If $\sigma > 1$, $\widetilde{W}(\ell, \sigma; \zeta) = 0$ for sufficiently large ℓ, because the Euler product expansion of $\zeta(s)$ is absolutely convergent. If $1/2 < \sigma < 1$, it is known that for sufficiently large ℓ,

$$\begin{aligned}(1.1)\quad & 2C_2^* \exp\{-C_3^*(1-\sigma)^{-1}(2\sigma - 1)(C_1^*)^{-1/(1-\sigma)}\ell^{1/(1-\sigma)} \\ & \qquad\qquad \times (\log \ell)^{\sigma/(1-\sigma)}(1 + o(1))\} \\ & \le \widetilde{W}(\ell, \sigma; \zeta) \\ & \le 4 \exp\big\{-\tfrac{3}{4}(1-\sigma)^{-1}(2\sigma-1)2^{-1/(1-\sigma)} \\ & \qquad\qquad \times \ell^{1/(1-\sigma)}(\log \ell)^{\sigma/(1-\sigma)}(1 + o(1))\big\},\end{aligned}$$

where C_1^*, C_2^*, and C_3^* are positive constants given in Theorem 0 below. If $\sigma = 1$, for sufficiently large ℓ,

$$\begin{aligned}(1.2)\quad 2C_2^* \exp\{-C_3^* \exp\exp((C_1^*)^{-1}\ell(1 + o(1)))\} & \le \widetilde{W}(\ell, 1; \zeta) \\ & \le 4 \exp\big\{-\tfrac{3}{4} \exp\exp(2^{-1}\ell(1 + o(1)))\big\}.\end{aligned}$$

The inequalities (1.1) are proved by Joyner [2, Chap. 5, p. 142, Theorem 4.3], and (1.2) is due to the second-named author [6]. These results determine the real magnitude of the quantity $\widetilde{W}(\ell, \sigma; \zeta)$ with respect to ℓ up to constant factors.

The present paper gives similar estimates for other zeta-functions (Theorem 1), and improves the constants in the above inequalities for the Riemann zeta-function (Theorem 2).

The basic tool of the proof of the above inequalities is Montgomery's theorem on the sums of independent random variables. Let $\mathbb{N}$ be the set of positive integers, and let $\boldsymbol{r} = \{r_n \mid n \in \mathbb{N}\}$ be a sequence of non-negative real numbers, with infinitely many non-zero terms, satisfying

$$\sum_{n=1}^{\infty} r_n^2 < \infty. \tag{1.3}$$

Let $\theta_1, \theta_2, \theta_3, \ldots$ be independent random variables with identical distribution on a probability space (Ω, P), where the law of θ_1 is a uniform distribution on the interval $[0,1]$. Put

$$X_n = \cos(2\pi\theta_n), \quad n \in \mathbb{N}, \quad \text{and} \quad X = \sum_{n=1}^{\infty} r_n X_n. \tag{1.4}$$

Note that (1.4) implies

$$E[X_n] = 0, \quad n \in \mathbb{N}, \tag{1.5}$$

where $E[\cdot]$ denotes the expectation value. Kolmogorov's theorem with (1.3) and (1.5) implies that X converges almost surely.

For $N \in \mathbb{N}$, put

$$A_N(\boldsymbol{r}) = \sum_{n=N+1}^{\infty} r_n^2 \quad \text{and} \quad B_N(\boldsymbol{r}) = \sum_{n=1}^{N} r_n.$$

The condition (1.3) implies that $A_N(\boldsymbol{r})$ is finite.

Montgomery proved the following upper and lower bounds for the probabilities that X takes large values. His upper bound is that, for any positive integer N,

$$P(X \geq 2B_N(\boldsymbol{r})) \leq \exp\left\{-\tfrac{3}{4} B_N(\boldsymbol{r})^2 A_N(\boldsymbol{r})^{-1}\right\}. \tag{1.6}$$

Also he showed

Theorem 0 (Montgomery [8]). *Let $\boldsymbol{r} = \{r_n\}$ be as above, and assume furthermore that $\{r_n\}$ decreases monotonically. Then there exist positive constants C_1^*, C_2^*, and C_3^*, for which*

$$P(X \geq C_1^* B_N(\boldsymbol{r})) \geq C_2^* \exp\{-C_3^* B_N(\boldsymbol{r})^2 A_N(\boldsymbol{r})^{-1}\} \tag{1.7}$$

holds for every positive integer N. *We can take* $C_1^* = 2^{-1}$, $C_2^* = 2^{-40}$, *and* $C_3^* = 100$.

The monotonicity assumption on $\{r_n\}$ in Theorem 0 is harmless in the application to the case of the Riemann zeta-function, but it is too restrictive to prove the results of the forms (1.1) and (1.2) for general zeta-functions. (Montgomery states both upper and lower bounds under the assumption of monotonic decrease of $\{r_n\}$, but the assumption is used only in the proof of the lower bound.)

Consider the following form of lower bound estimate for the probability that X takes large values:

$$(\#)\qquad P(X \geq C_1 B_N(\boldsymbol{r})) \geq C_2 \exp\{-C_3 B_N(\boldsymbol{r})^2 A_N(\boldsymbol{r})^{-1}\},$$

for sufficiently large N, with some positive constants C_1, C_2, and C_3. (The notations C_1^*, C_2^*, and C_3^* will be reserved for the monotonically decreasing case as in Theorem 0, while C_1, C_2, and C_3 will be used for general cases.) The main result of the present paper is Theorem 4, which gives a necessary and sufficient condition for existence of an estimate of the form (#).

Theorem 4 allows one to handle the case where $\{r_n\}$ does not decrease monotonically (Theorem 3). The second-named author [5] generalized Bohr and Jessen's theory to fairly general zeta-functions defined by certain Euler products. Denote such a zeta-function by $\phi(s)$, and put

$$V(T, R, \sigma; \phi) = \mu_1(\{t \in [-T, T] \mid \log \phi(\sigma + it) \in R\}).$$

In [5], the existence of the limit

$$W(R, \sigma; \phi) = \lim_{T\to\infty} (2T)^{-1} V(T, R, \sigma; \phi)$$

is proved not only in the domain of absolute convergence, but also in the critical strip under some moderate conditions. As is explained in [7], the inequality (#) plays the vital role in the study of the lower bound of

$$\widetilde{W}(\ell, \sigma; \phi) = 1 - W(R(\ell), \sigma; \phi)$$

in the critical strip.

Consider the case where $\phi = \phi_f$ is the Dirichlet series attached to a primitive form f of weight m (≥ 1) with respect to the full modular group $\mathrm{SL}(2, \mathbb{Z})$. In this case $W(R, \sigma; \phi_f)$ exists for any $\sigma > m/2$ (see [4]). Inequalities of the forms (1.1) and (1.2), which cannot be deduced from Montgomery's Theorem 0 in this case, are obtained from Theorem 3 in this paper:

Theorem 1. *For any sufficiently large* ℓ,

$$\begin{aligned}
&\alpha_1 \exp\{-\alpha_2(m, \sigma, f)\ell^{2/(m+1-2\sigma)}(\log \ell)^{(2\sigma-m+1)/(m+1-2\sigma)}(1 + o(1))\}\\
&\quad \leq \widetilde{W}(\ell, \sigma; \phi_f)\\
&\quad \leq 4 \exp\{-\alpha_3(m, \sigma)\ell^{2/(m+1-2\sigma)}(\log \ell)^{(2\sigma-m+1)/(m+1-2\sigma)}(1 + o(1))\}
\end{aligned}$$

for $m/2 < \sigma < (m+1)/2$, *and*

$$\alpha_1 \exp\{-\alpha_4(f) \exp\exp(\alpha_5 \ell(1+o(1)))\} \le \widetilde{W}(\ell, (m+1)/2; \phi_f) \le 4\exp\left\{-\tfrac{3}{8}\exp\exp(5(2+3\sqrt{6})^{-1}\ell(1+o(1)))\right\},$$

where α_1 *and* α_5 *are absolute positive constants*, *and* α_i, $i = 2,3,4$, *are positive constants depending only on the quantities written in the parentheses.*

The values of the constants $\alpha_1, \ldots, \alpha_5$ are explicitly written in Section 3.

Another interesting application of Theorem 4 is that it gives an improvement of constant factors, even in the monotonically decreasing case.

THEOREM 2. (i) *For any positive* C_2^*, *there exists* $N_0 = N_0(C_2^*)$ *for which* (1.7) *holds with this fixed* C_2^* *and* $(C_1^*, C_3^*) = (1/2, 47)$ *for any* $N \ge N_0$.

(ii) *In the case of the Riemann zeta-function*, *in* (1.1) *we can replace* C_1^* *and* C_3^* *by the improved values* $K(\sigma)/2$ *and* $47 - 6.31(K(\sigma)-1)$, *respectively*, *where*

$$K(\sigma) = \left(\frac{2\sigma-1}{1-\sigma}\right)^{(1-\sigma)/\sigma} + \left(\frac{2\sigma-1}{1-\sigma}\right)^{(1-2\sigma)/\sigma}.$$

Theorem 2(i) provides an improvement of the constants in (1.1), compared with the constants given by Theorem 0. But it is essentially included in Montgomery's argument, because the only novelty is the new choice (5.8) of the parameters. The argument which leads to Theorem 2(ii) is new. It is easy to see that $1 < K(\sigma) \le 2$ and $K(2/3) = 2$. Therefore Theorem 2(ii) gives a further improvement of the constants.

Theorems 1 and 2 are proved in Sections 3 and 6, respectively. In the following sections, ε denotes an arbitrarily small positive number, and is not necessarily the same at each occurrence.

The authors would like to thank Professor Akio Fujii for useful suggestions.

2. The main theorem. The following Theorem 3, a special case of the main Theorem 4, is in a form suitable for application to the proof of Theorem 1.

THEOREM 3. *Let* $\boldsymbol{r} = \{r_n\}$ *be a sequence of non-negative real numbers with infinitely many non-zero terms*, *and assume that it satisfies* (1.3) *and the following* (2.1) *and* (2.2) *with some positive constants* κ (≤ 1), C_4, C_5, *and* C_6:

$$r_n \le C_4 p_n^{-\kappa}, \qquad n \in \mathbb{N}, \tag{2.1}$$

where p_n *denotes the n-th prime number*, *and*

$$\#\{n \le x \mid r_n \ge C_5 p_n^{-\kappa}\} > C_6 x, \tag{2.2}$$

for sufficiently large x, where the symbol $\#S$ signifies the cardinality of the set S. Then, for any $\varepsilon > 0$ and any $C_2 > 0$, there exists an $N_0 \in \mathbb{N}$ for which (#) holds for any $N \geq N_0$, with $C_1 = 1/2$ and $C_3 = C + C^2$, $C = 6.31(C_4 C_5^{-1} C_6^{-\kappa})^2(1+\varepsilon)$.

The proof of this theorem is given in Section 4.

Remarks. 1. In the proof it is shown that $\kappa > 1/2$.

2. In [7], this theorem was quoted with a weaker value of C_3, obtained from Theorem 4 with $q = 1$. Consequently, the constants in Theorem 5 of [7] are weaker than those in Theorem 1 of the present paper.

The assumptions (2.1) and (2.2) are not necessary for (#) to hold. This can be seen in the following

Example 1. The sequence

$$\{r_n\} = 1, 1/2, e^{-3}, 1/3, e^{-5}, e^{-6}, e^{-7}, 1/4, e^{-9}, \ldots, e^{-15}, 1/5, e^{-17}, \ldots, e^{-31}, 1/6, e^{-33}, \ldots$$

satisfies neither (2.1) nor (2.2), but satisfies (#) with $C_1 = \frac{1}{2}(1 - 1/e)$, $C_2 = 1$, and $C_3 = 47$.

The following theorem gives a necessary and sufficient condition for (#) to hold.

Let $\{r_n\}$ be a sequence of non-negative real numbers, with infinitely many non-zero terms, satisfying (1.3). The rearrangement $\{\varrho_n\}$ of $\{r_n\}$ into a decreasing sequence is defined by a bijection $\lambda : \mathbb{N}_0 \to \mathbb{N}$, where $\mathbb{N}_0 = \{n \in \mathbb{N} \mid r_n \neq 0\}$, such that $r_n = \varrho_{\lambda(n)}$ for any $n \in \mathbb{N}_0$, and $\varrho_1 \geq \varrho_2 \geq \ldots$ Note that (1.3) implies $\lim_{n\to\infty} r_n = 0$, hence the rearrangement into a decreasing sequence is well-defined.

Theorem 4. *Let $\boldsymbol{r} = \{r_n\}$ be a sequence of non-negative real numbers with infinitely many non-zero terms, satisfying* (1.3), *and let $\boldsymbol{\rho} = \{\varrho_n\}$ be a rearrangement of $\{r_n\}$ into a decreasing sequence. Let*

$$f_{\mathsf{r},\boldsymbol{\rho}}(N_1, N_2; q) = B_{N_2}(\boldsymbol{\rho}) B_{N_1}(\boldsymbol{r})^{-1} + q A_{N_2}(\boldsymbol{\rho}) A_{N_1}(\boldsymbol{r})^{-1}, \quad N_1, N_2 \in \mathbb{N},\ q > 0. \tag{2.3}$$

(i) *A necessary and sufficient condition for the existence of an $N_0 \in \mathbb{N}$ and of positive constants C_1, C_2, and C_3 such that (#) holds for any $N \geq N_0$, is that there exist positive constants q, u_q, and N_0' such that*

$$f_{\mathsf{r},\boldsymbol{\rho}}(N_1, N_2; q) \geq u_q \quad \text{for all } N_1 \geq N_0' \text{ and } N_2 \geq N_0'. \tag{2.4}$$

If (2.4) *holds for some positive q, then it holds for any positive q.*

(ii) *When* (2.4) *holds for some $q > 0$, possible values of the constants are*:

$$C_1 = u_q/\nu_1 \quad \text{and} \quad C_3 = 2q\nu_2/\nu_1 + (q\nu_2/\nu_1)^2 - 2(q\nu_2/\nu_1^2)u_q,$$

and any positive C_2, *where* ν_1 *and* ν_2 *are constants satisfying* (5.6) *and* (5.7), *for example*, $\nu_1 = 2$ (*or* $2(1-\varepsilon)$ *for sufficiently small* $\varepsilon > 0$) *and* $\nu_2 = 12.62$. *The value of* N_0 *depends on* C_2.

The proof of this theorem is given in Section 5. The specific form (1.4) of the distribution of X_1 is not essential. The estimates needed in the proof of Theorem 4 are given in Proposition 5, which holds (with suitable change in the constants) for any independent random variables $X_1, X_2, \ldots$ with identical distribution, satisfying $E[X_1] = 0$, $E[X_1^2] > 0$, and $X_1 < M$ for a constant M almost surely.

A proof of the statement in Example 1 is as follows. Let $\boldsymbol{r} = \{r_n\}$ be as in Example 1. Then

$$\{\varrho_n\} = 1, 1/2, 1/3, \ldots, 1/20, e^{-3}, 1/21, 1/22, \ldots$$

Fix N_1 and define n_1 by $2^{n_1-1} \le N_1 < 2^{n_1}$. Put $N_2^0 = \max\{n \in \mathbb{N} \mid \varrho_n \ge 1/n_1\}$. It follows that $\{r_n \mid n > N_1\} \subset \{\varrho_n \mid n > N_2^0\}$. Hence if $N_2 \le N_2^0$, then $A_{N_2}(\boldsymbol{\rho}) \ge A_{N_1}(\boldsymbol{r})$, and if $N_2 > N_2^0$, then

$$B_{N_1}(\boldsymbol{r}) \le B_{N_2}(\boldsymbol{\rho}) + \sum_{n\ge 1} e^{-n} = B_{N_2}(\boldsymbol{\rho}) + \frac{1}{e-1} \le \left(1 + \frac{1}{e-1}\right) B_{N_2}(\boldsymbol{\rho}).$$

Therefore $B_{N_2}(\boldsymbol{\rho}) B_{N_1}(\boldsymbol{r})^{-1} \ge 1 - 1/e$, which implies $f_{\mathsf{r},\boldsymbol{\rho}}(N_1, N_2; 1) \ge 1 - 1/e$. Theorem 4 can be applied to obtain the assertion.

It may be worthwhile to note that not every sequence $\{r_n\}$ satisfies the condition (2.4).

Example 2. The sequence

$$\{r_n\} = \tau_1, \sigma_1, \tau_2, \sigma_2, \tau_3, \tau_4, \sigma_3, \tau_5, \ldots, \tau_8, \sigma_4, \tau_9, \ldots$$
$$\ldots, \tau_{2^{k-1}}, \sigma_k, \tau_{2^{k-1}+1}, \ldots, \tau_{2^k}, \sigma_{k+1}, \ldots,$$

where $\tau_k = \sigma_k = 1/k$, $k \in \mathbb{N}$, cannot have an estimate of the form (#). In fact,

$$\lim_{k\to\infty} f_{\mathsf{r},\boldsymbol{\rho}}(2^k + k, 2k^2; 1) = 0,$$

so that (2.4) does not hold. (We use τ_k and σ_k for single $1/k$ to guide the eyes.)

3. Deduction of Theorem 1 from Theorem 3. The upper bound part of Theorem 1 is proved in [6] and [7]. Therefore it is sufficient to prove the lower bound part.

Let ϕ_f be the Dirichlet series attached to a primitive form f of weight m with respect to $\mathrm{SL}(2, \mathbb{Z})$. Then ϕ_f has the Euler product expansion of the form

$$\phi_f(s)=\prod_{n=1}^{\infty}(1-\alpha_n p_n^{-s})^{-1}(1-\beta_n p_n^{-s})^{-1},$$

in the half-plane $\sigma>(m+1)/2$, and $\alpha_n+\beta_n=c(p_n)$, the p_nth Fourier coefficient of f.

Deligne's proof of Ramanujan–Petersson's conjecture asserts $|c(p_n)|\le 2p_n^{(m-1)/2}$ for any $n\in\mathbb{N}$. On the other hand, the inequality

$$|c(p_n)|>(\sqrt{2}-\varepsilon)p_n^{(m-1)/2},$$

for an arbitrarily small $\varepsilon>0$, is valid for a positive density of primes, as is shown in Corollary 2 of Ram Murty [10]. Hence we see that (2.1) and (2.2) are valid for $r_n=|c(p_n)|p_n^{-\sigma}$, $m/2<\sigma\le(m+1)/2$, with $\kappa=\sigma-(m-1)/2$, $C_4=2$, $C_5=\sqrt{2}-\varepsilon$, and some positive $C_6=C_6(\varepsilon)$. Theorem 3 asserts that (#) holds with $C_1=1/2$, $C_3=C+C^2$ with $C=12.62C_6^{-2\sigma+m-1}(1+3\varepsilon)$, and any C_2.

We choose $N=N(\ell)\in\mathbb{N}$ by the condition

$$\tfrac{1}{2}B_{N-1}(\boldsymbol{r})\le\ell+A<\tfrac{1}{2}B_N(\boldsymbol{r}),\tag{3.1}$$

where A is the positive constant, depending only on σ and ϕ_f, defined in Section 3 of [6]. Then we can deduce

$$\begin{aligned}\widetilde{W}(\ell,\sigma;\phi_f)&\ge\mu_\infty\big(X>\tfrac{1}{2}B_N(\boldsymbol{r})\text{ or }X<-\tfrac{1}{2}B_N(\boldsymbol{r})\big)\\&\ge 2C_2\exp(-C_3B_N(\boldsymbol{r})^2A_N(\boldsymbol{r})^{-1}),\end{aligned}\tag{3.2}$$

from (4.11) of [6] and (#).

Put $a(p_n)=c(p_n)p_n^{(1-m)/2}$. Rankin [11] proved

$$\sum_{p_n\le x}|a(p_n)|^2=\frac{x}{\log x}(1+o(1)),\tag{3.3}$$

and

$$\begin{aligned}\frac{1}{\sqrt{2}}\cdot\frac{x}{\log x}(1+o(1))&\le\sum_{p_n\le x}|a(p_n)|\\&\le\frac{2+3\sqrt{6}}{10}\cdot\frac{x}{\log x}(1+o(1)).\end{aligned}\tag{3.4}$$

Let x be a real number satisfying $p_N\le x<p_{N+1}$. By using partial summation, we obtain

$$\begin{aligned}B_N(\boldsymbol{r})=&\sum_{p_n\le x}|a(p_n)|x^{(m-1)/2-\sigma}\\&+\left(\sigma-\frac{m-1}{2}\right)\int_2^x\sum_{p_n\le\xi}|a(p_n)|\xi^{(m-1)/2-\sigma-1}d\xi.\end{aligned}$$

Hence, by using (3.4), it follows that

$$B_N(\boldsymbol{r}) \le \frac{2+3\sqrt{6}}{5(m+1-2\sigma)} \cdot \frac{x^{(m+1)/2-\sigma}}{\log x}(1+o(1)) \qquad \text{if } m/2 < \sigma < (m+1)/2, \tag{3.5}$$

$$B_N(\boldsymbol{r}) \le \frac{2+3\sqrt{6}}{10}(\log\log x)(1+o(1)) \qquad \text{if } \sigma = (m+1)/2. \tag{3.6}$$

Also, from (3.3) it follows that

$$A_N(\boldsymbol{r}) \ge \frac{1}{2\sigma - m} \cdot \frac{x^{m-2\sigma}}{\log x}(1+o(1)).$$

Hence,

$$\begin{aligned} &B_N(\boldsymbol{r})^2 A_N(\boldsymbol{r})^{-1} \\ &\le \frac{(2+3\sqrt{6})^2(2\sigma-m)}{25(m+1-2\sigma)^2} \cdot \frac{x}{\log x}(1+o(1)) \qquad \text{if } m/2 < \sigma < (m+1)/2, \end{aligned} \tag{3.7}$$

$$\begin{aligned} &B_N(\boldsymbol{r})^2 A_N(\boldsymbol{r})^{-1} \\ &\le \frac{(2+3\sqrt{6})^2}{100} x \log x (\log\log x)^2 (1+o(1)) \qquad \text{if } \sigma = (m+1)/2. \end{aligned} \tag{3.8}$$

Lower bounds for $B_N(\boldsymbol{r})$ follow from arguments similar to those for (3.5) and (3.6). These lower bounds with (3.1) imply upper bounds of x and $x/\log x$ in terms of ℓ. Substituting such upper bounds in (3.7) and (3.8) and then substituting them in the right-hand side of (3.2), we obtain the assertion of Theorem 1.

The above proof gives the explicit values of the constants: First, an arbitrary positive number can be chosen as α_1. Then, for any large $\ell \ge \ell_0 = \ell_0(\alpha_1)$, we can choose

$$\alpha_2(m,\sigma,f) = 2C_3\left(\frac{2+3\sqrt{6}}{10}\right)^2 \frac{2\sigma-m}{m+1-2\sigma} 2^{3/(m+1-2\sigma)},$$

$$\alpha_3(m,\sigma) = \frac{3}{4} \cdot \frac{2\sigma-m}{m+1-2\sigma}\left(\frac{2+3\sqrt{6}}{5}\right)^{-2/(m+1-2\sigma)},$$

$$\alpha_4(f) = C_3\left(\frac{2+3\sqrt{6}}{10}\right)^2 \quad \text{and} \quad \alpha_5 = 2\sqrt{2},$$

where $C_3 = C + C^2$ with $C = 12.62 C_6^{-2\sigma+m-1}(1+\varepsilon)$. (See also Theorem 5 of [7].)

The above proof of Theorem 1 is similar to that developed in Sections 4 and 5 of [6]. A general form of the argument is given in Lemma 3 of [7], in which the roles of (#) and "prime-number-theorem type" results (such as Rankin's (3.3) and (3.4)) are clarified in a more general situation. (The notation μ_∞ is used instead of P in [6] and [7].) Roughly speaking, a "prime-number-theorem type" estimate such as (3.4) gives estimate on $B_N(\boldsymbol{r})$ such as $B_N(\boldsymbol{r}) = O(N^{1-\kappa})$, with $1/2 < \kappa < 1$. It should be noted, however, that such an estimate is not sufficient for (#) to hold. This can be seen in the following

Example 3. The sequence

$$\{r_n\} = \tau_1, \sigma_1, \tau_2, \sigma_2, \tau_3, \tau_4, \sigma_3, \tau_5, \ldots, \tau_8, \sigma_4, \tau_9, \ldots$$
$$\ldots, \tau_{2^{k-1}}, \sigma_k, \tau_{2^{k-1}+1}, \ldots, \tau_{2^k}, \sigma_{k+1}, \ldots,$$

where $\tau_k = \sigma_k = k^{-\kappa}$, $k = 1, 2, 3, \ldots$, and $1/2 < \kappa < 1$, cannot have an estimate of the form (#). The proof is similar to that of Example 2.

By using the notion of the rearrangement, Montgomery–Odlyzko [9] gives another general lower-bound. Let $X_1, X_2, \ldots$ be independent random variables such that $E[X_n] = 0$, $|X_n| \le 1$, and $E[X_n^2] \ge C$ for a constant $C > 0$. They state that if $B_N(\boldsymbol{\rho}) \ge 2V$, then

$$P(X \ge V) \ge a_1 \exp(-a_2 V^2 A_N(\boldsymbol{\rho})^{-1}) \tag{3.9}$$

with positive constants a_1 and a_2 depending only on C. This rearrangement method is useful in the case of Dedekind zeta-functions; the lower-bound part of the Theorem in [6] also uses the rearrangement method, which is essentially the same as (3.9). However, in general, it is difficult to know the arithmetic properties of $A_N(\boldsymbol{\rho})$ and $B_N(\boldsymbol{\rho})$, so (3.9) is not sufficient for other arithmetic applications.

4. Deduction of Theorem 3 from Theorem 4. Define a subsequence $\boldsymbol{s} = \{s_n\}$ of $\boldsymbol{r} = \{r_n\}$ by selecting all terms that satisfy

$$r_n \ge C_5 p_n^{-\kappa}. \tag{4.1}$$

Define a one-to-one increasing map $f : \mathbb{N} \to \mathbb{N}$ by writing $n = f(m)$ if r_n is the mth element in $\{r_n\}$ that satisfies (4.1); we have $s_m = r_{f(m)}$. Then (2.2) implies that there exists an $M_1 \in \mathbb{N}$ for which

$$m > C_6 f(m) \tag{4.2}$$

holds for any $m \ge M_1$. On the other hand, it is known (§57 of [3]) that for any $\varepsilon > 0$, there exists an $M_2 = M_2(\varepsilon) \in \mathbb{N}$ for which

$$(1-\varepsilon) n \log n < p_n < (1+\varepsilon) n \log n \tag{4.3}$$

holds for any $n \ge M_2$. Hence, if $m \ge M_3 = \max(M_1, M_2)$, then from (4.2) and (4.3) we have

$$p_{f(m)} < (1+\varepsilon) f(m) \log f(m) < (1+\varepsilon) C_6^{-1} m \log(C_6^{-1} m). \tag{4.4}$$

Next, let $\boldsymbol{\sigma} = \{\sigma_n\}$ be a rearrangement of $\boldsymbol{s} = \{s_n\}$ into a decreasing sequence. Then we claim $\sigma_n \ge C_5 p_{f(n)}^{-\kappa}$ for any n. In fact, let us assume the contrary, and let n_0 be the smallest positive integer for which $\sigma_{n_0} < C_5 p_{f(n_0)}^{-\kappa}$ holds. Then there are exactly $n_0 - 1$ elements in $\{\sigma_n\}$ such that $\sigma_n \geq C_5 p_{f(n_0)}^{-\kappa}$, while the definition of $\{s_n\}$ implies that there are at least n_0 elements in $\{s_n\}$ satisfying $s_n \geq C_5 p_{f(n_0)}^{-\kappa}$. This is a contradiction, hence our claim follows. From this claim and (4.4), we have

$$(4.5)\qquad \begin{aligned} A_N(\boldsymbol{\rho}) \geq A_N(\boldsymbol{\sigma}) &\geq C_5^2 \sum_{n>N} p_{f(n)}^{-2\kappa} \\ &\geq (C_5 C_6^\kappa)^2 (1-\varepsilon) \sum_{n>N} (n \log n)^{-2\kappa}, \end{aligned}$$

for sufficiently large N. Since (1.3) implies $A_N(\boldsymbol{\rho}) \leq A_N(\boldsymbol{r}) < \infty$, the sum on the right-hand side of (4.5) must converge, therefore it is required that $\kappa > 1/2$. From (2.1) and (4.3) we have

$$(4.6)\qquad A_N(\boldsymbol{r}) \leq C_4^2 \sum_{n>N} p_n^{-2\kappa} < C_4^2 (1+\varepsilon) \sum_{n>N} (n \log n)^{-2\kappa}.$$

From (4.5) and (4.6), it follows that for sufficiently large N we have

$$(4.7)\qquad A_N(\boldsymbol{r}) \leq (C_4 C_5^{-1} C_6^{-\kappa})^2 (1+\varepsilon) A_N(\boldsymbol{\rho}).$$

If $N_2 < N_1$, then from (4.7) we have

$$A_{N_2}(\boldsymbol{\rho}) \geq (C_4^{-1} C_5 C_6^\kappa)^2 (1-\varepsilon) A_{N_2}(\boldsymbol{r}) \geq (C_4^{-1} C_5 C_6^\kappa)^2 (1-\varepsilon) A_{N_1}(\boldsymbol{r})$$

for sufficiently large N_1 and N_2. If $N_2 \geq N_1$, then we have

$$B_{N_2}(\boldsymbol{\rho}) \geq B_{N_1}(\boldsymbol{\rho}) \geq B_{N_1}(\boldsymbol{r}).$$

Hence, (2.4) is satisfied with $q = (C_4 C_5^{-1} C_6^{-\kappa})^2 (1+\varepsilon)$ and $u_q = 1$. Theorem 4 with $\nu_1 = 2$ therefore implies Theorem 3.

Remark. We can prove a result slightly weaker than Theorem 3 directly without Theorem 4. From the inequalities

$$B_N(\boldsymbol{r}) \geq \sum_{m \leq C_6 N} s_m \geq C_5 C_6^\kappa (1-\varepsilon) \sum_{2 \leq m \leq C_6 N} (m \log m)^{-\kappa},$$

and

$$B_N(\boldsymbol{\rho}) \leq C_4 (1+\varepsilon) \sum_{2 \leq n \leq N} (n \log n)^{-\kappa},$$

we have

$$(4.8)\qquad B_N(\boldsymbol{r}) \geq C_4^{-1} C_5 C_6 (1-\varepsilon) B_N(\boldsymbol{\rho}).$$

The distribution of $\sum \varrho_n X_n$ is equal to that of $\sum r_n X_n$ (see Lemma 3 of [6]). Therefore, Theorem 0 together with (4.7) and (4.8) implies (#), with $C_1 = C_1^* = \nu_1^{-1}$ and $C_3 = C_3^* C_4^4 C_5^{-4} C_6^{-2(1+\kappa)} (1+\varepsilon)$ (see (6.1) for the value of C_3^*). These are the values quoted in [7]. The assumptions (2.1) and (2.2) imply

$C_6 \le 1 \le C_4 C_5^{-1}$, from which it follows that the above value C_3 is weaker (larger) than that in Theorem 3.

5. Proof of Theorem 4. Denote by $I_0(t)$ the modified Bessel function defined by

$$I_0(t) = \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \left(\frac{t}{2}\right)^{2n}. \tag{5.1}$$

It is known that

$$I_0(t) = \int_0^1 \exp(t\cos(2\pi\theta))\,d\theta. \tag{5.2}$$

(See, for example, Section 17.23 of [12].) Hence (1.4) implies

$$E[\exp(tX_n)] = I_0(t), \qquad n \in \mathbb{N}. \tag{5.3}$$

Proposition 5. (i) *The following upper bounds hold for* $I_0(t)$:

$$I_0(t) \le e^t, \qquad t \ge 0, \tag{5.4}$$

and

$$I_0(t) \le \exp(t^2/4), \qquad t \ge 0. \tag{5.5}$$

(ii) *The following lower bounds hold*:

$$I_0(t) \ge (1+\nu_0)\exp(t/\nu_1), \qquad t \ge \alpha, \tag{5.6}$$

and

$$I_0(t) \ge \exp(t^2/\nu_2), \qquad 0 \le t < \alpha, \tag{5.7}$$

with some positive constants α, ν_0, ν_1, *and* ν_2. *For example*, *we can take*

$$(\nu_0, \nu_1, \nu_2, \alpha) = (700^{-1/2}, 2 \text{ or } 2(1-\varepsilon), 12.62, 5). \tag{5.8}$$

Proof. The bound (5.4) is trivial. The bound (5.5) is proved in Montgomery [8]. In the same paper, Montgomery gives a proof of (5.6) and (5.7) with

$$(\nu_0, \nu_1, \nu_2, \alpha) = (1, 2, 19, 7). \tag{5.9}$$

The following is a slight modification of his argument. Since $\exp(t\cos(2\pi\theta)) \ge \exp(\sqrt{3}t/2)$, for $0 \le \theta \le 1/12$ and $11/12 \le \theta \le 1$, from (5.2) we see $I_0(t) \ge (1/6)\exp(\sqrt{3}t/2)$. To prove (5.6) with $\nu_1 = 2$ or $2(1-\varepsilon)$, it is therefore sufficient to show

$$\frac{1}{6}\exp\left(\frac{\sqrt{3}}{2}t\right) \ge (1+\nu_0)\exp(t/(2(1-\varepsilon))) \quad \text{for } t \ge \alpha.$$

This is equivalent to

$$\nu_0 \le \frac{1}{6}\exp\left(\frac{\sqrt{3}-1}{2}\alpha - \varepsilon\right) - 1. \tag{5.10}$$

Next, (5.1) implies $I_0(t) \ge 1 + t^2/4$, hence, to prove (5.7), it is sufficient to show $1 + t^2/4 \ge \exp(t^2/\nu_2)$ for $0 < t < \alpha$. This is equivalent to

$$\nu_2 \ge \alpha^2/\log(1+\alpha^2/4). \tag{5.11}$$

With any values of ν_0, ν_2, and α satisfying (5.10) and (5.11), the inequalities (5.6) and (5.7) hold with $\nu_1 = 2$ or $\nu_1 = 2(1-\varepsilon)$. We can check that the choices (5.8) and (5.9) satisfy (5.10) and (5.11) for sufficiently small $\varepsilon \ge 0$. This completes the proof of Proposition 5.

Proof of Theorem 4. First we prove that (2.4) implies (#). Assume that (2.4) holds for some positive constants q, u_q, N_0', and for all $N_1 \ge N_0'$ and $N_2 \ge N_0'$. By definitions and (1.3), $A_N(\boldsymbol{r})$ decreases monotonically to zero and $B_N(\boldsymbol{r})$ increases monotonically as N increases. Hence for any $C_2 > 0$, there exists an $\widetilde{N}_0 \in \mathbb{N}$ for which

$$\#\{n \in \mathbb{N} \mid q\nu_1^{-1}\nu_2 r_n B_N(\boldsymbol{r}) A_N(\boldsymbol{r})^{-1} \ge \alpha\} \times \log(1+\nu_0) \ge \log(1+\sqrt{C_2}) \tag{5.12}$$

holds for every $N > \widetilde{N}_0$. Put

$$\mathcal{N}_1 = \{n \in \mathbb{N} \mid q\nu_1^{-1}\nu_2 r_n B_N(\boldsymbol{r}) A_N(\boldsymbol{r})^{-1} \ge \alpha\} \quad \text{and} \quad \mathcal{N}_2 = \mathbb{N} - \mathcal{N}_1.$$

We then have

$$\begin{aligned}\sum_{n=1}^{\infty} &\log\{I_0(q\nu_1^{-1}\nu_2 r_n B_N(\boldsymbol{r}) A_N(\boldsymbol{r})^{-1})\} \\ &\ge \sum_{n\in\mathcal{N}_1}\{\log(1+\nu_0) + (q\nu_1^{-1}\nu_2 r_n B_N(\boldsymbol{r}) A_N(\boldsymbol{r})^{-1}/\nu_1)\} \\ &\quad + \sum_{n\in\mathcal{N}_2}(q\nu_1^{-1}\nu_2 r_n B_N(\boldsymbol{r}) A_N(\boldsymbol{r})^{-1})^2/\nu_2 \\ &\ge \log(1+\sqrt{C_2}) + q\nu_1^{-2}\nu_2 B_N(\boldsymbol{r}) A_N(\boldsymbol{r})^{-1}\sum_{n\in\mathcal{N}_1} r_n \\ &\quad + q^2\nu_1^{-2}\nu_2 B_N(\boldsymbol{r})^2 A_N(\boldsymbol{r})^{-2}\sum_{n\in\mathcal{N}_2} r_n^2,\end{aligned} \tag{5.13}$$

for $N > \widetilde{N}_0$. Put

$$N_2 = \max\{n \in \mathbb{N} \mid \varrho_n \ge \alpha q^{-1}\nu_1\nu_2^{-1} A_N(\boldsymbol{r}) B_N(\boldsymbol{r})^{-1}\}.$$

Since $\{\varrho_n\}$ is a rearrangement of $\{r_n\}$, we have

$$B_{N_2}(\boldsymbol{\rho}) = \sum_{n\in\mathcal{N}_1} r_n \quad \text{and} \quad A_{N_2}(\boldsymbol{\rho}) = \sum_{n\in\mathcal{N}_2} r_n^2.$$

Note that for sufficiently large N, we have $N_2 \ge N_0'$. Using (2.4) and (5.13) we therefore obtain, for sufficiently large N,

$$\sum_{n=1}^{\infty} \log\{I_0(\lambda r_n)\} \ge \log(1+\sqrt{C_2}) + u_q\nu_1^{-1}\lambda B_N(\boldsymbol{r}), \tag{5.14}$$

where $\lambda = q\nu_1^{-1}\nu_2 B_N(\boldsymbol{r})A_N(\boldsymbol{r})^{-1}$. Put $G = \exp(\lambda X)$ and $g = \exp\{u_q\nu_1^{-1}\lambda B_N(\boldsymbol{r})\}$. Then, (5.3) and (5.14) imply

$$E[G] = E[\exp(\lambda X)] = \prod_{n\ge 1} I_0(\lambda r_n) \ge (1+\sqrt{C_2})g. \tag{5.15}$$

On the other hand, by Schwarz's inequality, we have

$$\begin{aligned} \Big(\int_{G\ge g} G\,dP\Big)^2 &\le \Big(\int_{G\ge g} G^2\,dP\Big)\Big(\int_{G\ge g} dP\Big) \\ &\le E[G^2]P(G\ge g) \\ &= \Big(\prod_{n\ge 1} I_0(2\lambda r_n)\Big)P(X \ge u_q\nu_1^{-1}B_N(\boldsymbol{r})). \end{aligned} \tag{5.16}$$

By definition, $g \ge 1 > 0$. With (5.15) we have

$$\int_{G\ge g} G\,dP \ge \int_{G\ge g} (G-g)\,dP \ge E[G-g] = E[G]-g \ge \sqrt{C_2}g.$$

Substituting this into (5.16) and using (5.4) and (5.5), we have

$$\begin{aligned} &P(X \ge u_q\nu_1^{-1}B_N(\boldsymbol{r})) \\ &\qquad \ge C_2 g^2\Big\{\prod_{n=1}^{N}\exp(-2\lambda r_n)\Big\}\Big\{\prod_{n=N+1}^{\infty}\exp(-\lambda^2 r_n^2)\Big\} \\ &\qquad = C_2 g^2 \exp\{-(2q\nu_1^{-1}\nu_2 + q^2\nu_1^{-2}\nu_2^2)B_N(\boldsymbol{r})^2 A_N(\boldsymbol{r})^{-1}\} \\ &\qquad = C_2\exp\{-C_3 B_N(\boldsymbol{r})^2 A_N(\boldsymbol{r})^{-1}\}, \end{aligned}$$

where

$$C_3 = 2q\nu_1^{-1}\nu_2 + q^2\nu_1^{-2}\nu_2^2 - 2q\nu_1^{-2}\nu_2 u_q. \tag{5.17}$$

This implies (#) with the constants as claimed in Theorem 4(ii). If $q' > q$ then $f_{\mathsf{r},\boldsymbol{\rho}}(N_1,N_2;q') > f_{\mathsf{r},\boldsymbol{\rho}}(N_1,N_2;q)$, and if $q' < q$ then $f_{\mathsf{r},\boldsymbol{\rho}}(N_1,N_2;q') > q'q^{-1}f_{\mathsf{r},\boldsymbol{\rho}}(N_1,N_2;q)$, hence if (2.4) holds for some $q>0$, then it holds for any $q>0$.

Next we prove that (#) implies (2.4) with $q = 1$. Assume that (#) holds. For any $\lambda \geq 0$ and for any real number x we have

$$P(X \geq x) \leq \exp(-\lambda x)E[\exp(\lambda X)].$$

Therefore for any $\lambda \geq 0$,

$$\begin{aligned} P(X \geq C_1 B_{N_1}(\boldsymbol{r})) &\leq \exp(-\lambda C_1 B_{N_1}(\boldsymbol{r}))E[\exp(\lambda X)] \\ &= \exp(-\lambda C_1 B_{N_1}(\boldsymbol{r})) \prod_{n\geq 1} I_0(\lambda r_n) \\ &= \exp(-\lambda C_1 B_{N_1}(\boldsymbol{r})) \prod_{n\geq 1} I_0(\lambda \varrho_n) \\ &\leq \exp\{-\lambda C_1 B_{N_1}(\boldsymbol{r}) + \lambda B_{N_2}(\boldsymbol{\rho}) + 4^{-1}\lambda^2 A_{N_2}(\boldsymbol{\rho})\}, \end{aligned}$$

$N_1 \in \mathbb{N}$, $N_2 \in \mathbb{N}$, where the last inequality comes from (5.4) and (5.5). Combining with (#), we see that there exists an integer N_0 such that for $N_1 > N_0$ and $N_2 > N_0$,

$$\begin{aligned} -\lambda C_1 B_{N_1}(\boldsymbol{r}) + \lambda B_{N_2}(\boldsymbol{\rho}) + 4^{-1}\lambda^2 A_{N_2}(\boldsymbol{\rho}) \\ + C_3 B_{N_1}(\boldsymbol{r})^2 A_{N_1}(\boldsymbol{r})^{-1} - \log C_2 \geq 0. \end{aligned} \tag{5.18}$$

Take arbitrary integers N_1 and N_2 satisfying $N_1 > N_0$ and $N_2 > N_0$. Consider first the case

$$B_{N_2}(\boldsymbol{\rho}) \leq C_1 B_{N_1}(\boldsymbol{r}). \tag{5.19}$$

Put

$$\lambda = 2A_{N_2}(\boldsymbol{\rho})^{-1}(C_1 B_{N_1}(\boldsymbol{r}) - B_{N_2}(\boldsymbol{\rho})) \quad (\geq 0). \tag{5.20}$$

Substituting (5.20) into (5.18) and multiplying by $A_{N_2}(\boldsymbol{\rho})B_{N_1}(\boldsymbol{r})^{-2}$, we have

$$\begin{aligned} 2C_1 B_{N_2}(\boldsymbol{\rho})B_{N_1}(\boldsymbol{r})^{-1} + C_3 A_{N_2}(\boldsymbol{\rho})A_{N_1}(\boldsymbol{r})^{-1} \\ \geq C_1^2 + B_{N_2}(\boldsymbol{\rho})^2 B_{N_1}(\boldsymbol{r})^{-2} + A_{N_2}(\boldsymbol{\rho})B_{N_1}(\boldsymbol{r})^{-2}\log C_2 \\ \geq C_1^2 + A_{N_2}(\boldsymbol{\rho})B_{N_1}(\boldsymbol{r})^{-2}\log C_2. \end{aligned} \tag{5.21}$$

If $C_2 \geq 1$ then $A_{N_2}(\boldsymbol{\rho})B_{N_1}(\boldsymbol{r})^{-2}\log C_2 \geq 0$. If $0 < C_2 < 1$ then from (1.3) and the monotonicity of $A_N(\boldsymbol{\rho})$ and $B_N(\boldsymbol{r})$, there exists an integer $\widetilde{N}_0$, depending only on $\{r_n\}$, $\{\varrho_n\}$, C_1, and C_2, such that if $N_1 > \widetilde{N}_0$ and $N_2 > \widetilde{N}_0$, we have

$$2^{-1}C_1^2 > -(\log C_2)A_{N_2}(\boldsymbol{\rho})B_{N_1}(\boldsymbol{r})^{-2} > 0.$$

Therefore, for any positive C_2 we have, from (5.21),

$$2C_1 B_{N_2}(\boldsymbol{\rho})B_{N_1}(\boldsymbol{r})^{-1} + C_3 A_{N_2}(\boldsymbol{\rho})A_{N_1}(\boldsymbol{r})^{-1} \geq 2^{-1}C_1^2, \tag{5.22}$$

if $N_1 > \max\{N_0, \widetilde{N}_0\}$ and $N_2 > \max\{N_0, \widetilde{N}_0\}$, hence

$$f_{\mathsf{r},\boldsymbol{\rho}}(N_1, N_2; 1) \geq 2^{-1}C_1^2 \max(2C_1, C_3)^{-1} \quad (> 0). \tag{5.23}$$

It remains to consider the case $B_{N_2}(\boldsymbol{\rho}) > C_1 B_{N_1}(\boldsymbol{r})$, for which we have

$$f_{\mathsf{r},\boldsymbol{\rho}}(N_1, N_2; 1) \geq B_{N_2}(\boldsymbol{\rho}) B_{N_1}(\boldsymbol{r})^{-1} > C_1 \quad (> 0). \tag{5.24}$$

From (5.23) and (5.24) we have (2.4). This completes the proof of Theorem 4.

6. Proof of Theorem 2. Let $q = 1$. If $\{r_n\}$ is a monotonically decreasing sequence, then $u = u_q \geq 1$, because $B_{N_2}(\boldsymbol{\rho}) \geq B_{N_1}(\boldsymbol{r})$ if $N_1 \leq N_2$, and $A_{N_2}(\boldsymbol{\rho}) \geq A_{N_1}(\boldsymbol{r})$ if $N_1 \geq N_2$. Hence Theorem 4 implies that we may take $C_1^* = \nu_1^{-1} = 2^{-1}$ and

$$C_3^* = 2\nu_1^{-1}\nu_2 + \nu_1^{-2}\nu_2^2 - 2\nu_1^{-2}\nu_2 = 46.12\ldots < 47, \tag{6.1}$$

which proves Theorem 2(i).

For a monotonically decreasing sequence in general, it is not easy to go beyond the claim $u \geq 1$. However, in the case of the Riemann zeta-function $\zeta(s)$, we can improve the value of C_3^*. In this case, $r_n = \varrho_n = p_n^{-\sigma}$ (p_n denotes the nth prime number, $2^{-1} < \sigma < 1$), hence by using partial summation, we have

$$A_N(\boldsymbol{r}) = (2\sigma - 1)^{-1} N^{1-2\sigma} (\log N)^{-2\sigma} (1 + o(1))$$

and

$$B_N(\boldsymbol{r}) = (1 - \sigma)^{-1} N^{1-\sigma} (\log N)^{-\sigma} (1 + o(1)).$$

Hence

$$\begin{aligned} f_{\mathsf{r},\boldsymbol{\rho}}(N_1, N_2; 1) &= (N_2/N_1)^{1-\sigma} (\log N_1/\log N_2)^{\sigma} (1 + o(1)) \\ &\quad + (N_2/N_1)^{1-2\sigma} (\log N_1/\log N_2)^{2\sigma} (1 + o(1)). \end{aligned}$$

If $N_1 \geq N_2$, then

$$f_{\mathsf{r},\boldsymbol{\rho}}(N_1, N_2; 1) \geq \phi(N_2/N_1)(1 - \varepsilon), \tag{6.2}$$

where $\phi(x) = x^{1-\sigma} + x^{1-2\sigma}$. If $N_1 < N_2$, then we put $\alpha = N_2/N_1$ to obtain

$$\begin{aligned} f_{\mathsf{r},\boldsymbol{\rho}}(N_1, N_2; 1) &= \alpha^{1-\sigma} (\log N_1/(\log N_1 + \log \alpha))^{\sigma} (1 + o(1)) \\ &\quad + \alpha^{1-2\sigma} (\log N_1/(\log N_1 + \log \alpha))^{2\sigma} (1 + o(1)). \end{aligned} \tag{6.3}$$

To consider the lower bound of $f_{\mathsf{r},\boldsymbol{\rho}}(N_1, N_2; 1)$ for large N_1 and N_2, it is sufficient to restrict ourselves to the case $\log \alpha \leq \eta \log N_1$ for any small $\eta > 0$, because otherwise the first term on the right-hand side of (6.3) tends to infinity as $N_1 \to \infty$. Hence we may assume (6.2) also for the case $N_1 < N_2$. The function $\phi(x)$ attains its minimum

$$K(\sigma) = \left(\frac{2\sigma - 1}{1 - \sigma}\right)^{(1-\sigma)/\sigma} + \left(\frac{2\sigma - 1}{1 - \sigma}\right)^{(1-2\sigma)/\sigma}$$

at $x = \left(\frac{2\sigma-1}{1-\sigma}\right)^{1/\sigma}$. Therefore Theorem 4 can be applied to obtain Theorem 2(ii).

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Received on 13.4.1994
and in revised form on 26.7.1994 (2594)