Algebraic independence of the values of generalized Mahler functions

by

THOMAS TÖPFER (Köln)

1. Introduction and results. In the last years arithmetic properties of holomorphic functions were studied which satisfy a functional equation of the shape

\[(1)\] \[P(z, f(z), f(T(z))) = 0,\]

where \(P(z, u, w)\) is a polynomial with coefficients in \(\overline{\mathbb{Q}}\), the field of all algebraic numbers, and \(T(z)\) is an algebraic function. This generalizes investigations of Mahler [M1], [M2], [M3], which dealt with functional equations of the form

\[(2)\] \[f(z^d) = R(z, f(z))\]

with \(d \in \mathbb{N}, d \geq 2\), and a rational function \(R(z, u)\) (resp. generalizations of these functional equations to several variables and several functions). Certain cases of (1) were studied extensively by different authors. For a survey of results about the transformations considered by Mahler see [M4], [K1], [L], [LP]. If \(T(z)\) is a polynomial, the transcendence of \(f(\alpha)\) for algebraic \(\alpha\) was proved by Nishioka [Ni1]. This was generalized to algebraic functions \(T(z)\) by Becker in [B3]. Applications to Böttcher functions were given by Becker and Bergweiler [BB], and transcendence measures for these functions can be found in [B4] (see also [NT]). The algebraic independence of several values \(f_1(\alpha), \ldots, f_m(\alpha)\) was proved by Becker [B2] for certain rational transformations \(T(z)\) under additional technical assumptions.

Since a general zero order estimate for functions satisfying (2) with \(z^d\) replaced by rational functions \(T(z)\) was proved in [T3], we will give an application of the zero order estimate in this paper and derive measures for the algebraic independence of the values of the functions considered by Becker in [B2]. Furthermore we give lower bounds for the transcendence degree of \(\mathbb{Q}(f_1(\alpha), \ldots, f_m(\alpha))\) over \(\mathbb{Q}\), if \(f_1, \ldots, f_m\) satisfy functional equations with more general rational transformations \(T(z)\).

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Theorem 1. Let $f_1, \ldots, f_m : U \to \mathbb{C}$ be holomorphic in a neighborhood $U$ of $\omega \in \mathbb{C}$, algebraically independent over $\mathbb{C}(z)$, and suppose the power series coefficients of $f_1, \ldots, f_m$ in the expansion at $\omega$ are algebraic. Suppose that $T(z) = T_1(z)/T_2(z)$ with $T_1, T_2 \in \mathbb{Q}[z]$, $\deg T = \max\{\deg T_1, \deg T_2\} = d \geq 2$, $\omega$ is a fixed point of $T$ of order $\ord_{\omega} T = d$, and $f = (f_1, \ldots, f_m)$ satisfies the functional equation
\begin{equation}
 a(z)f(z) = A(z)f(T(z)) + B(z),
\end{equation}
where $A(z)$ is a regular $m \times m$ matrix with entries in $\mathbb{Q}[z]$, $B(z) \in (\mathbb{Q}[z])^m$, and $a(z) \in \mathbb{Q}[z]$. Let $\alpha \in U$ be an algebraic number with $\lim_{k \to \infty} T^k(\alpha) = \omega$, where $T^k(\alpha)$ denotes the $k$-th iterate of $T$ at $\alpha$, and suppose for $k \in \mathbb{N}_0$ that $T^k(\alpha) \in U \setminus \{\omega, \infty\}$, and $T^k(\alpha)$ is neither a zero of $a(z)$ nor a zero of $\det A(z)$. Then for each polynomial $Q \in \mathbb{Z}[y_1, \ldots, y_m] \setminus \{0\}$ with $\deg Q \leq D$, where $\deg Q$ denotes the total degree of $Q$ in all variables, and $H(Q) \leq H$, where $H(Q)$ denotes the height of $Q$, i.e. the maximum of the moduli of the coefficients of $Q$, the inequality
\begin{align*}
 |Q(f(\alpha))| > \exp(-c_1 D^m (D^{m+2} + \log H))
\end{align*}
holds with a constant $c_1 \in \mathbb{R}_+$ depending only on $f$ and $\alpha$.

Remarks. (i) For $\omega = 0$, $T(z) = p(z^{-1})^{-1}$ with a polynomial $p \in \mathbb{Q}[z]$, and a diagonal matrix $A(z)$, Theorem 1 is the quantitative analogue of the theorem in [B2], where the algebraic independence of the function values under consideration was proved.

(ii) With $T(z) = z^d$, $d \in \mathbb{N}$, $d \geq 2$, and $\omega = 0$, Theorem 1 includes an earlier result of Becker (Theorem 1 in [B1]) and the improvement of Nishioka (Theorem 1 in [Ni2]).

Theorem 2. Let $f_1, \ldots, f_m : U \to \mathbb{C}$ be holomorphic in a neighborhood $U$ of $\omega \in \mathbb{C}$, algebraically independent over $\mathbb{C}(z)$, and suppose the power series coefficients of $f_1, \ldots, f_m$ in the expansion at $\omega$ are algebraic. Suppose that $T(z) = T_1(z)/T_2(z)$ with $T_1, T_2 \in \mathbb{Q}[z]$, $\deg T = d$, $\omega$ is a fixed point of $T$ with $\ord_{\omega} T = \delta \geq 2$, and $f = (f_1, \ldots, f_m)$ satisfies
\begin{equation}
 a(z)f(z) = A(z)f(T(z)) + B(z),
\end{equation}
where $A(z)$ is a regular $m \times m$ matrix with entries in $\mathbb{Q}[z]$, $B(z) \in (\mathbb{Q}[z])^m$, and $a(z) \in \mathbb{Q}[z]$. Let $\alpha \in U$ be an algebraic number with $\lim_{k \to \infty} T^k(\alpha) = \omega$, and suppose for $k \in \mathbb{N}_0$ that $T^k(\alpha) \in U \setminus \{\omega, \infty\}$, and $T^k(\alpha)$ is neither a zero of $a(z)$ nor a zero of $\det A(z)$. Let $m_0$ be the greatest integer satisfying
\begin{align*}
 m_0 < \min \left( \frac{2 \log \delta}{\log d} - 1 \right) + \frac{\log \delta}{\log d}.
\end{align*}
Then
\begin{align*}
 \trdeg_{Q} Q(f(\alpha)) \geq m_0.
\end{align*}
Corollary 1. Suppose the assumptions of Theorem 2 are fulfilled with $d < \delta^{1+1/2m}$. Then $f_1(\alpha), \ldots, f_m(\alpha)$ are algebraically independent. In particular, for $m = 1$ and $d < \delta^{3/2}$ we have $f(\alpha) \notin \mathbb{Q}$.

Remark. The case $m = 1$ is Becker’s result in [B3] in the special case of rational transformations and the functional equation (3).

Theorem 3. Let $f_1, \ldots, f_m : U \to \mathbb{C}$ be holomorphic in a neighborhood $U$ of $\omega \in \mathbb{C}$, algebraically independent over $\mathbb{C}(z)$, and suppose $f_1(\omega), \ldots, f_m(\omega)$ are algebraic. Suppose that $T \in \mathbb{C}[z]$, $\deg T = d$, $\omega$ is a fixed point of $T$ with $\text{ord}_\omega T = \delta \geq 2$, and $f = (f_1, \ldots, f_m)$ satisfies
\begin{equation}
(4) \quad f(z) = A(z)f(T(z)) + B(z),
\end{equation}
where $A(z)$ is a regular $m \times m$ matrix with entries in $\mathbb{Q}[z]$, and $B(z) \in (\mathbb{Q}[z])^m$. Let $\alpha \in U$ be an algebraic number with $\lim_{k \to \infty} T^k(\alpha) = \omega$, and suppose for $k \in \mathbb{N}_0$ that $T^k(\alpha) \in U \setminus \{\omega\}$, and $\det A(T^k(\alpha)) \neq 0$. Let $m_0$ be the greatest integer satisfying
\[ m_0 < (m + 1) \frac{\log \delta}{\log d}. \]

Then
\[ \text{trdeg}_\mathbb{Q} \mathbb{Q}(f(\alpha)) \geq m_0. \]

Corollary 2. Suppose the assumptions of Theorem 3 are fulfilled and $d < \delta^{1+1/m}$. Then $f_1(\alpha), \ldots, f_m(\alpha)$ are algebraically independent. In particular, for $m = 1$ and $d < \delta^2$ we get $f(\alpha) \notin \mathbb{Q}$.

Remark. Since the condition $d < \delta^{3/2}$ in Corollary 1 coincides with the condition given in the theorem of Becker in [B3] in the special case of rational transformations and functional equations of type (3), the weaker condition of Corollary 2 for polynomial transformations and the more restricted functional equations of type (4) gives a first answer to a question posed by Becker (p. 119 in [B3]). He asked for weaker technical assumptions of this form to extend the range of applications of Mahler’s method.

2. Examples and applications. Our first example deals with series of the form
\[ \chi_i(z) = \sum_{h=0}^{\infty} q_i(T^h(z)) \quad (i = 1, \ldots, m), \]
where $T(z) = T_1(z)/T_2(z) \in \mathbb{Q}[z]$, $d_j = \deg T_j$ ($j = 1, 2$), $\omega \in \mathbb{C}$ is a fixed point of $T$ of order $\delta \geq 2$, $q_i \in \mathbb{Q}[z]$ with $\deg q_i \geq 1$ and $q_i(\omega) = 0$ for $i = 1, \ldots, m$. Then all $\chi_i$ are holomorphic in a neighborhood $U$ of $\omega$ and satisfy the functional equation
\[ \chi_i(z) = \chi_i(T(z)) + q_i(z) \quad (i = 1, \ldots, m). \]
Corollary 3. Suppose $q_1, \ldots, q_m$ are $\mathbb{C}$-linearly independent, $0 < d_2 < d_1 = d$, and $\alpha \in \overline{\mathbb{Q}}$ satisfies $\lim_{k \to \infty} T^k(\alpha) = \omega$ and $T^k(\alpha) \neq \omega$ for $k \in \mathbb{N}_0$. Then

$$\text{trdeg}_{\mathbb{Q}} \mathbb{Q}(\chi_1(\alpha), \ldots, \chi_m(\alpha)) \geq m_0,$$

where $m_0$ denotes the greatest integer satisfying

$$m_0 < (m + 1) \frac{\log \delta}{\log d} - \left(1 - \frac{\log \delta}{\log d}\right)m.$$

Proof. For the application of Theorem 2 we have to show that $\chi_1, \ldots, \chi_m$ are algebraically independent. In the next paragraph this will be derived from Lemma 6 of Section 3.

Suppose that $\chi_1, \ldots, \chi_m$ are algebraically dependent. By Lemma 6 there exist $g_i \in \mathbb{C}(z)$ with $\deg g_i = \gamma_i$ ($i = 1, 2$), $\gamma = \max\{\gamma_1, \gamma_2\}$, and $s_1, \ldots, s_m \in \mathbb{C}$, not all zero, such that

$$g_1(z) - g_2(T(z)) + \sum_{i=1}^{m} s_i q_i(z).$$

Since the sum on the right is nonzero, we know that $\gamma \geq 1$. From this equation we get the polynomial identity

$$g_1(z)h_2(z) = g_2(z)h_1(z) + g_2(z)h_2(z)\sum_{i=1}^{m} s_i q_i(z)$$

with $h_i(z) = T_2(z)g_i(T(z)) \in \mathbb{C}[z]$ ($i = 1, 2$). Since $g_1, g_2$ resp. $T_1, T_2$ are coprime, we see that $h_1, h_2$ are also coprime. Thus $h_2 | g_2$, and the condition $d_2 < d_1$ implies

$$\deg h_2 = (\gamma - \gamma_2)d_2 + \gamma_2d_1 \leq \gamma_2 = \deg g_2.$$

But $d_2 \geq 1$, $d_1 \geq 2$ and $\gamma \geq 1$. Hence we get a contradiction, and so $\chi_1, \ldots, \chi_m$ must be algebraically independent. Then application of Theorem 2 completes the proof. ■

Corollary 4. Suppose that $1, q_1, \ldots, q_m$ are $\mathbb{C}$-linearly independent, $T(z) \in \overline{\mathbb{Q}}[z]$ with $2 \leq \delta \leq d$, $d \nmid \deg(\sum_{i=1}^{m} s_i q_i(z))$ for arbitrary $(s_1, \ldots, s_m) \in \mathbb{C}^m \setminus \{0\}$, and $\alpha \in \overline{\mathbb{Q}}$ satisfies $\lim_{k \to \infty} T^k(\alpha) = \omega$ and $T^k(\alpha) \neq \omega$ for $k \in \mathbb{N}_0$. Then

$$\text{trdeg}_{\mathbb{Q}} \mathbb{Q}(\chi_1(\alpha), \ldots, \chi_m(\alpha)) \geq m_0,$$

where $m_0$ denotes the greatest integer satisfying

$$m_0 < (m + 1) \frac{\log \delta}{\log d}.$$

Proof. Under the assumption that $\chi_1, \ldots, \chi_m$ are algebraically dependent, we get analogously to the proof of Corollary 3 the polynomial identity
(notice that \(T_2 = 1\), hence \(h_2 = g_2\))

\[
g_1(z)g_2(T(z)) = g_2(z)g_1(T(z)) + g_2(z)g_2(T(z)) \sum_{i=1}^{m} s_i q_i(z). \tag{5}
\]

The coprimality of \(g_1, g_2\) implies \(g_2(T(z)) \mid g_2(z), \) hence \(\gamma_2 = 0\). Now we compare the degrees in (5). The degree on the left side is \(\gamma_1\), and the two terms on the right have degrees \(\gamma_1 d\) and \(\text{deg}(\sum_{i=1}^{m} s_i q_i(z)) = \Delta\), respectively. Since \(d \geq 2\), this forces \(\gamma_1 d = \Delta\). But \(\Delta\) is not divisible by \(d\) except for \(\Delta = 0\). Then \(\gamma_1 = 0\), and we get the contradiction \(\sum_{i=1}^{m} s_i q_i(z) = 0\). Therefore \(\chi_1, \ldots, \chi_m\) are algebraically independent. Now application of Theorem 3 yields the assertion. \(\blacksquare\)

**Corollary 5.** Suppose \(q_1, \ldots, q_m\) are \(\mathbb{C}\)-linearly independent, \(T(z) = T_1(z)/T_2(z) \in \overline{\mathbb{Q}}(z), 0 < d_2 < d_1 = d = \delta, \) and \(\alpha \in \overline{\mathbb{Q}}\) satisfies \(\lim_{k \to \infty} T^k(\alpha) = \omega\) and \(T^k(\alpha) \in U \setminus \{\omega\} \) for \(k \in \mathbb{N}_0\). Then for each polynomial \(Q \in \mathbb{Z}[y] \setminus \{0\}\) with \(\text{deg} Q \leq D\) and \(H(Q) \leq H\),

\[
|Q(\chi_1(\alpha), \ldots, \chi_m(\alpha))| > \exp(-c_1 D^m (D^{m+2} + \log H)).
\]

**Proof.** From the proof of Corollary 3 we know that \(\chi_1, \ldots, \chi_m\) are algebraically independent. Since \(\delta = d\), the assertion follows from Theorem 1. \(\blacksquare\)

**Remark.** The same quantitative result can be derived under the assumptions of Corollary 4 for \(\delta = d\).

Now we consider certain Cantor series introduced by Tamura [Ta]. Let

\[
\theta_i(z) = \sum_{h=0}^{\infty} \frac{1}{q_i(z) q_i(T(z)) \cdots q_i(T^h(z))} \quad (i = 1, \ldots, m) \tag{6}
\]

with \(T(z) = T_1(z)/T_2(z) \in \overline{\mathbb{Q}}(z), \) \(\deg T_j = d_j \) \((j = 1, 2)\), \(\omega \in \hat{\mathbb{C}}\) is a fixed point of \(T\) of order \(\delta \geq 2\), \(q_i \in \overline{\mathbb{Q}}[z] \) with \(\deg q_i \geq 1\) and \(|q_i(\omega)| > 1\) for \(i = 1, \ldots, m\) (notice that \(\omega = \infty\) and \(q_i(\infty) = \infty\) is possible). The functions \(\theta_i\) are holomorphic in a neighborhood of \(\omega \in \hat{\mathbb{C}}\) and satisfy the functional equation

\[
\theta_i(T(z)) = q_i(z) \theta_i(z) - 1 \quad (i = 1, \ldots, m).
\]

Tamura proved the transcendence of \(\theta(\alpha)\) for certain \(\alpha\) in the special case \(q(z) = z, T(z) \in \mathbb{Z}[z]\) and \(\deg T \geq 3\). The more general case of polynomials \(q_i, T \in \overline{\mathbb{Q}}[z] \) \((i = 1, \ldots, m)\) was treated by Becker [B2]. He derived algebraic independence results for \(\theta_1(\alpha), \ldots, \theta_m(\alpha)\) at algebraic points \(\alpha\) and discussed in detail the transcendence of \(\theta(\alpha)\) for linear polynomials \(q\) and algebraic \(\alpha\). Here we study rational transformations and give qualitative and quantitative generalizations of Becker’s results.
Corollary 6. Suppose \( q_1, \ldots, q_m \) are pairwise distinct, \( \max\{2, d_2\} < d_1 = d, 1 \leq \deg q_i < d - 1 \) for \( i = 1, \ldots, m \). Let \( \alpha \) be an algebraic number with \( \lim_{k \to \infty} T^k(\alpha) = \omega \) and \( q_i(T^k(\alpha)) \neq 0 \) for \( k \in \mathbb{N}_0 \) and \( i = 1, \ldots, m \). If \( m_0 \) is the greatest integer satisfying

\[
m_0 < (m + 1) \left( \frac{\delta}{\log 3} - \left( 1 - \frac{\log \delta}{\log d} \right) m, \right.
\]

then

\[
\trdeg Q(\theta_1(\alpha), \ldots, \theta_m(\alpha)) \geq m_0.
\]

If \( \delta = d \), then \( \theta_1(\alpha), \ldots, \theta_m(\alpha) \) are algebraically independent, and for all polynomials \( Q \in \mathbb{Z}[y] \setminus \{0\} \) with \( \deg Q \leq D \) and \( H(Q) \leq H \),

\[
|Q(\theta_1(\alpha), \ldots, \theta_m(\alpha))| > \exp(-c_1 D^m(D^{m+2} + \log H)).
\]

Proof. The assertions are obvious consequences of Theorems 1 and 2, if the algebraic independence of \( \theta_1, \ldots, \theta_m \) is verified. Thus we assume that \( \theta_1, \ldots, \theta_m \) are algebraically dependent, and apply Lemma 6. First we must show that \( q_i(z)/q_j(z) \) for \( i \neq j \) is not of the form \( q(T(z))/g(z) \) for some \( g \in \mathbb{C}(z) \). With \( g(z) = g_1(z)/g_2(z), \deg g_i = \gamma_i \) \( (i = 1, 2) \), and \( \gamma = \max\{\gamma_1, \gamma_2\} \) we suppose on the contrary that

\[
q_i(z)g_1(z)h_2(z) = q_j(z)g_2(z)h_1(z),
\]

where \( h_i(z) = T_2(z)^\gamma g_i(T(z)) \in \mathbb{C}[z] \). Since \( g_1, g_2 \) resp. \( T_1, T_2 \) are coprime, we see that \( h_1, h_2 \) are also coprime. Thus \( h_1 | q_i g_1, h_2 | q_j g_2 \), and this implies (notice that \( d_2 < d_1 \))

\[
\deg h_i = \gamma d_2 + \gamma_i (d_1 - d_2) = \gamma_i d_1 + (\gamma - \gamma_i) d_2 \leq d_1 - 2 + \gamma_i \quad (i = 1, 2).
\]

Since \( d_1 \geq 3 \), we must have \( \gamma_1 = \gamma_2 = 0 \), but this leads to the contradiction \( q_i = q_j \). Now all conditions of Lemma 6 are fulfilled, and then there exist \( i \in \{1, \ldots, m\} \) and a rational function \( g \) (with \( g_i, h_i, \gamma_i, \gamma \) as above) such that

\[
g_2(z)h_1(z) = g_2(z)h_2(z) + q_i(z)g_1(z)h_2(z).
\]

Hence \( h_2 | g_2 \), and this yields

\[
\deg h_2 = \gamma_2 d_1 + (\gamma - \gamma_2) d_2 \leq \gamma_2.
\]

But \( d_1 \geq 3 \), and so \( \gamma_2 = d_2 = 0 \). Now we compare the degrees on both sides of (7) and get \( d_1 \gamma_1 \leq \gamma_1 + d_1 - 2 \). Since \( d_1 \geq 3 \), we must have \( \gamma_1 = 0 \), but then \( q_i(z) \) is a constant, and this is excluded. Thus \( \theta_1, \ldots, \theta_m \) cannot be algebraically dependent. 

Corollary 7. Suppose that \( T \in \mathbb{Q}[z] \) is a polynomial with \( d \geq 2 \), and \( q \in \mathbb{Q}[z] \) is a linear polynomial with \( q(T(z))^2 \neq q(z)^2 - 2 \). Let \( \alpha \) be an algebraic number with \( \lim_{k \to -\infty} T^k(\alpha) = \infty \) and \( q(T^k(\alpha)) \neq 0 \) for \( k \in \mathbb{N}_0 \).
Then for each polynomial \( Q \in \mathbb{Z}[y] \setminus \{0\} \) with \( \deg Q \leq D, H(Q) \leq H \) the inequality
\[
|Q(\theta(\alpha))| > \exp(-c_1 D(D^3 + \log H))
\]
holds for \( \theta(z) \) as in (6). In particular, \( \theta(\alpha) \) is an \( S \)-number in Mahler’s classification of transcendental numbers.

**Proof.** In Corollary 2 of [B2] Becker showed that \( \theta(z) \) is a transcendental function for \( q(z), T(z) \) as above. Then Theorem 1 with \( \omega = \infty \) yields the assertion (notice that \( \deg T = d = \text{ord}_\infty T \)).

The next example deals with the series
\[
\Omega(z) = \sum_{h=0}^{\infty} \frac{(-1)^h}{q(T^h(z))}
\]
with \( q, T \in \overline{\mathbb{Q}}[z] \) and \( \deg q \geq 1, d \geq 2 \), which was introduced by Becker [B2]. Then \( \Omega(z) \) is holomorphic in a neighborhood of \( \omega = \infty \) and satisfies
\[
\Omega(T(z)) = -\Omega(z) + 1/q(z).
\]

**Corollary 8.** Suppose \( q(T(z)) \neq \lambda^{-1} q(z)^2 + q(z) - \lambda \) for any \( \lambda \in \mathbb{C} \setminus \{0\} \), and \( \alpha \) is an algebraic number with \( \lim_{k \to \infty} T^k(\alpha) = \infty \) and \( q(T^k(\alpha)) \neq 0 \) for \( k \in \mathbb{N}_0 \). Then for each \( Q \in \mathbb{Z}[y] \setminus \{0\} \) with \( \deg Q \leq D \) and \( H(Q) \leq H \),
\[
|Q(\Omega(\alpha))| > \exp(-c_1 D(D^3 + \log H))
\]
In particular, this transcendence measure is valid for Cahen’s constant
\[
C = \sum_{h=0}^{\infty} \frac{(-1)^h}{S_h - 1},
\]
where \( S_0 = 2 \) and \( S_{h+1} = S_h^2 - S_h + 1 \) for \( h \geq 0 \).

**Remark.** The transcendence of \( C \) was proved by Davison and Shallit [DS] with continued fractions and later by Becker in [B2] using the identity \( C = \Omega(2) \) for \( q(z) = z - 1, T(z) = z^2 - z + 1 \). Corollary 8 implies that \( C \) is a \( S \)-number in Mahler’s classification of transcendental numbers.

**Proof of Corollary 8.** In Corollary 3 of [B2] the transcendence of the function \( \Omega(z) \) was proved. Then Theorem 1 yields the assertion.

The last example was studied by Becker in [B3], Corollary 1. Let
\[
\sigma(z) = \prod_{h=0}^{\infty} q(T^h(z)),
\]
where \( q \in \overline{\mathbb{Q}}[z], \deg q \geq 1, \) and \( T(z) = T_1(z)/T_2(z) \in \overline{\mathbb{Q}}(z), \deg T_i = d, \) \( (i = 1, 2) \), and \( \omega \in \hat{\mathbb{C}} \) is a fixed point of \( T \) of order \( \delta \). Assume that \( q(\omega) = 1 \).
Then \( \sigma(z) \) is holomorphic in a neighborhood of \( \omega \) and satisfies the functional equation

\[
\sigma(z) = q(z)\sigma(T(z)).
\]

**Corollary 9.** Suppose \( 0 < d_2 < d_1 = \delta, \) and \( \alpha \) is an algebraic number with \( \lim_{k \to \infty} T^k(\alpha) = \omega \) and \( q(T^k(\alpha)) \neq 0, T^k(\alpha) \neq \omega, \infty \) for \( k \in \mathbb{N}_0. \) Then for any polynomial \( Q \in \mathbb{Z}[y] \setminus \{0\} \) with \( \deg Q \leq D, H(Q) \leq H, \)

\[
|Q(\sigma(\alpha))| > \exp(-c_1D(D^3 + \log H)).
\]

**Proof.** The transcendence of \( \sigma(z) \) was proved in Corollary 1 of [B3]. Then the assertion follows from Theorem 1. ■

**3. Preliminaries and auxiliary results.** Throughout the paper let \( K \) denote an algebraic number field, and \( \mathcal{O}_K \) is the ring of integers in \( K. \) Define \( \alpha, \) the *house* of the algebraic number \( \alpha, \) as the maximum of the moduli of the conjugates of \( \alpha. \) A *denominator* of an algebraic number \( \alpha \) is a positive integer \( d \) such that \( d\alpha \in \mathcal{O}_K. \) For a polynomial \( P \) with algebraic coefficients the *height* \( H(P) \) is defined as the maximum of the houses of the coefficients, and the *length* \( L(P) \) is the sum of the houses of the coefficients.

**Lemma 1.** Suppose the rational function \( g(z) = r(z)/s(z) \in K(z) \) is holomorphic in a neighborhood of \( z = 0. \) Then for each \( h \in \mathbb{N}_0 \) the power series coefficients \( g_h \) of

\[
g(z) = \sum_{h=0}^{\infty} g_h z^h
\]
satisfy

(i) \( g_h \in K(g_0), \)

(ii) \( |g_h| \leq \exp(c_2(h + 1)), \)

(iii) \( D^{c_2(h+1)}|g_h| \in \mathcal{O}_K \)

with suitable \( D \in \mathbb{N} \) and \( c_2 \in \mathbb{R}_+ \) depending only on \( g. \)

**Proof.** From \( r(z) = s(z) \sum_{h=0}^{\infty} g_h z^h \) with \( r(z) = \sum_{i=0}^{l} r_i z^i, \) \( s(z) = \sum_{i=0}^{l} s_i z^i \) we get the following recurrence relation for the coefficients \( g_h \) (with \( r_h = 0 \) for \( h > l \)), \( h \in \mathbb{N}_0: \)

\[
g_h = \frac{r_h}{s_0} - \sum_{\mu=1}^{\min\{l,h\}} \frac{s_\mu}{s_0} g_{h-\mu}.
\]

This implies the assertion. ■
LEMMA 2. Suppose \( T(z) = T_1(z)/T_2(z) \) is a rational function with \( \delta = \text{ord}_0 T \geq 2 \), and \( \alpha \in \mathbb{C} \) satisfies \( T^k(\alpha) \neq 0 \) for \( k \in \mathbb{N}_0 \) and \( \lim_{k \to \infty} T^k(\alpha) = 0 \). Then for all \( k \geq \overline{k} \),
\[
-c_3 \delta^k \leq \log |T^k(\alpha)| \leq -c_4 \delta^k
\]
with \( c_3, c_4 \in \mathbb{R}_+ \), \( \overline{k} \in \mathbb{N} \) depending on \( T \) and \( \alpha \).

Proof. Since 0 is a zero of \( T \) of order \( \delta \geq 2 \), we have \( T(z) = z^\delta g(z) \), where \( g(z) \) is holomorphic in a neighborhood of \( z = 0 \) and \( g(0) \neq 0 \). Then there exists a constant \( \varepsilon \in \mathbb{R}_+ \) depending only on \( T \) such that for all \( \beta \in \mathbb{C} \) with \( 0 < |\beta| < \varepsilon \) (< 1),
\[
\gamma_0 |\beta|^\delta \leq |T(\beta)| \leq \gamma_1 |\beta|^\delta,
\]
where \( \gamma_0, \gamma_1 \in \mathbb{R}_+ \) depend on \( T \). Thus
\[
\exp(-\gamma_2 \delta^k) \leq \gamma_0 |\beta|^\delta^k \leq |T^k(\beta)| \leq \gamma_1 |\beta|^\delta^k \leq \exp(-\gamma_3 \delta^k)
\]
with \( \gamma_2, \gamma_3 \in \mathbb{R}_+ \) depending on \( T \) and \( \beta \). Since \( \lim_{k \to \infty} T^k(\alpha) = 0 \), we know \( 0 < |T^k(\alpha)| < \varepsilon \) for \( k \geq \overline{k} \) with \( \overline{k} \in \mathbb{N} \) depending on \( T \) and \( \alpha \), and together with (8) this yields the assertion. \[ \square \]

The proofs of the theorems depend on the following results from elimination theory.

LEMMA 3. Suppose \( \omega \in \mathbb{C}^m \). Then there exists a constant \( c_5 = c_5(\omega, K) \in \mathbb{R}_+ \) with the following property: If there exist increasing functions \( \Psi_1, \Psi_2 : \mathbb{N} \to \mathbb{R}_+ \), numbers \( \Phi_1, \Phi_2, \Lambda \in \mathbb{R}_+ \), positive integers \( k_0, k_1 \) with \( k_0 < k_1 \), \( m_0 \in \{0, \ldots, m\} \) and polynomials \( (Q_k)_{k_0 \leq k \leq k_1} \), such that the following assumptions are satisfied:

(i) \( \Phi_2 \geq \Phi_1 \geq c_5, \Lambda \geq \Psi_1(k+1)/\Psi_2(k) \geq 1 \) for \( k \in \{k_0, \ldots, k_1\} \),
(ii) \( \Psi_2(k) \geq c_5(\log H(Q_k) + \deg Q_k) \) for \( k \in \{k_0, \ldots, k_1\} \),
(iii) the polynomials \( Q_k \in O_k[y_1, \ldots, y_m] \) \( (k_0 \leq k \leq k_1) \) satisfy
   (a) \( \deg Q_k \leq \Phi_1 \),
   (b) \( \log H(Q_k) \leq \Phi_2 \),
   (c) \( \exp(-\Psi_1(k)) \leq |Q_k(\omega)| \leq \exp(-\Psi_2(k)) \),
(iv) \( \Psi_2(k_1) \geq c_5 \Lambda^{m_0-1} \Phi_1^{m_0-1} \max\{\Psi_1(k_0), \Phi_2\} \),

then
\[
\text{trdeg}_Q Q(\omega) \geq m_0.
\]

Proof. This is Theorem 1 in [T1] with slight modifications. \[ \square \]

LEMMA 4. Suppose \( \omega \in \mathbb{C}^m \). Then there exists a constant \( c_6 = c_6(\omega, K) \in \mathbb{R}_+ \) with the following property: If there exist functions \( \Psi_1, \Psi_2 : \mathbb{N}^2 \to \mathbb{R}_+ \), which are increasing in the first variable, numbers \( \Phi_1, \Phi_2, \Lambda, U, \tau \in \mathbb{R}_+ \), positive integers \( N_0, N_1 \) with \( N_0 \leq N_1 \), for each \( N \in \{N_0, \ldots, N_1\} \) positive integers \( k_0(N), k_1(N) \) with \( k_0(N) \leq k_1(N) \), and polynomials \( Q_{k,N} \) for
N \in \{N_0, \ldots, N_1\} and k \in \{k_0(N), \ldots, k_1(N)\}$, such that the following assumptions are satisfied for positive integers $D, H$ and all $N \in \{N_0, \ldots, N_1\}$, $k \in \{k_0(N), \ldots, k_1(N)\}$:

(i) 
\begin{align*}
&\text{(a) } \Phi_2 \geq \Phi_1 \geq c_6, \quad A \geq \Psi_1(k + 1, N)/\Psi_2(k, N) \geq 1, \\
&\text{(b) } \Psi_1(k_1(N), N) \geq \Psi_1(k_0(N + 1), N + 1), \\
&\text{(c) } U \leq \max\{\Psi_2(k, N) \mid N_0 \leq N \leq N_1, k_0(N) \leq k \leq k_1(N)\}, \\
&\tau \geq \min\{\Psi_1(k, N) \mid N_0 \leq N \leq N_1, k_0(N) \leq k \leq k_1(N)\},
\end{align*}

(ii) 
$\Psi_2(k, N) \geq c_6(\log H(Q_k,N) + \deg Q_k,N)$.

(iii) 
the polynomials $Q_{k,N} \in O_K[y_1, \ldots, y_m]$ satisfy
\begin{align*}
\text{(a) } &\deg Q_{k,N} \leq \Phi_1, \\
\text{(b) } &\log H(Q_{k,N}) \leq \Phi_2, \\
\text{(c) } &\exp(-\Psi_1(k, N)) \leq |Q_{k,N}(\omega)| \leq \exp(-\Psi_2(k, N)), \\
\text{(iv) } &U \geq c_6 A^{m-1} \Phi_1^{n-1} \max\{\tau D, A(\Phi_1 \log H + \Phi_2 D)\},
\end{align*}

then for all polynomials $R \in Z[y_1, \ldots, y_m] \setminus \{0\}$ with $\deg R \leq D, H(R) \leq H$,

\[ |R(\omega)| \geq \exp(-U). \]

Proof. Lemma 4 can be derived from Jabbouri's criterion [J] analogous to the proof of the proposition in [T2].

Lemma 5. Let $f_1, \ldots, f_m \in \mathbb{C}[[z]]$ be formal power series which satisfy

\[ A_0(z, f(z))f(T(z)) = A(z, f(z)), \]

where $f(z) = (f_1(z), \ldots, f_m(z))$, $T(z) = T_1(z)/T_2(z)$ is a rational function with $T_1, T_2 \in \mathbb{C}[z]$, $d = \max\{\deg T_1, \deg T_2\}$, $\delta = \ord_0 T \geq 2$, $A(z, y) = (A_1(z, y), \ldots, A_m(z, y))$, and $A_i(z, y) \in \mathbb{C}[z, y_1, \ldots, y_m] \setminus \{0\}$ ($0 \leq i \leq m$) are polynomials with $\deg_z A_i \leq s$ and $\deg_{y_1, \ldots, y_m} A_i \leq t$. Suppose that $t^m < \delta$ and $Q \in \mathbb{C}[z, y_1, \ldots, y_m]$ with $\deg_z Q \leq M$, $\deg_{y_1, \ldots, y_m} Q \leq N$ and $M \geq N \geq 1$. If $Q(z, f(z)) \neq 0$, then

\[ \ord_0 Q(z, f(z)) \leq c_7 M N^m \log d / (\log \delta - m \log t) \]

with a constant $c_7 \in \mathbb{R}_+$ depending on $f$.

Proof. See Theorem 1 in [T3].

The following result of Kubota is often useful to verify the algebraic independence of the functions $f_1, \ldots, f_m$.

Lemma 6. Suppose $f_{i,j} \in \mathbb{C}[[z]]$ ($1 \leq i \leq m, 1 \leq j \leq n(i)$) are formal power series satisfying the functional equations

\[ f_{i,j}(z) = a_{i,j} f_{i,j}(T(z)) + b_{i,j}(z) \quad (1 \leq i \leq m, 1 \leq j \leq n(i)) \]

with $a_i, b_i \in \mathbb{C}(z)$, $T \in \mathbb{C}(z)$ is not constant, $a_i \neq 0$, and $a_{i_1}/a_{i_2}$ is not of the form $g(T(z))/g(z)$ with $g \in \mathbb{C}(z)$ for $i_1 \neq i_2$. If $f_{1,1}, \ldots, f_{m,n(m)}$ are algebraically dependent, then there exist indices $1 \leq i_1 < \ldots < i_R \leq m$,
complex numbers $c_{i,r,j}$ for $1 \leq r \leq R$ and $1 \leq j \leq n(i_r)$, not all zero, and functions $g_1, \ldots, g_R \in \mathbb{C}(z)$ with the following properties:

(i) $g_r(z) = a_{i_r}(z)g_r(T(z)) + \sum_{j=1}^{n(i_r)} c_{i_r,j}b_{i_r,j}(z)$ for $1 \leq r \leq R$,

(ii) there exist $m_1, \ldots, m_R \in \mathbb{Z}$, not all zero, such that

$$
\prod_{r=1}^{R} \left( \sum_{j=1}^{n(i_r)} c_{i_r,j}f_{i_r,j}(z) - g_r(z) \right)^{m_r} \in \mathbb{C}(z).
$$

**Proof.** See Theorem 2 in [K2].

4. Proof of Theorem 1. The first step in the proof of the theorems is the reduction to the case $\omega = 0$, as shown in [B3]. This is done by means of a suitable Möbius transformation $\Phi(z)$, which is defined as

$$
\Phi(z) = \begin{cases} 
    z - \omega & \text{for } \omega \in \mathbb{C}, \\
    \frac{1}{z - \beta} & \text{for } \omega = \infty \text{ with an algebraic number } \beta \neq T^k(\alpha) \text{ for } k \in \mathbb{N}.
\end{cases}
$$

Then we consider the functions $f_i^*(z) = f_i(\Phi^{-1}(z))$ and the transformation $T^*(z) = \Phi(T(\Phi^{-1}(z)))$ (notice that $\deg T^* = \deg T$ and $\ord_0 T^* = \ord_0 T$). Since the functional equations

$$
a^*(z)f^*(z) = A^*(z)f^*(T^*(z)) + B^*(z)
$$

with $a^*(z) = a(\Phi^{-1}(z))$, $A^*(z) = A(\Phi^{-1}(z))$, $B^*(z) = B(\Phi^{-1}(z))$ hold, the assumptions of Theorem 1 are fulfilled for $f^*$, $d(z)a^*(z)$, $d(z)A^*(z)$, $d(z)B^*(z)$, where $d(z) \in \mathbb{Q}[z]$ is a common denominator for the rational functions in $A^*, B^*, a^*$, and further $\omega = 0$.

The next step in the proof of Theorem 1 is the estimate of the power series coefficients of the functions $f_i$ and the construction of an auxiliary function with high vanishing order at $z = 0$. This yields a sequence of auxiliary polynomials in $f_1(\alpha), \ldots, f_m(\alpha)$. Application of Lemmas 3 and 5 and a suitable choice of the parameters completes the proof.

For the proof of Lemmas 7–9 we suppose that $T(z) = T_1(z)/T_2(z)$ with $T_1, T_2 \in \mathbb{Q}[z]$, $\omega = 0$, $d = \deg T \geq \delta = \ord_0 T \geq 2$. Further we define for $f_i(z) = \sum_{h=0}^{\infty} f_{i,h}z^h$ the power series coefficients of the $j$th power $f_i^j(z)$ by

$$
f_i^j(z) = \sum_{h=0}^{\infty} \left( \sum_{h_1 + \ldots + h_j = h} f_{i_1,h_1} \ldots f_{i_j,h_j} \right) z^h = \sum_{h=0}^{\infty} \sum_{h_{j_1} + \ldots + h_{j_m} = h} f_{i_{j_1},h}z^h
$$

and for $j = (j_1, \ldots, j_m) \in \mathbb{N}_0^m$. 

|
\[(10) \quad f(z)^j = f_1^j(z) \ldots f_m^j(z) = \sum_{h=0}^{\infty} \left( \sum_{h_1 + \ldots + h_m = h} f_{1,h_1}^{(j_1)} \ldots f_{m,h_m}^{(j_m)} \right) z^h = \sum_{h=0}^{\infty} f_h^{(j)} z^h. \]

**Lemma 7.** Suppose the above mentioned assumptions are fulfilled, and \( f \) satisfies (3). Then for all \( h \in \mathbb{N}_0 \) and \( j \in \mathbb{N}, \ j \in \mathbb{N}_0^m \) with \( |j| = j_1 + \ldots + j_m \),

(i) \( f_{i,h} \in K \),

(ii) \( |\mathcal{F}_{i,h}| \leq \exp(c_8(1+h)) \), \( D^{[c_8(1+h)]} f_{i,h} \in O_K \),

(iii) \( |\mathcal{F}_{i,h}^{(j)}| \leq \exp(c_9(j+h)) \), \( D^{[c_9(j+h)]} f_{i,h}^{(j)} \in O_K \),

(iv) \( |\mathcal{F}_{i,h}^{(j)}| \leq \exp(c_{10}(|j|+h)) \), \( D^{[c_{10}(|j|+h)]} f_{i,h}^{(j)} \in O_K \),

where \( D \in \mathbb{N}, c_8, c_9, c_{10} \in \mathbb{R}_+ \), and the algebraic number field \( K \) depend on \( f_1, \ldots, f_m \).

**Proof.** Without loss of generality we may assume that \( f_i(0) = 0 \) for all \( i \) (otherwise we consider \( f_i(z) - f_i(0) \)), and the entries of \( a(z)^{-1}A(z) \) (hence of \( a(z)^{-1}B(z) \)) are regular in \( z = 0 \). If there exist entries of \( a(z)^{-1}A(z) \) which are not regular in \( z = 0 \), and the pole order is at most \( s \), we put

\[
R_i(z) = \sum_{h=0}^{s-1} f_{i,h} z^h \quad (1 \leq i \leq m), \quad R(z) = (R_1(z), \ldots, R_m(z)),
\]

and consider the functions \( g_i(z) = (f_i(z) - R_i(z))z^{-s} \), which satisfy the functional equation

\[
g(z) = T(z)^s z^{-s} a(z)^{-1} A(z) g(T(z))
= z^{-s} (B(z) - a(z)^{-1} A(z) B(T(z)) + B(z)),
\]

and then \( T(z)^s z^{-s} a(z)^{-1} A(z) \) is regular in \( z = 0 \) because of \( \delta \geq 2 \). Now let \( K \) denote the algebraic number field which is generated by the coefficients of the power series expansion of the entries of \( a(z)^{-1}A(z) \) and \( a(z)^{-1}B(z) \), the fixed point \( \omega \) (remember the Möbius transformation \( \Phi \)), the coefficients of \( T \), finitely many power series coefficients of \( f_1, \ldots, f_m \) (if necessary, see above), and the point \( \beta \) from the beginning of this section (if necessary). With \( a(z)^{-1}A(z) = (a_{i,j}(z))_{1 \leq i,j \leq m}, \ a(z)^{-1}B(z) = (b_i(z))_{1 \leq i \leq m} \) and

\[
a_{i,j}(z) = \sum_{h=0}^{\infty} a_{i,j,h} z^h, \quad b_i(z) = \sum_{h=0}^{\infty} b_{i,h} z^h,
\]

\[
T(z) = \sum_{h=0}^{\infty} p_h z^h, \quad (T(z))^l = \sum_{h=0}^{\infty} p_h^{(l)} z^h,
\]

the functional equation implies
\[
\sum_{h=1}^{\infty} f_{i,h} z^h = \sum_{j=1}^{m} \left( \sum_{h=0}^{\infty} a_{i,j,h} z^h \right) \left( \sum_{l=1}^{\infty} \sum_{h=0}^{\infty} p_{k}^{(l)} z^h \right) + \sum_{h=0}^{\infty} b_{i,h} z^h
\]
\[
= \sum_{h=1}^{\infty} \left( \sum_{j=1}^{m} \sum_{k=1}^{\infty} a_{i,j,h-k} \left( \sum_{l=1}^{\infty} f_{j,l} p_{k}^{(l)} \right) \right) z^h + \sum_{h=0}^{\infty} b_{i,h} z^h,
\]
and we get the identity
\[
(11) \quad f_{i,h} = \sum_{k=1}^{\min\{h,N\}} \left( \sum_{\nu=0}^{N} r_{\nu} z^\nu \right) f(z)^\nu = \sum_{h=0}^{\infty} \beta_h z^h,
\]
Now assertion (i) is obvious. According to Lemma 1(ii) the power series coefficients \( p_h \) of \( T \) are bounded by \( p_h \leq \exp(\gamma_0 (h + 1)) \) with \( \gamma_0 \in \mathbb{R}^+ \), and then
\[
\left| p_h^{(l)} \right| \leq \sum_{h_1 + \ldots + h_l = h} \prod_{h_i} \leq \exp(\gamma_1 (l + h)).
\]
Together with (11) and the bounds of Lemma 1(ii) for the power series coefficients of \( a_{i,j} \) and \( b_i \) this yields the first part of (ii) by induction, and with suitable \( D \in \mathbb{N} \) the second part of (ii) follows from Lemma 1(iii).

Assertions (iii) and (iv) are consequences of (ii) and the identities (9), (10) (notice that the number of \( h \in \mathbb{N}_0 \) with \( |h| = h \) is bounded by \( \binom{h+j-1}{j-1} \leq 2^{h+j} \)).

**Lemma 8.** For \( N \in \mathbb{N} \) there exists a polynomial \( R_N(z, y) \in O_K[z, y_1, \ldots, y_m] \setminus \{0\} \) with the following properties:

(i) \( \deg_z R_N \leq N, \deg_y R_N \leq N \),

(ii) \( H(R_N) \leq \exp(c_{11} N^{1+m}) \),

(iii) \( c_{12} N^{1+m} \leq \nu(N) = \text{ord}_0 R_N(f(z)) \leq c_{13} N^{1+m} \log d / \log \delta \).

**Proof.** Put
\[
R_N(z, y) = \sum_{\nu=0}^{N} \sum_{|\mu| \leq N} r_{\nu,\mu} z^\nu y^\mu
\]
with unknown coefficients \( r_{\nu,\mu} \). Then
\[
R_N(z, f(z)) = \sum_{\nu=0}^{N} \sum_{|\mu| \leq N} r_{\nu,\mu} z^\nu f(z)^\mu = \sum_{h=0}^{\infty} \beta_h z^h
\]
with
\[
\beta_h = \sum_{\nu=0}^{\min\{h,N\}} \sum_{|\mu| \leq N} r_{\nu,\mu} f^{(\mu)}_{h-\nu}.
\]
The left-hand inequality of assertion (iii) is equivalent to the condition \( \beta_h = 0 \) for \( 0 \leq h < c_{12}N^{1+m} \). This yields at most \([c_{12}N^{1+m}] + 1\) linear equations in the \((N+1)\binom{N+m}{m}\) unknowns \( r_{\nu,\mu} \). After multiplication with \( D[c_{12}N^{1+m}] \) (see Lemma 7) the coefficients of the linear equations are algebraic integers, and the houses are bounded by \( \exp(\gamma_0N^{1+m}) \). Since \((N+1)\binom{N+m}{m} \geq \frac{1}{m!}N^{1+m} > 2c_{12}N^{1+m} + 1 \) for suitable \( c_{12} \in \mathbb{R}_+ \), Siegel’s lemma yields the assertion of Lemma 8 apart from the upper bound for the zero order \( \nu(N) \) in (iii), but this is a consequence of Lemma 5.

**Lemma 9.** For \( k \in \mathbb{N} \) with \( \delta^k \geq c_{14}\nu(N) \),

\[
\exp(-c_{15}\nu(N)\delta^k) \leq |R_N(T^k(\alpha), f(T^k(\alpha)))| \leq \exp(-c_{16}\nu(N)\delta^k),
\]

where the constants \( c_{14}, c_{15}, c_{16} \in \mathbb{R}_+ \) depend only on \( f \) and \( \alpha \).

**Proof.** From Lemma 7 and (12) we get (notice that \( h \geq c_{12}N^{1+m} \))

\[
|\beta_h| \leq |\overline{\beta_h}| \leq \exp(\gamma_0h), \quad D[\gamma_0h]\beta_h \in O_K.
\]

Then we consider

\[
R_N(T^k(\alpha), f(T^k(\alpha))) = \beta_\nu(N) T^k(\alpha) \nu(N) \left( 1 + \sum_{h=1}^{\infty} \frac{\beta_{h+\nu(N)}}{\beta_\nu(N)} (T^k(\alpha))^h \right).
\]

Since

\[
|\beta_\nu(N)| \geq (D[\gamma_0\nu(N)]|\overline{\beta_\nu(N)}|)^{-[K:Q]}
\]

and

\[
\left| \frac{\beta_{h+\nu(N)}}{\beta_\nu(N)} \right| \leq \exp(\gamma_1(h + \nu(N)))
\]

for \( h \in \mathbb{N} \), Lemma 2 implies for \( k \in \mathbb{N} \) with \( \delta^k \geq \gamma_2\nu(N) \),

\[
\left| \sum_{h=1}^{\infty} \frac{\beta_{h+\nu(N)}}{\beta_\nu(N)} (T^k(\alpha))^h \right| \leq \sum_{h=1}^{\infty} \exp(\gamma_1(h + \nu(N)) - \gamma_3h\delta^k) < \frac{1}{2},
\]

hence

\[
\frac{1}{2} |\beta_\nu(N)||T^k(\alpha)|^{\nu(N)} \leq |R_N(T^k(\alpha), f(T^k(\alpha)))| \leq \frac{3}{2} |\beta_\nu(N)||T^k(\alpha)|^{\nu(N)}.
\]

Now (13), (14) together with Lemma 2 complete the proof.

From now on we suppose in addition that \( \delta = \ord_0 T = \deg T = d \), i.e., the assumptions of Theorem 1 are fulfilled with \( \omega = 0 \). For the application of Lemma 4 we define polynomials \( R_{k,N} \in K[z, y] \) for \( k, N \in \mathbb{N} \) with \( \delta^k \geq c_{14}\nu(N) \) by

\[
R_{0,N}(z, y) = R_N(z, y),
\]

\[
R_{k+1,N}(z, y) = (\det A(z))^N T_2(z)^{d_k N} R_{k,N}(T(z), A(z)^{-1}(a(z)y - B(z))) \text{,}
\]

where the coefficients of the linear equations are algebraic integers, and the houses are bounded by \( \exp(\gamma_0N^{1+m}) \). Since \((N+1)\binom{N+m}{m} \geq \frac{1}{m!}N^{1+m} > 2c_{12}N^{1+m} + 1 \) for suitable \( c_{12} \in \mathbb{R}_+ \), Siegel’s lemma yields the assertion of Lemma 8 apart from the upper bound for the zero order \( \nu(N) \) in (iii), but this is a consequence of Lemma 5.
where the degree of the entries of $A(z)$ and $B(z)$ is at most $s \in \mathbb{N}$, and $d_k = c_{17} (d^k - 1) / (d - 1) + d^k$ with $c_{17} = m s$.

**Lemma 10.** Suppose $k, N \in \mathbb{N}$. Then

(i) $R_{k, N} \in K[z, y]$, 
(ii) $\deg_z R_{k, N} \leq d_k N \leq 2 c_{17} d^k N$, $\deg_y R_{k, N} \leq N$, 
(iii) $H(R_{k, N}) \leq \exp(c_{18} (d^k + N m))$, 
and if $d^k \geq c_{19} \nu(N)$, then

(iv) $\exp(-c_{20} \nu(N)d^k) \leq |R_{k, N}(\alpha, f(\alpha))| \leq \exp(-c_{21} \nu(N)d^k)$.

**Proof.** (i), (ii) are proved by induction; (i) follows from the fact that the matrix $\det A(z)A(z)^{-1}$ has entries in $K[z]$, and (ii) is a consequence of $\deg T = d$ and the definition of $c_{17}$. Suppose that $L$ is an upper bound for the length of $a(z)$ and the entries of $A(z)$ and $B(z)$. Then assertion (iii) follows from

$$H(R_{k+1, N}) \leq L(R_{k+1, N}) \leq L(R_{k, N}) \max\{1, L\}^{m N} \max\{1, L(T_1), L(T_2)\}^{d_k N} \leq L(R_N) \exp\left(\gamma_0 \sum_{l=0}^{k} d_l N\right) \leq \exp(\gamma_1 d^k N + \gamma_2 N^{1 + m}).$$

The last assertion is a consequence of $d = \delta$, Lemma 8, and

$$R_{k, N}(\alpha, f(\alpha)) = \prod_{j=0}^{k-1} (\det A(T_j(\alpha)))^N \prod_{j=0}^{k-1} (T_2(T_j(\alpha)))^{d_k - 1 - j} N \prod_{j=0}^{k-1} (T_2(T_j(\alpha)))^{d_k - 1 - j} N R_N(T^k(\alpha), f(T^k(\alpha))),$$

since

$$\exp(-\gamma_3 d^k N) \leq \prod_{j=0}^{k-1} |\det A(T_j(\alpha))|^N \leq \exp(\gamma_4 d^k N)$$

and

$$\exp(-\gamma_5 d^k N) \leq \prod_{j=0}^{k-1} |T_2(T^j(\alpha))|^{d_k - 1 - j} N \leq \exp(\gamma_6 d^k N).$$

Suppose that $D_1$ is a denominator of $\alpha$, $D_2$ is a common denominator of the coefficients of $T(z)$, and $D_3$ is a common denominator of the coefficients of $a(z)$ and the entries of $A(z)$ and $B(z)$. Then we put

$$Q_{k, N}(y) = (D_1 D_2)^{2 c_{17} d^k N + 1} D_3^{m k N} R_{k, N}(\alpha, y).$$
Thus for $N \geq N_0$ and $k \in \mathbb{N}$ with $d^k \geq c_{22}N^{1+m}$ (cf. Lemma 8(iii)),

$$Q_{k,N} \in O_K[y], \quad \deg Q_{k,N} \leq N, \quad H(Q_{k,N}) \leq \exp(c_{23}d^kN),$$

$$\exp(-c_{24}d^kN^{1+m}) \leq |Q_{k,N}(f(\alpha))| \leq \exp(-c_{25}d^kN^{1+m}).$$

With sufficiently large constants $\gamma_0, \gamma_1 \in \mathbb{R}_+$, which depend only on $f, \alpha, N_0,$ and the constant $c_6$ of Lemma 4, we choose $N_1 = \lceil \gamma_0 D \rceil$ and the parameters $k_0(N), k_1(N)$ for $N \in \{N_0, \ldots, N_1\}$ such that

$$d^{k_0(N)} - 1 < c_{22}N^{1+m} \leq d^{k_0(N)},$$

$$k_1 = k_1(N) = \left[ \frac{1}{\log d} \log \left( \frac{D^{m+1} + \log H}{D} \right) + \gamma_1 \right].$$

$D$ and $H$ as in the assumptions of Theorem 1. Hence $k_0(N) \leq k_1$, and for the application of Lemma 4 we define

$$\Phi_1 = N_1, \quad \Phi_2 = c_{23}N_1d^{k_1},$$

$$\Psi_1(k, N) = c_{24}d^kN^{1+m}, \quad \Psi_2(k, N) = c_{25}d^kN^{1+m}.$$  

Then obviously (i), (ii), (iii) of Lemma 4 are fulfilled with $\Lambda = d\Phi_2/c_{25}$ and

$$U = c_{24}d^{k_1}N_1^{1+m}, \quad \tau = c_{24}d^{k_0(N)}N_0^{1+m}.$$  

Furthermore, we see that

$$U \geq \gamma_2N_1^{m}\max\{\log H + d^{k_1}D, \tau D/N_1\} \geq d^{k_1}N_1^{m}\max\{\tau D, A(\Phi_1 \log H + \Phi_2 D)\},$$

and Lemma 4 implies

$$|Q(f(\alpha))| > \exp(-U) \geq \exp(-\gamma_3d^{k_1}N_1^{1+m}) \geq \exp\left(-\gamma_4D^{m+1}\left(\frac{D^{m+1} + \log H}{D}\right)\right). \blacksquare$$

5. Proof of Theorem 2. The first part of the proof up to Lemma 9 and the definition of the polynomials $R_{k,N}$ in the paragraph after Lemma 9 is identical with the proof of Theorem 1. Since $2 \leq \delta \leq d$, Lemma 10 must be slightly modified.

**LEMMA 11.** Suppose $k, N \in \mathbb{N}$. Then

(i) $R_{k,N} \in K[z, y],$

(ii) $\deg_x R_{k,N} \leq d_kN \leq 2c_{17}d^kN, \quad \deg_y R_{k,N} \leq N,$

(iii) $H(R_{k,N}) \leq \exp(c_{18}(d^k + N^m)),$

and if $\delta^k \geq c_{26}\nu(N)$ and $Nd^k \leq c_{27}\nu(N)\delta^k$, then

(iv) $\exp(-c_{28}\nu(N)\delta^k) \leq |R_{k,N}(\alpha, f(\alpha))| \leq \exp(-c_{29}\nu(N)\delta^k).$
Proof. The additional assumption in (iv) is necessary to compensate the bounds of Lemma 9 and (15), (16).

With denominators $D_1, D_2, D_3$ as in (17) we define polynomials $Q_{k,N}$ by
\[ Q_{k,N}(y) = (D_1 D_2)^{|2c_{17}d^{k}N|+1} D_3^{nkN} R_{k,N}(\alpha, y). \]
Thus for $k \in \mathbb{N}$ with $Nd^k \leq c_{30}\nu(N)\delta^k$ and $\delta^k \geq c_{31}\nu(N)$ we have
\[ Q_{k,N} \in \mathcal{O}_K[y], \quad \deg Q_{k,N} \leq N, \quad H(Q_{k,N}) \leq \exp(c_{32}d^kN), \]
\[ \exp(-c_{33}\delta^k\nu(N)) \leq |Q_{k,N}(f(\alpha))| \leq \exp(-c_{34}\delta^k\nu(N)). \]
With sufficiently large $\gamma_0, \gamma_1 \in \mathbb{R}_+$, which depend on $f$ and $\alpha$, we define
\[ k_0 = \left[ \frac{\log \nu(N)}{\log \delta} + \gamma_0 \right], \quad k_1 = \left[ \frac{\log \nu(N) - m_0 \log N}{\log d - \log \delta} - \gamma_1 \right] \]
(notice that $c_{30} \in \mathbb{R}_+$ may be very small). Then obviously $Nd^k \leq c_{30}\nu(N)\delta^k$ and $\delta^k \geq c_{31}\nu(N)$ for $k_0 \leq k \leq k_1$ (without loss of generality $m_0 \geq 1$), and $k_0 \leq k_1$ is shown in (19). Furthermore,
\[ \nu(N)\delta^{k_1} \geq \gamma_2 N^{m_0}d^{k_1}, \]
and the definition of $m_0, k_0, k_1$ together with $\nu(N) \geq c_{12}N^{1+m}$ yields
\[ \delta^{k_1} \geq \gamma_3 N^{m_0-1}\delta^{k_0} \]
with $\gamma_2, \gamma_3 \in \mathbb{R}_+$ for $N \geq N_0(\gamma_0, \ldots, \gamma_3)$. Thus we define
\[ \Phi_1 = N, \quad \Phi_2 = c_{32}d^{k_1}N, \]
\[ \Psi_1(k) = c_{33}\delta^k\nu(N), \quad \Psi_2(k) = c_{34}\delta^k\nu(N), \quad \Lambda = \delta c_{33}/c_{34}, \]
and if we now fix $N \in \mathbb{N}$ sufficiently large with respect to $\gamma_0, \ldots, \gamma_3, \delta, f, \alpha$, and $c_5$, we put $Q_k = Q_{k,N}$ for $k_0 \leq k \leq k_1$ and this value of $N$. Then (18), (19) imply
\[ \Psi_2(k) \geq c_5 A^{m_0-1}\Phi_1^{m_0-1} \max \{\Psi_1(k_0), \Phi_2\}, \]
and the other assumptions of Lemma 3 are also fulfilled for this choice of parameters. The application of Lemma 3 completes the proof of Theorem 2.

6. Proof of Theorem 3. Under the assumptions of Theorem 3 we can give sharper bounds for the power series coefficients of $f_1, \ldots, f_m$ in the expansion at $\omega$. This yields a weaker condition for $k_0$, hence a better bound for $m_0$.

Analogously to Section 4 we apply the Möbius transformation $\Phi$ to get $\omega = 0$. Then the sharper estimates for the power series coefficients depend on the fact that $a(z) = 1$, and $T(z)$ and the entries of $A(z)$ and $B(z)$ are polynomials. For the sake of simplicity the case $\omega = \infty$ is excluded, because then $\Phi$ transforms the functional equation into another system, where in general $a(z)$ is not constant, and $T(z)$ is rational.
Since the proof of Theorem 3 is analogous to the proof of Theorem 2 apart from the estimates for the power series coefficients, most proofs are shortened or omitted.

**Lemma 12.** Suppose that the assumptions of Theorem 3 are fulfilled with \( \omega = 0 \). Then for all \( h \in \mathbb{N}_0 \) and \( j \in \mathbb{N}_0 \),

(i) \( f_{i,h} \in K \),  
(ii) \( \left| f_{i,h} \right| \leq \exp(c_{34}\log(h + 2)) \), \( D^{c_{34}\log(h+2)} \) \( f_{i,h} \in O_K \),  
(iii) \( \left| f_{i,h}^{(j)} \right| \leq \exp(c_{35}\log(h + 2)) \), \( D^{c_{35}\log(h+2)} \) \( f_{i,h}^{(j)} \in O_K \),

(iv) \( \left| f_{i,h}^{(j)} \right| \leq \exp(c_{36}|j|\log(h + 2)) \), \( D^{c_{36}|j|\log(h+2)} \) \( f_{i,h}^{(j)} \in O_K \),

where \( D, c_{34}, c_{35}, c_{36} \in \mathbb{R}_+ \), and the algebraic number field \( K \) depend on \( f \).

**Proof.** Without loss of generality \( f_i(0) = 0 \) for all \( i \) (since \( f_1(0), \ldots, f_m(0) \in \mathbb{C}, \) the functions \( f_i(z) - f_i(0), 1 \leq i \leq m, \) satisfy functional equations of the required form). Then with \( A(z) = (a_{i,j}(z))_{1 \leq i,j \leq m}, B(z) = (B_i(z))_{1 \leq i \leq m} \) and

\[
a_{i,j}(z) = \sum_{h=0}^{s} a_{i,j,h} z^h, \quad B_i(z) = \sum_{h=0}^{s} b_{i,h} z^h,
\]

\[
T(z) = \sum_{h=0}^{d} p_h z^h, \quad (T(z))^l = \sum_{h=0}^{d^l} p_h^{(l)} z^h,
\]

the functional equation implies

\[
\sum_{h=1}^{\infty} f_{i,h} z^h = \sum_{j=1}^{m} \left( \sum_{h=0}^{s} a_{i,j,h} z^h \right) \left( \sum_{l=1}^{d^l} p_l^{(j)} z^h \right) + \sum_{h=0}^{s} b_{i,h} z^h
\]

\[
= \sum_{h=0}^{\infty} \left( \sum_{j=1}^{m} \sum_{h=\delta}^{k=\max(\delta, h-s)} a_{i,j,h-k} \sum_{\log k / \log d \leq l \leq \log k / \log \delta} f_{j,l} p_k^{(l)} \right) z^h
\]

\[
+ \sum_{h=0}^{s} b_{i,h} z^h,
\]

and from the identity

\[
f_{i,h} = \sum_{k=\max(\delta, h-s)}^{h} \sum_{j=1}^{m} a_{i,j,h-k} \left( \sum_{\log k / \log d \leq l \leq \log k / \log \delta} f_{j,l} p_k^{(l)} \right) + b_{i,h}
\]

(with \( b_{i,h} = 0 \) for \( h > s \)) assertion (i) follows immediately. Since

\[
\sum_{h_1 + \ldots + h_l = h} \prod_{h_i} \leq \exp(\gamma_0 l)
\]
(notice that $\delta \leq h_i \leq d$ for $i = 1, \ldots, l$), the first part of (ii) follows from (20), if we choose $D \in \mathbb{N}$ as a suitable denominator for the coefficients of $T(z)$ and the entries of $A(z)$ and $B(z)$. Then (iii), (iv) can be derived from (9), (10) respectively (notice that the number of $h \in \mathbb{N}$ with $|h| = h$ is bounded by $(|h| + j - 1) \leq \exp(j \log(h + 1))$. ■

**Lemma 13.** For $N \in \mathbb{N}$ there exists a polynomial $R_N(z, y) \in O_K[z, y_1, \ldots, y_m \setminus \{0\}$ with the following properties:

(i) $\deg_z R_N \leq N$, $\deg_y R_N \leq N$,
(ii) $H(R_N) \leq \exp(c_{37} N \log(N + 1)),
(iii) c_{38} N^{1+m} \leq \nu(N) = \text{ord}_0 R_N(z, f(z)).$

**Proof.** Analogous to Lemma 8. ■

**Lemma 14.** For $k \in \mathbb{N}$ with $\delta^k \geq c_{39} N \log \nu(N)$,

$$\exp(-c_{40} \nu(N)\delta^k) \leq |R_N(T^k(\alpha), f(T^k(\alpha)))| \leq \exp(-c_{41} \nu(N)\delta^k),$$

where $c_{39}, c_{40}, c_{41} \in \mathbb{R}_+$ depend only on $f$ and $\alpha$.

**Proof.** Analogous to Lemma 9. Notice that

$$|\beta_h| \leq \frac{1}{|b_h|} \leq \exp(\gamma_0 N \log h), \quad D^{[\gamma_0 N \log |h|}] \beta_h \in O_K$$

and $h \geq \nu(N)$. ■

Now we define polynomials $R_{k,N}$ by

$$R_{0,N}(z, y) = R_N(z, y),
R_{k+1,N}(z, y) = (\det A(z))^N R_N(T(z), A(z)^{-1}(y - B(z)))$$

where the degree of the entries of $A(z)$ and $B(z)$ is at most $s$.

**Lemma 15.** Suppose $k, N \in \mathbb{N}$. Then

(i) $R_{k,N} \in K[z, y]$,
(ii) $\deg_z R_{k,N} \leq c_{42} (d^k - 1)/(d - 1) + d^k \leq 2c_{42}d^k$, $\deg_y R_{k,N} \leq N$,
(iii) $H(R_{k,N}) \leq \exp(c_{43} N (\log(N + 1) + d^k))$

with $c_{42} = sm$, $c_{43} \in \mathbb{R}_+$.

If $\delta^k \geq c_{44} N \log \nu(N)$ and $N d^k \leq c_{45} \nu(N)\delta^k$, then

(iv) $\exp(-c_{46} \nu(N)\delta^k) \leq |R_{k,N}(\alpha, f(\alpha))| \leq \exp(-c_{47} \nu(N)\delta^k)$.

**Proof.** Analogous to Lemma 10 resp. Lemma 11. ■

Suppose that $D_1$ is a denominator of $\alpha$, $D_2$ is a common denominator of the coefficients of $T(z)$, and $D_3$ is a common denominator of the coefficients of the entries of $A(z)$ and $B(z)$. Then we define

$$Q_{k,N}(y) = (D_1 D_2)^{[2c_{42}d^k N] + 1} D_3^{mkN} R_{k,N}(\alpha, y).$$
Thus for $N \geq N_0$ and $\delta^k \geq c_{48} N \log \nu(N)$ and $Nd^k \leq c_{49} \nu(N)\delta^k$ we have
\[ Q_{k,N} \in O_K[y], \quad \deg Q_{k,N} \leq N, \quad H(Q_{k,N}) \leq \exp(c_{50} d^k N), \]
\[ \exp(-c_{51} \delta^k \nu(N)) \leq |Q_{k,N}(f(\alpha))| \leq \exp(-c_{52} \delta^k \nu(N)). \]

With sufficiently large $\gamma_0, \gamma_1 \in \mathbb{R}_+$, which depend on $f$ and $\alpha$, we choose
\[ k_0 = \left[ \frac{\log(N \log \nu(N))}{\log \delta} + \gamma_0 \right], \quad k_1 = \left[ \frac{\log \nu(N) - m_0 \log N}{\log d - \log \delta} - \gamma_1 \right]. \]

This implies $\delta^k \geq c_{48} N \log \nu(N)$ and $Nd^k \leq c_{49} \nu(N)\delta^k$. Furthermore,
\[ \nu(N)\delta^{k_1} \geq \gamma_2 N^{m_0}d^{k_1} \]
for $N \geq N_0(\gamma_2)$. Since $m_0 \log d < (1 - \varepsilon)(m + 1) \log \delta$ for some $\varepsilon \in \mathbb{R}_+$ and $\nu(N) \geq c_{38} N^{1+m}$, we have for all $N \geq N_0(\gamma_0, \ldots, \gamma_3, \varepsilon)$,
\[ \delta^{k_1} \geq \gamma_3 N^{m_0 - 1} \delta^{k_0}. \]

Thus let
\[ \Phi_1 = N, \quad \Phi_2 = c_{50} Nd^{k_1}, \quad \Psi_1(k) = c_{51} \delta^k \nu(N), \quad \Psi_2(k) = c_{52} \delta^k \nu(N), \quad \Lambda = \delta c_{51}/c_{52}, \]
where $N$ is fixed sufficiently large with respect to $\gamma_0, \ldots, \gamma_3, \varepsilon, \delta, f, \alpha$, and $c_5$, and put
\[ Q_k(y) = Q_{k,N}(y) \]
for $k_0 \leq k \leq k_1$ and this value of $N$. Then
\[ \Psi_2(k_1) \geq c_5 \Lambda^{m_0 - 1} \Phi_1^{m_0 - 1} \max\{\Psi_1(k_0), \Phi_2\}, \]
and since all other assumptions of Lemma 3 are fulfilled, the assertion of Theorem 3 now follows from Lemma 3. ■

References

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MATHEMATISCHES INSTITUT
UNIVERSIT¨AT ZU K¨OLN
WEYERTAL 86–90
D-50931 K¨OLN, GERMANY

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