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On a multiplicative hybrid problem

by

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1. Main results. In [2], H. Iwaniec and A. Sárközy considered the following multiplicative hybrid problem. Let N be a natural number large enough, G_1 and G_2 be subsets of $\{N+1,\ldots,2N\}$, $|G_1| \gg N, |G_2| \gg N$. They proved that there exist integers n_1, n_2, b with $n_1 \in G_1$, $n_2 \in G_2$, and

$$n_1 n_2 = b^2 + O(b^{1/2}\log^{1/2} b). \tag{1}$$

In this paper, we consider a more general case. Let $k \geq 3$ be a fixed integer and N be an integer large enough. $G_1,\ldots,G_k$ are k subsets of $\{N+1,\ldots,2N\}$. Suppose Δ is a real number satisfying $0 < \Delta \leq 1/2$. Let S_k denote the number of solutions to the inequality

$$\|(n_1\ldots n_k)^{1/k}\| \leq \Delta, \quad n_1 \in G_1,\ldots,n_k \in G_k.$$

We shall estimate S_k. Our main result is

Theorem 1. *If $T_k = |G_1|\ldots|G_k|$, then*

$$S_k = 2\Delta T_k + O(T_k^{1/2} N^{(k-1)/2}\log^{(k-2)/2} N) \tag{2}$$

for $k \geq 4$ and

$$S_3 = 2\Delta T_3 + O(N^{5/2}\log^{1/2} N). \tag{3}$$

The constant implied in (2) *depends on k.*

As an application of Theorem 1 we have immediately

Theorem 2. *If $|G_1| \gg N,\ldots,|G_k| \gg N$, then there exist integers $n_1,\ldots,n_k$, b with $n_1 \in G_1,\ldots,n_k \in G_k$ such that*

$$n_1\ldots n_k = b^k + O(b^{k-3/2}\log^{(k-2)/2} b) \quad (k \geq 3). \tag{4}$$

The constant implied in (4) *depends on k.*

Remark. Theorem 2 of [1] implies our Theorem 2 with a weak log factor in the error term for $k \geq 4$. So we use a little different method to get a slightly better result.

Notations. Throughout this paper, $\|x\| = \min\{|x-n| \mid n \text{ is an integer}\}$, $\{x\}$ means the fractional part of $x, b(x) = \{x\} - 1/2$ and $e(x) = \exp(2\pi i x)$. $|G|$ stands for the number of elements of G. As usual, $d_t(n)$ denotes the number of ways n can be written as a product of t factors.

2. Some lemmas. To complete our proof we need some lemmas.

LEMMA 1 (Lemma 1 of [2]). *Let A and B be two finite sets of real numbers, $A \subset [-X, X]$, $B \subset [-Y, Y]$. Then for any complex functions $u(x)$ and $v(y)$, we have*

$$\Big|\sum_{x\in A}\sum_{y\in B} u(x)v(y)e(xy)\Big|^2 \le 2\pi^2(1+XY)\sum_{x\in A}\sum_{\substack{x'\in A\\ 2Y|x-x'|\le 1}} |u(x)u(x')| \sum_{y\in B}\sum_{\substack{y'\in B\\ 2X|y-y'|\le 1}} |v(y)v(y')|.$$

LEMMA 2 (Theorem 12.2 of [3]). *Suppose $t \ge 2$ is an integer and write*

$$D_t(x) = \sum_{n\le x} d_t(n) = xP_t(\log x) + \Delta_t(x),$$

where $P_t(u)$ is a polynomial of degree $t-1$ in u. Then $\Delta_t(x) = O(x^{\theta_t})$ with $\theta_t = (t-1)/(t+1)$.

LEMMA 3. *Suppose t is an integer, $2 \le t \le k$, $Q_1, \ldots, Q_t$ are t subsets of $\{N+1, \ldots, 2N\}$, δ is a real number, $0 < \delta < 1$. Let $A(Q_1, \ldots, Q_t; \delta)$ be the number of solutions to the inequality*

$$|(n_1 \ldots n_t)^{1/k} - (\overline{n}_1 \ldots \overline{n}_t)^{1/k}| \le \delta, \quad n_i, \overline{n}_i \in Q_i,\ i = 1, \ldots, t. \tag{5}$$

Then

$$A(Q_1, \ldots, Q_t; \delta) \ll (N^{t\theta_t} + \delta N^{t(k-1)/k} \log^{k-1} N)|Q_1| \ldots |Q_t|.$$

Proof. The inequality (5) implies

$$|(n_1 \ldots n_t) - (\overline{n}_1 \ldots \overline{n}_t)| \le \delta k (2N)^{t(k-1)/k}, \quad n_i, \overline{n}_i \in Q_i,\ i = 1, \ldots, t. \tag{6}$$

For any fixed $(\overline{n}_1, \ldots, \overline{n}_t)$, the number of solutions of (6) is

$$S(\overline{n}_1, \ldots, \overline{n}_t) \le D_t(\overline{n}_1 \ldots \overline{n}_t + \delta k(2N)^{t(k-1)/k}) - D_t(\overline{n}_1 \ldots \overline{n}_t - \delta k(2N)^{t(k-1)/k}). \tag{7}$$

For simplicity, we put $x_0 = \overline{n}_1 \ldots \overline{n}_t$, $y_0 = \delta k(2N)^{t(k-1)/k}$. Then by Lemma 2 we get

$$(8)\quad S(\overline{n}_1,\ldots,\overline{n}_t)$$
$$\ll (x_0+y_0)P_t(\log(x_0+y_0))-(x_0-y_0)P_t(\log(x_0-y_0))+O((x_0+y_0)^{\theta_t})$$
$$=2y_0(P_t(\log x')+P_t'(\log x'))+O((x_0+y_0)\theta_t)$$
$$\ll \delta N^{t(k-1)/k}\log^{t-1}N+N^{t\theta_t}.$$

So we get

$$A(Q_1,\ldots,Q_t;\delta)\le \sum_{\overline{n}_i\in Q_i} S(\overline{n}_1,\ldots,\overline{n}_t)$$
$$\ll (N^{t\theta_t}+\delta N^{t(k-1)/k}\log^{t-1}N)|Q_1|\ldots|Q_t|.$$

This completes the proof of Lemma 3.

LEMMA 4. *Suppose k_1, k_2, N_1, N_2 are natural numbers, $\delta>0$, and Q_1 and Q_2 are subsets of $\{N_1,\ldots,2N_1\}$ and $\{N_2,\ldots,2N_2\}$ respectively. Let A denote the number of solutions to the inequality*

$$|n_1^{1/k_1}n_2^{1/k_2}-\overline{n}_1^{1/k_1}\overline{n}_2^{1/k_2}|\le\delta,\qquad n_1,\overline{n}_1\in Q_1,\ n_2,\overline{n}_2\in Q_2.$$

Then

$$A\ll N_1N_2\log N_2+\delta N_1^{2-1/k_1}N_2^{2-1/k_2}. \tag{9}$$

Proof. The idea of the proof of Lemma 4 comes from [2]. Given $r\le 2N_1$ and $s\le 2N_2$ let V_{rs} stand for the number of solutions to

$$|n_1^{1/k_1}n_2^{1/k_2}-\overline{n}_1^{1/k_1}\overline{n}_2^{1/k_2}|\le\delta \tag{10}$$

in $n_1,\overline{n}_1\in Q_1$, $n_2,\overline{n}_2\in Q_2$ such that $(n_1,\overline{n}_1)=r$ and $(n_2,\overline{n}_2)=s$.

By (10) we get

$$\left|\frac{n_1}{\overline{n}_1}-\left(\frac{\overline{n}_2}{n_2}\right)^{k_1/k_2}\right|\ll \delta N_1^{-1/k_1}N_2^{-1/k_2}. \tag{11}$$

Since the points $n_1/\overline{n}_1$ are $\left(\frac{r}{2N_1}\right)^2$-spaced, we get

$$V_{rs}\ll (1+\delta N_1^{-1/k_1}N_2^{-1/k_2}N_1^2r^{-2})T_s^2 \tag{12}$$

by the Dirichlet box principle, where

$$T_s=|\{n_2\in Q_2\mid n_2\equiv 0 \pmod s\}|\ll N_2/s.$$

Thus we obtain

$$V_{rs}\ll N_2^2s^{-2}+\delta N_1^{2-1/k_1}N_2^{2-1/k_2}r^{-2}s^{-2}. \tag{13}$$

Similarly, we have

$$V_{rs}\ll N_1^2r^{-2}+\delta N_1^{2-1/k_1}N_2^{2-1/k_2}r^{-2}s^{-2}. \tag{14}$$

So

$$V_{rs}\ll \min(N_2^2/s^2,N_1^2/r^2)+\delta N_1^{2-1/k_1}N_2^{2-1/k_2}r^{-2}s^{-2}. \tag{15}$$

Summing over r and s we complete the proof.

3. Proof of Theorem 1 ($k \ge 4$). It is easy to check that

$$[n^{1/k}+\varDelta]-[n^{1/k}-\varDelta]=\begin{cases}1, & \|n^{1/k}\|\le \varDelta,\\ 0, & \text{otherwise.}\end{cases}$$

So we have

$$S_k=\sum_{n_i\in G_i}([(n_1\ldots n_k)^{1/k}+\varDelta]-[(n_1\ldots n_k)^{1/k}-\varDelta]) \tag{16}$$
$$=2\varDelta T_k+\sum_{n_i\in G_i}(b((n_1\ldots n_k)^{1/k}-\varDelta)-b((n_1\ldots n_k)^{1/k}+\varDelta)).$$

It is well known that

$$b(t)=-\sum_{0<|h|\le H}\frac{e(ht)}{2\pi ih}+O\Big(\min\Big(1,\frac{1}{H\|t\|}\Big)\Big) \tag{17}$$

and

$$\min\Big(1,\frac{1}{H\|t\|}\Big)=\sum_{h=-\infty}^{\infty}a_h e(ht) \tag{18}$$

with

$$a_0\ll\frac{\log H}{H},\qquad a_h\ll\min\Big(\frac{1}{|h|},\frac{H}{h^2}\Big)\qquad \text{if } h\ne 0. \tag{19}$$

Using (17)–(19), we have

$$\sum_{n_i\in G_i}b((n_1\ldots n_k)^{1/k}\pm\varDelta) \tag{20}$$
$$=-\sum_{0<|h|\le H}\frac{1}{2\pi ih}\sum_{n_i\in G_i}e(h(n_1\ldots n_k)^{1/k}\pm h\varDelta)$$
$$+O\Big(\sum_{n_i\in G_i}\min\Big(1,\frac{1}{H\|(n_1\ldots n_k)^{1/k}\pm\varDelta\|}\Big)\Big)$$
$$\ll\frac{T_k\log H}{H}+\sum_{h=1}^{\infty}\min\Big(\frac{1}{h},\frac{H}{h^2}\Big)\Big|\sum_{n_i\in G_i}e(h(n_1\ldots n_k)^{1/k})\Big|.$$

So the problem is now reduced to the estimation of the exponential sum

$$\varSigma(h)=\sum_{n_i\in G_i}e(h(n_1\ldots n_k)^{1/k}).$$

Now suppose $k\ge 4$ and $t=[k/2]$, then $t\ge 2$. Applying Lemma 1 to the sequences $A=\{h(n_1\ldots n_t)^{1/k}\mid n_i\in G_i,\ i=1,\ldots,t\}$ and $B=\{(n_{t+1}\ldots n_k)^{1/k}\mid n_i\in G_i,\ i=t+1,\ldots,k\}$, we get

$$\varSigma(h)\ll(hNV_1V_2)^{1/2}, \tag{21}$$

where $V_1 = A(G_1, \ldots, G_t; 2^{-1}(2N)^{-(k-t)/k}h^{-1})$ and $V_2 = A(G_{t+1}, \ldots, G_k; 2^{-1}(2N)^{-t/k}h^{-1})$. V_1 and V_2 can then be estimated by Lemma 3. We have

$$V_1 \ll (N^{t\theta_t} + h^{-1}N^{t-1}\log^{t-1} N)|G_1| \ldots |G_t| \tag{22}$$

and

$$V_2 \ll (N^{(k-t)\theta_{k-t}} + h^{-1}N^{k-t-1}\log^{k-t-1} N)|G_{t+1}| \ldots |G_k|. \tag{23}$$

Combining (20)–(23), we get

$$\begin{aligned}(24)\qquad &\sum_{n_i \in G_i} b((n_1 \ldots n_k)^{1/k} \pm \varDelta)\\ &\ll \frac{T_k \log H}{H} + H^{1/2}T_k^{1/2}N^{(1+t\theta_t+(k-t)\theta_{k-t})/2} + T_k^{1/2}N^{(k-1)/2}\log^{(k-2)/2} N.\end{aligned}$$

Choosing H such that the first two terms in (24) are equal, we get

$$\begin{aligned}(25)\qquad &\sum_{n_i \in G_i} b((n_1 \ldots n_k)^{1/k} \pm \varDelta)\\ &\ll T_k^{2/3}N^{(1+t\theta_t+(k-t)\theta_{k-t})/3}\log^{1/3} N + T_k^{1/2}N^{(k-1)/2}\log^{(k-2)/2} N\\ &\ll T_k^{1/2}N^{(k-1)/2}\log^{(k-2)/2} N.\end{aligned}$$

Hence Theorem 1 for the case $k \ge 4$ follows from (16) and (25).

4. Proof of Theorem 1 ($k = 3$). Choosing $H = N/\log N$, we have

$$\begin{aligned}(26)\qquad &\sum_{n_i \in G_i} b((n_1n_2n_3)^{1/3} \pm \varDelta)\\ &= -\sum_{0<|h|\le H} \frac{1}{2\pi ih} \sum_{n_i \in G_i} e(h(n_1n_2n_3)^{1/3} \pm h\varDelta)\\ &\quad + O\Big(\sum_{n_i \in G_i} \min\Big(1, \frac{1}{H\|(n_1n_2n_3)^{1/3} \pm \varDelta\|}\Big)\Big)\\ &\ll \frac{T_3 \log H}{H} + S_1 + S_2,\end{aligned}$$

where

$$S_1 = \Big|\sum_{h \le H} \frac{1}{h} \sum_{n_i \in G_i} e(h(n_1n_2n_3)^{1/3} \pm h\varDelta)\Big|$$

and

$$S_2 = \Big|\sum_{h \le H^2} a_h \sum_{n_i \in G_i} e(h(n_1n_2n_3)^{1/3} \pm h\varDelta)\Big|.$$

We only estimate S_2 and we can estimate S_1 in the same way.

We have

$$\sum_{h\le H^2} a_h \sum_{n_i\in G_i} e(h(n_1n_2n_3)^{1/3} \pm h\Delta) \tag{27}$$
$$= \sum_{L=2^l\ge 1} \sum_{\substack{L\le h<2L\\ n_i\in G_i}} a_h e(h(n_1n_2n_3)^{1/3} \pm h\Delta)$$
$$\ll \sum_{L=2^l} c(L)\Bigg| \sum_{\substack{L\le h<2L\\ n_i\in G_i}} \frac{a_h e(\pm h\Delta)}{c(L)} e(h(n_1n_2n_3)^{1/3})\Bigg|,$$

where $c(h) = \min(1/h, H/h^2)$.

Now we only need to estimate

$$\Sigma(L) = \sum_{\substack{L\le h<2L\\ n_i\in G_i}} \frac{a_h e(\pm h\Delta)}{c(L)} e(h(n_1n_2n_3)^{1/3}).$$

Applying Lemma 1 to the sequences $A = \{hn_1^{1/3} \mid L \le h < 2L,\ n_1 \in G_1\}$ and $B = \{(n_2n_3)^{1/3} \mid n_2 \in G_2,\ n_3 \in G_3\}$, we get

$$\Sigma(L) \ll (LNV_3V_4)^{1/2}, \tag{28}$$

where

$$V_3 = \sum_{\substack{x,x'\in A\\ 2(2N)^{2/3}|x-x'|\le 1}} 1 \quad\text{and}\quad V_4 = \sum_{\substack{y,y'\in B\\ 2L(2N)^{1/3}|y-y'|\le 1}} 1.$$

By Lemma 4 we have

$$V_3 \ll LN\log N \tag{29}$$

and

$$V_4 \ll N^2\log N + N^3/L. \tag{30}$$

Combining (27)–(30), we obtain

$$S_2 \ll N^{5/2}\log^{1/2} N. \tag{31}$$

Similarly,

$$S_1 \ll N^{5/2}\log^{1/2} N. \tag{32}$$

Hence Theorem 1 for the case $k = 3$ follows from (16), (26), (31) and (32).

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