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Algebraic independence of the values of generalized Mahler functions

by

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1. Introduction and results. In the last years arithmetic properties of holomorphic functions were studied which satisfy a functional equation of the shape

$$P(z, f(z), f(T(z))) = 0, \tag{1}$$

where $P(z,u,w)$ is a polynomial with coefficients in $\overline{\mathbb{Q}}$, the field of all algebraic numbers, and $T(z)$ is an algebraic function. This generalizes investigations of Mahler [M1], [M2], [M3], which dealt with functional equations of the form

$$f(z^d) = R(z, f(z)) \tag{2}$$

with $d \in \mathbb{N}$, $d \geq 2$, and a rational function $R(z,u)$ (resp. generalizations of these functional equations to several variables and several functions). Certain cases of (1) were studied extensively by different authors. For a survey of results about the transformations considered by Mahler see [M4], [K1], [L], [LP]. If $T(z)$ is a polynomial, the transcendence of $f(\alpha)$ for algebraic α was proved by Nishioka [Ni1]. This was generalized to algebraic functions $T(z)$ by Becker in [B3]. Applications to Böttcher functions were given by Becker and Bergweiler [BB], and transcendence measures for these functions can be found in [B4] (see also [NT]). The algebraic independence of several values $f_1(\alpha), \ldots, f_m(\alpha)$ was proved by Becker [B2] for certain rational transformations $T(z)$ under additional technical assumptions.

Since a general zero order estimate for functions satisfying (2) with z^d replaced by rational functions $T(z)$ was proved in [T3], we will give an application of the zero order estimate in this paper and derive measures for the algebraic independence of the values of the functions considered by Becker in [B2]. Furthermore we give lower bounds for the transcendence degree of $\mathbb{Q}(f_1(\alpha), \ldots, f_m(\alpha))$ over $\mathbb{Q}$, if $f_1, \ldots, f_m$ satisfy functional equations with more general rational transformations $T(z)$.

Theorem 1. *Let* $f_1, \ldots, f_m : U \to \mathbb{C}$ *be holomorphic in a neighborhood* U *of* $\omega \in \widehat{\mathbb{C}}$, *algebraically independent over* $\mathbb{C}(z)$, *and suppose the power series coefficients of* $f_1, \ldots, f_m$ *in the expansion at* ω *are algebraic. Suppose that* $T(z) = T_1(z)/T_2(z)$ *with* $T_1, T_2 \in \overline{\mathbb{Q}}[z]$, $\deg T = \max\{\deg T_1, \deg T_2\} = d \geq 2$, ω *is a fixed point of* T *of order* $\operatorname{ord}_\omega T = d$, *and* $\underline{f} = (f_1, \ldots, f_m)$ *satisfies the functional equation*

$$a(z)\underline{f}(z) = A(z)\underline{f}(T(z)) + \underline{B}(z), \tag{3}$$

where $A(z)$ *is a regular* $m \times m$ *matrix with entries in* $\overline{\mathbb{Q}}[z]$, $\underline{B}(z) \in (\overline{\mathbb{Q}}[z])^m$, *and* $a(z) \in \overline{\mathbb{Q}}[z]$. *Let* $\alpha \in U$ *be an algebraic number with* $\lim_{k\to\infty} T^k(\alpha) = \omega$, *where* $T^k(\alpha)$ *denotes the* k*-th iterate of* T *at* α, *and suppose for* $k \in \mathbb{N}_0$ *that* $T^k(\alpha) \in U \setminus \{\omega, \infty\}$, *and* $T^k(\alpha)$ *is neither a zero of* $a(z)$ *nor a zero of* $\det A(z)$. *Then for each polynomial* $Q \in \mathbb{Z}[y_1, \ldots, y_m] \setminus \{0\}$ *with* $\deg Q \leq D$, *where* $\deg Q$ *denotes the total degree of* Q *in all variables*, *and* $H(Q) \leq H$, *where* $H(Q)$ *denotes the height of* Q, *i.e. the maximum of the moduli of the coefficients of* Q, *the inequality*

$$|Q(\underline{f}(\alpha))| > \exp(-c_1 D^m (D^{m+2} + \log H))$$

holds with a constant $c_1 \in \mathbb{R}_+$ *depending only on* $\underline{f}$ *and* α.

Remarks. (i) For $\omega = 0$, $T(z) = p(z^{-1})^{-1}$ with a polynomial $p \in \overline{\mathbb{Q}}[z]$, and a diagonal matrix $A(z)$, Theorem 1 is the quantitative analogue of the theorem in [B2], where the algebraic independence of the function values under consideration was proved.

(ii) With $T(z) = z^d$, $d \in \mathbb{N}$, $d \geq 2$, and $\omega = 0$, Theorem 1 includes an earlier result of Becker (Theorem 1 in [B1]) and the improvement of Nishioka (Theorem 1 in [Ni2]).

Theorem 2. *Let* $f_1, \ldots, f_m : U \to \mathbb{C}$ *be holomorphic in a neighborhood* U *of* $\omega \in \widehat{\mathbb{C}}$, *algebraically independent over* $\mathbb{C}(z)$, *and suppose the power series coefficients of* $f_1, \ldots, f_m$ *in the expansion at* ω *are algebraic. Suppose that* $T(z) = T_1(z)/T_2(z)$ *with* $T_1, T_2 \in \overline{\mathbb{Q}}[z]$, $\deg T = d$, ω *is a fixed point of* T *with* $\operatorname{ord}_\omega T = \delta \geq 2$, *and* $\underline{f} = (f_1, \ldots, f_m)$ *satisfies*

$$a(z)\underline{f}(z) = A(z)\underline{f}(T(z)) + \underline{B}(z),$$

where $A(z)$ *is a regular* $m \times m$ *matrix with entries in* $\overline{\mathbb{Q}}[z]$, $\underline{B}(z) \in (\overline{\mathbb{Q}}[z])^m$, *and* $a(z) \in \overline{\mathbb{Q}}[z]$. *Let* $\alpha \in U$ *be an algebraic number with* $\lim_{k\to\infty} T^k(\alpha) = \omega$, *and suppose for* $k \in \mathbb{N}_0$ *that* $T^k(\alpha) \in U \setminus \{\omega, \infty\}$, *and* $T^k(\alpha)$ *is neither a zero of* $a(z)$ *nor a zero of* $\det A(z)$. *Let* m_0 *be the greatest integer satisfying*

$$m_0 < m\left(\frac{2\log\delta}{\log d} - 1\right) + \frac{\log\delta}{\log d}.$$

Then

$$\operatorname{trdeg}_{\mathbb{Q}} \mathbb{Q}(\underline{f}(\alpha)) \geq m_0.$$

COROLLARY 1. *Suppose the assumptions of Theorem* 2 *are fulfilled with* $d < \delta^{1+1/2m}$. *Then* $f_1(\alpha), \ldots, f_m(\alpha)$ *are algebraically independent. In particular*, *for* $m = 1$ *and* $d < \delta^{3/2}$ *we have* $f(\alpha) \notin \overline{\mathbb{Q}}$.

Remark. The case $m = 1$ is Becker's result in [B3] in the special case of rational transformations and the functional equation (3).

THEOREM 3. *Let* $f_1, \ldots, f_m : U \to \mathbb{C}$ *be holomorphic in a neighborhood* U *of* $\omega \in \mathbb{C}$, *algebraically independent over* $\mathbb{C}(z)$, *and suppose* $f_1(\omega), \ldots, f_m(\omega)$ *are algebraic. Suppose that* $T \in \overline{\mathbb{Q}}[z]$, $\deg T = d$, ω *is a fixed point of* T *with* $\mathrm{ord}_\omega T = \delta \geq 2$, *and* $\underline{f} = (f_1, \ldots, f_m)$ *satisfies*

$$\underline{f}(z) = A(z)\underline{f}(T(z)) + \underline{B}(z), \tag{4}$$

where $A(z)$ *is a regular* $m \times m$ *matrix with entries in* $\overline{\mathbb{Q}}[z]$, *and* $\underline{B}(z) \in (\overline{\mathbb{Q}}[z])^m$. *Let* $\alpha \in U$ *be an algebraic number with* $\lim_{k\to\infty} T^k(\alpha) = \omega$, *and suppose for* $k \in \mathbb{N}_0$ *that* $T^k(\alpha) \in U \setminus \{\omega\}$, *and* $\det A(T^k(\alpha)) \neq 0$. *Let* m_0 *be the greatest integer satisfying*

$$m_0 < (m+1)\frac{\log \delta}{\log d}.$$

Then

$$\mathrm{trdeg}_{\mathbb{Q}}\, \mathbb{Q}(\underline{f}(\alpha)) \geq m_0.$$

COROLLARY 2. *Suppose the assumptions of Theorem* 3 *are fulfilled and* $d < \delta^{1+1/m}$. *Then* $f_1(\alpha), \ldots, f_m(\alpha)$ *are algebraically independent. In particular*, *for* $m = 1$ *and* $d < \delta^2$ *we get* $f(\alpha) \notin \overline{\mathbb{Q}}$.

Remark. Since the condition $d < \delta^{3/2}$ in Corollary 1 coincides with the condition given in the theorem of Becker in [B3] in the special case of rational transformations and functional equations of type (3), the weaker condition of Corollary 2 for polynomial transformations and the more restricted functional equations of type (4) gives a first answer to a question posed by Becker (p. 119 in [B3]). He asked for weaker technical assumptions of this form to extend the range of applications of Mahler's method.

2. Examples and applications. Our first example deals with series of the form

$$\chi_i(z) = \sum_{h=0}^{\infty} q_i(T^h(z)) \qquad (i = 1, \ldots, m),$$

where $T(z) = T_1(z)/T_2(z) \in \overline{\mathbb{Q}}(z)$, $d_j = \deg T_j$ $(j = 1, 2)$, $\omega \in \mathbb{C}$ is a fixed point of T of order $\delta \geq 2$, $q_i \in \overline{\mathbb{Q}}[z]$ with $\deg q_i \geq 1$ and $q_i(\omega) = 0$ for $i = 1, \ldots, m$. Then all χ_i are holomorphic in a neighborhood U of ω and satisfy the functional equation

$$\chi_i(z) = \chi_i(T(z)) + q_i(z) \qquad (i = 1, \ldots, m).$$

Corollary 3. *Suppose* $q_1, \ldots, q_m$ *are* $\mathbb{C}$*-linearly independent*, $0 < d_2 < d_1 = d$, *and* $\alpha \in \overline{\mathbb{Q}}$ *satisfies* $\lim_{k\to\infty} T^k(\alpha) = \omega$ *and* $T^k(\alpha) \neq \omega$ *for* $k \in \mathbb{N}_0$. *Then*

$$\operatorname{trdeg}_{\mathbb{Q}} \mathbb{Q}(\chi_1(\alpha), \ldots, \chi_m(\alpha)) \geq m_0,$$

where m_0 *denotes the greatest integer satisfying*

$$m_0 < (m+1)\frac{\log \delta}{\log d} - \left(1 - \frac{\log \delta}{\log d}\right)m.$$

P r o o f. For the application of Theorem 2 we have to show that $\chi_1, \ldots, \chi_m$ are algebraically independent. In the next paragraph this will be derived from Lemma 6 of Section 3.

Suppose that $\chi_1, \ldots, \chi_m$ are algebraically dependent. By Lemma 6 there exist $g_i \in \mathbb{C}(z)$ with $\deg g_i = \gamma_i$ $(i = 1, 2)$, $\gamma = \max\{\gamma_1, \gamma_2\}$, and $s_1, \ldots, s_m \in \mathbb{C}$, not all zero, such that

$$\frac{g_1(z)}{g_2(z)} = \frac{g_1(T(z))}{g_2(T(z))} + \sum_{i=1}^{m} s_i q_i(z).$$

Since the sum on the right is nonzero, we know that $\gamma \geq 1$. From this equation we get the polynomial identity

$$g_1(z)h_2(z) = g_2(z)h_1(z) + g_2(z)h_2(z)\sum_{i=1}^{m} s_i q_i(z)$$

with $h_i(z) = T_2(z)^{\gamma} g_i(T(z)) \in \mathbb{C}[z]$ $(i = 1, 2)$. Since g_1, g_2 resp. T_1, T_2 are coprime, we see that h_1, h_2 are also coprime. Thus $h_2 \mid g_2$, and the condition $d_2 < d_1$ implies

$$\deg h_2 = (\gamma - \gamma_2)d_2 + \gamma_2 d_1 \leq \gamma_2 = \deg g_2.$$

But $d_2 \geq 1$, $d_1 \geq 2$ and $\gamma \geq 1$. Hence we get a contradiction, and so $\chi_1, \ldots, \chi_m$ must be algebraically independent. Then application of Theorem 2 completes the proof. ■

Corollary 4. *Suppose that* $1, q_1, \ldots, q_m$ *are* $\mathbb{C}$*-linearly independent*, $T(z) \in \overline{\mathbb{Q}}[z]$ *with* $2 \leq \delta \leq d$, $d \nmid \deg(\sum_{i=1}^{m} s_i q_i(z))$ *for arbitrary* $(s_1, \ldots, s_m) \in \mathbb{C}^m \setminus \{\underline{0}\}$, *and* $\alpha \in \overline{\mathbb{Q}}$ *satisfies* $\lim_{k\to\infty} T^k(\alpha) = \omega$ *and* $T^k(\alpha) \neq \omega$ *for* $k \in \mathbb{N}_0$. *Then*

$$\operatorname{trdeg}_{\mathbb{Q}} \mathbb{Q}(\chi_1(\alpha), \ldots, \chi_m(\alpha)) \geq m_0,$$

where m_0 *denotes the greatest integer satisfying*

$$m_0 < (m+1)\frac{\log \delta}{\log d}\,.$$

P r o o f. Under the assumption that $\chi_1, \ldots, \chi_m$ are algebraically dependent, we get analogously to the proof of Corollary 3 the polynomial identity

(notice that $T_2 = 1$, hence $h_2 = g_2$)

$$g_1(z)g_2(T(z)) = g_2(z)g_1(T(z)) + g_2(z)g_2(T(z))\sum_{i=1}^{m} s_i q_i(z). \tag{5}$$

The coprimality of g_1, g_2 implies $g_2(T(z)) \mid g_2(z)$, hence $\gamma_2 = 0$. Now we compare the degrees in (5). The degree on the left side is γ_1, and the two terms on the right have degrees $\gamma_1 d$ and $\deg(\sum_{i=1}^m s_i q_i(z)) = \Delta$, respectively. Since $d \ge 2$, this forces $\gamma_1 d = \Delta$. But Δ is not divisible by d except for $\Delta = 0$. Then $\gamma_1 = 0$, and we get the contradiction $\sum_{i=1}^m s_i q_i(z) = 0$. Therefore $\chi_1, \ldots, \chi_m$ are algebraically independent. Now application of Theorem 3 yields the assertion. ∎

Corollary 5. *Suppose $q_1, \ldots, q_m$ are $\mathbb{C}$-linearly independent, $T(z) = T_1(z)/T_2(z) \in \overline{\mathbb{Q}}(z)$, $0 < d_2 < d_1 = d = \delta$, and $\alpha \in \overline{\mathbb{Q}}$ satisfies $\lim_{k\to\infty} T^k(\alpha) = \omega$ and $T^k(\alpha) \in U\setminus\{\omega\}$ for $k \in \mathbb{N}_0$. Then for each polynomial $Q \in \mathbb{Z}[\underline{y}]\setminus\{0\}$ with* $\deg Q \le D$ *and* $H(Q) \le H$,

$$|Q(\chi_1(\alpha), \ldots, \chi_m(\alpha))| > \exp(-c_1 D^m (D^{m+2} + \log H)).$$

Proof. From the proof of Corollary 3 we know that $\chi_1, \ldots, \chi_m$ are algebraically independent. Since $\delta = d$, the assertion follows from Theorem 1. ∎

Remark. The same quantitative result can be derived under the assumptions of Corollary 4 for $\delta = d$.

Now we consider certain Cantor series introduced by Tamura [Ta]. Let

$$\theta_i(z) = \sum_{h=0}^{\infty} \frac{1}{q_i(z)q_i(T(z))\ldots q_i(T^h(z))} \quad (i = 1, \ldots, m) \tag{6}$$

with $T(z) = T_1(z)/T_2(z) \in \overline{\mathbb{Q}}(z)$, $\deg T_j = d_j$ $(j = 1, 2)$, $\omega \in \widehat{\mathbb{C}}$ is a fixed point of T of order $\delta \ge 2$, $q_i \in \overline{\mathbb{Q}}[z]$ with $\deg q_i \ge 1$ and $|q_i(\omega)| > 1$ for $i = 1, \ldots, m$ (notice that $\omega = \infty$ and $q_i(\infty) = \infty$ is possible). The functions θ_i are holomorphic in a neighborhood of $\omega \in \widehat{\mathbb{C}}$ and satisfy the functional equation

$$\theta_i(T(z)) = q_i(z)\theta_i(z) - 1 \quad (i = 1, \ldots, m).$$

Tamura proved the transcendence of $\theta(\alpha)$ for certain α in the special case $q(z) = z$, $T(z) \in \mathbb{Z}[z]$ and $\deg T \ge 3$. The more general case of polynomials $q_i, T \in \overline{\mathbb{Q}}[z]$ $(i = 1, \ldots, m)$ was treated by Becker [B2]. He derived algebraic independence results for $\theta_1(\alpha), \ldots, \theta_m(\alpha)$ at algebraic points α and discussed in detail the transcendence of $\theta(\alpha)$ for linear polynomials q and algebraic α. Here we study rational transformations and give qualitative and quantitative generalizations of Becker's results.

Corollary 6. *Suppose* $q_1, \ldots, q_m$ *are pairwise distinct*, $\max\{2, d_2\} < d_1 = d$, $1 \le \deg q_i < d-1$ *for* $i = 1, \ldots, m$. *Let* α *be an algebraic number with* $\lim_{k\to\infty} T^k(\alpha) = \omega$ *and* $q_i(T^k(\alpha)) \ne 0$, $T^k(\alpha) \ne \omega$ *for* $k \in \mathbb{N}_0$ *and* $i = 1, \ldots, m$. *If* m_0 *is the greatest integer satisfying*

$$m_0 < (m+1)\frac{\log\delta}{\log d} - \left(1 - \frac{\log\delta}{\log d}\right)m,$$

then

$$\operatorname{trdeg}_{\mathbb{Q}} \mathbb{Q}(\theta_1(\alpha), \ldots, \theta_m(\alpha)) \ge m_0.$$

If $\delta = d$, *then* $\theta_1(\alpha), \ldots, \theta_m(\alpha)$ *are algebraically independent*, *and for all polynomials* $Q \in \mathbb{Z}[\underline{y}] \setminus \{0\}$ *with* $\deg Q \le D$ *and* $H(Q) \le H$,

$$|Q(\theta_1(\alpha), \ldots, \theta_m(\alpha))| > \exp(-c_1 D^m(D^{m+2} + \log H)).$$

Proof. The assertions are obvious consequences of Theorems 1 and 2, if the algebraic independence of $\theta_1, \ldots, \theta_m$ is verified. Thus we assume that $\theta_1, \ldots, \theta_m$ are algebraically dependent, and apply Lemma 6. First we must show that $q_i(z)/q_j(z)$ for $i \ne j$ is not of the form $g(T(z))/g(z)$ for some $g \in \mathbb{C}(z)$. With $g(z) = g_1(z)/g_2(z)$, $\deg g_i = \gamma_i$ $(i = 1, 2)$, and $\gamma = \max\{\gamma_1, \gamma_2\}$ we suppose on the contrary that

$$q_i(z)g_1(z)h_2(z) = q_j(z)g_2(z)h_1(z),$$

where $h_i(z) = T_2(z)^\gamma g_i(T(z)) \in \mathbb{C}[z]$. Since g_1, g_2 resp. T_1, T_2 are coprime, we see that h_1, h_2 are also coprime. Thus $h_1 \mid q_i g_1$, $h_2 \mid q_j g_2$, and this implies (notice that $d_2 < d_1$)

$$\deg h_i = \gamma d_2 + \gamma_i(d_1 - d_2) = \gamma_i d_1 + (\gamma - \gamma_i)d_2 \le d_1 - 2 + \gamma_i \qquad (i = 1, 2).$$

Since $d_1 \ge 3$, we must have $\gamma_1 = \gamma_2 = 0$, but this leads to the contradiction $q_i = q_j$. Now all conditions of Lemma 6 are fulfilled, and then there exist $i \in \{1, \ldots, m\}$ and a rational function g (with $g_i, h_i, \gamma_i, \gamma$ as above) such that

$$g_2(z)h_1(z) = h_2(z)g_2(z) + q_i(z)g_1(z)h_2(z). \tag{7}$$

Hence $h_2 \mid g_2$, and this yields

$$\deg h_2 = \gamma_2 d_1 + (\gamma - \gamma_2)d_2 \le \gamma_2.$$

But $d_1 \ge 3$, and so $\gamma_2 = d_2 = 0$. Now we compare the degrees on both sides of (7) and get $d_1\gamma_1 \le \gamma_1 + d_1 - 2$. Since $d_1 \ge 3$, we must have $\gamma_1 = 0$, but then $q_i(z)$ is a constant, and this is excluded. Thus $\theta_1, \ldots, \theta_m$ cannot be algebraically dependent. ■

Corollary 7. *Suppose that* $T \in \overline{\mathbb{Q}}[z]$ *is a polynomial with* $d \ge 2$, *and* $q \in \overline{\mathbb{Q}}[z]$ *is a linear polynomial with* $q(T(z))^2 \ne q(z)^2 - 2$. *Let* α *be an algebraic number with* $\lim_{k\to\infty} T^k(\alpha) = \infty$ *and* $q(T^k(\alpha)) \ne 0$ *for* $k \in \mathbb{N}_0$.

Then for each polynomial $Q \in \mathbb{Z}[y] \setminus \{0\}$ *with* $\deg Q \le D$, $H(Q) \le H$ *the inequality*

$$|Q(\theta(\alpha))| > \exp(-c_1 D(D^3 + \log H))$$

holds for $\theta(z)$ *as in* (6). *In particular*, $\theta(\alpha)$ *is an* S*-number in Mahler's classification of transcendental numbers.*

Proof. In Corollary 2 of [B2] Becker showed that $\theta(z)$ is a transcendental function for $q(z)$, $T(z)$ as above. Then Theorem 1 with $\omega = \infty$ yields the assertion (notice that $\deg T = d = \operatorname{ord}_\infty T$). ∎

The next example deals with the series

$$\Omega(z) = \sum_{h=0}^{\infty} \frac{(-1)^h}{q(T^h(z))}$$

with $q, T \in \overline{\mathbb{Q}}[z]$ and $\deg q \ge 1$, $d \ge 2$, which was introduced by Becker [B2]. Then $\Omega(z)$ is holomorphic in a neighborhood of $\omega = \infty$ and satisfies

$$\Omega(T(z)) = -\Omega(z) + 1/q(z).$$

Corollary 8. *Suppose* $q(T(z)) \ne \lambda^{-1} q(z)^2 + q(z) - \lambda$ *for any* $\lambda \in \mathbb{C} \setminus \{0\}$, *and* α *is an algebraic number with* $\lim_{k\to\infty} T^k(\alpha) = \infty$ *and* $q(T^k(\alpha)) \ne 0$ *for* $k \in \mathbb{N}_0$. *Then for each* $Q \in \mathbb{Z}[y] \setminus \{0\}$ *with* $\deg Q \le D$ *and* $H(Q) \le H$,

$$|Q(\Omega(\alpha))| > \exp(-c_1 D(D^3 + \log H)).$$

In particular, *this transcendence measure is valid for Cahen's constant*

$$C = \sum_{h=0}^{\infty} \frac{(-1)^h}{S_h - 1},$$

where $S_0 = 2$ *and* $S_{h+1} = S_h^2 - S_h + 1$ *for* $h \ge 0$.

Remark. The transcendence of C was proved by Davison and Shallit [DS] with continued fractions and later by Becker in [B2] using the identity $C = \Omega(2)$ for $q(z) = z - 1$, $T(z) = z^2 - z + 1$. Corollary 8 implies that C is a S-number in Mahler's classification of transcendental numbers.

Proof of Corollary 8. In Corollary 3 of [B2] the transcendence of the function $\Omega(z)$ was proved. Then Theorem 1 yields the assertion. ∎

The last example was studied by Becker in [B3], Corollary 1. Let

$$\sigma(z) = \prod_{h=0}^{\infty} q(T^h(z)),$$

where $q \in \overline{\mathbb{Q}}[z]$, $\deg q \ge 1$, and $T(z) = T_1(z)/T_2(z) \in \overline{\mathbb{Q}}(z)$, $\deg T_i = d_i$ $(i = 1, 2)$, and $\omega \in \widehat{\mathbb{C}}$ is a fixed point of T of order δ. Assume that $q(\omega) = 1$.

Then $\sigma(z)$ is holomorphic in a neighborhood of ω and satisfies the functional equation

$$\sigma(z) = q(z)\sigma(T(z)).$$

COROLLARY 9. *Suppose* $0 < d_2 < d_1 = \delta$, *and* α *is an algebraic number with* $\lim_{k\to\infty} T^k(\alpha) = \omega$ *and* $q(T^k(\alpha)) \neq 0$, $T^k(\alpha) \neq \omega, \infty$ *for* $k \in \mathbb{N}_0$. *Then for any polynomial* $Q \in \mathbb{Z}[y] \setminus \{0\}$ *with* $\deg Q \le D$, $H(Q) \le H$,

$$|Q(\sigma(\alpha))| > \exp(-c_1 D(D^3 + \log H)).$$

Proof. The transcendence of $\sigma(z)$ was proved in Corollary 1 of [B3]. Then the assertion follows from Theorem 1. ∎

3. Preliminaries and auxiliary results. Throughout the paper let K denote an algebraic number field, and O_K is the ring of integers in K. Define $\overline{|\alpha|}$, the *house* of the algebraic number α, as the maximum of the moduli of the conjugates of α. A *denominator* of an algebraic number α is a positive integer d such that $d\alpha \in O_K$. For a polynomial P with algebraic coefficients the *height* $H(P)$ is defined as the maximum of the houses of the coefficients, and the *length* $L(P)$ is the sum of the houses of the coefficients.

LEMMA 1. *Suppose the rational function* $g(z) = r(z)/s(z) \in K(z)$ *is holomorphic in a neighborhood of* $z = 0$. *Then for each* $h \in \mathbb{N}_0$ *the power series coefficients* g_h *of*

$$g(z) = \sum_{h=0}^{\infty} g_h z^h$$

satisfy

(i) $g_h \in K(g_0)$,
(ii) $\overline{|g_h|} \le \exp(c_2(h+1))$,
(iii) $D^{[c_2(h+1)]} g_h \in O_K$

with suitable $D \in \mathbb{N}$ *and* $c_2 \in \mathbb{R}_+$ *depending only on* g.

Proof. From $r(z) = s(z)\sum_{h=0}^{\infty} g_h z^h$ with $r(z) = \sum_{i=0}^{l} r_i z^i$, $s(z) = \sum_{i=0}^{l} s_i z^i$ we get the following recurrence relation for the coefficients g_h (with $r_h = 0$ for $h > l$), $h \in \mathbb{N}_0$:

$$g_h = \frac{r_h}{s_0} - \sum_{\mu=1}^{\min\{l,h\}} \frac{s_\mu}{s_0} g_{h-\mu}.$$

This implies the assertion. ∎

Lemma 2. *Suppose $T(z) = T_1(z)/T_2(z)$ is a rational function with $\delta = \operatorname{ord}_0 T \ge 2$, and $\alpha \in \mathbb{C}$ satisfies $T^k(\alpha) \ne 0$ for $k \in \mathbb{N}_0$ and $\lim_{k\to\infty} T^k(\alpha) = 0$. Then for all $k \ge \overline{k}$,*

$$-c_3\delta^k \le \log|T^k(\alpha)| \le -c_4\delta^k$$

with $c_3, c_4 \in \mathbb{R}_+$, $\overline{k} \in \mathbb{N}$ depending on T and α.

Proof. Since 0 is a zero of T of order $\delta \ge 2$, we have $T(z) = z^\delta g(z)$, where $g(z)$ is holomorphic in a neighborhood of $z = 0$ and $g(0) \ne 0$. Then there exists a constant $\varepsilon \in \mathbb{R}_+$ depending only on T such that for all $\beta \in \mathbb{C}$ with $0 < |\beta| < \varepsilon$ (< 1),

$$\gamma_0|\beta|^\delta \le |T(\beta)| \le \gamma_1|\beta|^\delta,$$

where $\gamma_0, \gamma_1 \in \mathbb{R}_+$ depend on T. Thus

$$\exp(-\gamma_2\delta^k) \le \gamma_0^k|\beta|^{\delta^k} \le |T^k(\beta)| \le \gamma_1^k|\beta|^{\delta^k} \le \exp(-\gamma_3\delta^k) \tag{8}$$

with $\gamma_2, \gamma_3 \in \mathbb{R}_+$ depending on T and β. Since $\lim_{k\to\infty} T^k(\alpha) = 0$, we know $0 < |T^k(\alpha)| < \varepsilon$ for $k \ge \overline{k}$ with $\overline{k} \in \mathbb{N}$ depending on T and α, and together with (8) this yields the assertion. ∎

The proofs of the theorems depend on the following results from elimination theory.

Lemma 3. *Suppose $\underline{\omega} \in \mathbb{C}^m$. Then there exists a constant $c_5 = c_5(\underline{\omega}, K) \in \mathbb{R}_+$ with the following property: If there exist increasing functions $\Psi_1, \Psi_2 : \mathbb{N} \to \mathbb{R}_+$, numbers $\Phi_1, \Phi_2, \Lambda \in \mathbb{R}_+$, positive integers k_0, k_1 with $k_0 < k_1$, $m_0 \in \{0, \ldots, m\}$ and polynomials $(Q_k)_{k_0 \le k \le k_1}$, such that the following assumptions are satisfied:*

(i) $\Phi_2 \ge \Phi_1 \ge c_5$, $\Lambda \ge \Psi_1(k+1)/\Psi_2(k) \ge 1$ *for* $k \in \{k_0, \ldots, k_1\}$,
(ii) $\Psi_2(k) \ge c_5(\log H(Q_k) + \deg Q_k)$ *for* $k \in \{k_0, \ldots, k_1\}$,
(iii) *the polynomials* $Q_k \in O_K[y_1, \ldots, y_m]$ $(k_0 \le k \le k_1)$ *satisfy*
  (a) $\deg Q_k \le \Phi_1$,
  (b) $\log H(Q_k) \le \Phi_2$,
  (c) $\exp(-\Psi_1(k)) \le |Q_k(\underline{\omega})| \le \exp(-\Psi_2(k))$,
(iv) $\Psi_2(k_1) \ge c_5\Lambda^{m_0-1}\Phi_1^{m_0-1}\max\{\Psi_1(k_0), \Phi_2\}$,

then

$$\operatorname{trdeg}_{\mathbb{Q}} \mathbb{Q}(\underline{\omega}) \ge m_0.$$

Proof. This is Theorem 1 in [T1] with slight modifications. ∎

Lemma 4. *Suppose $\underline{\omega} \in \mathbb{C}^m$. Then there exists a constant $c_6 = c_6(\underline{\omega}, K) \in \mathbb{R}_+$ with the following property: If there exist functions $\Psi_1, \Psi_2 : \mathbb{N}^2 \to \mathbb{R}_+$, which are increasing in the first variable, numbers $\Phi_1, \Phi_2, \Lambda, U, \tau \in \mathbb{R}_+$, positive integers N_0, N_1 with $N_0 \le N_1$, for each $N \in \{N_0, \ldots, N_1\}$ positive integers $k_0(N), k_1(N)$ with $k_0(N) \le k_1(N)$, and polynomials $Q_{k,N}$ for*

$N \in \{N_0, \ldots, N_1\}$ and $k \in \{k_0(N), \ldots, k_1(N)\}$, such that the following assumptions are satisfied for positive integers D, H and all $N \in \{N_0, \ldots, N_1\}$, $k \in \{k_0(N), \ldots, k_1(N)\}$:

(i) (a) $\Phi_2 \ge \Phi_1 \ge c_6$, $\Lambda \ge \Psi_1(k+1, N)/\Psi_2(k, N) \ge 1$,
(b) $\Psi_1(k_1(N), N) \ge \Psi_1(k_0(N+1), N+1)$,
(c) $U \le \max\{\Psi_2(k, N) \mid N_0 \le N \le N_1, k_0(N) \le k \le k_1(N)\}$,
$\tau \ge \min\{\Psi_1(k, N) \mid N_0 \le N \le N_1, k_0(N) \le k \le k_1(N)\}$,

(ii) $\Psi_2(k, N) \ge c_6(\log H(Q_{k,N}) + \deg Q_{k,N})$,

(iii) *the polynomials* $Q_{k,N} \in O_K[y_1, \ldots, y_m]$ *satisfy*
(a) $\deg Q_{k,N} \le \Phi_1$,
(b) $\log H(Q_{k,N}) \le \Phi_2$,
(c) $\exp(-\Psi_1(k, N)) \le |Q_{k,N}(\underline{\omega})| \le \exp(-\Psi_2(k, N))$,

(iv) $U \ge c_6 \Lambda^{m-1} \Phi_1^{m-1} \max\{\tau D, \Lambda(\Phi_1 \log H + \Phi_2 D)\}$,

then for all polynomials $R \in \mathbb{Z}[y_1, \ldots, y_m] \setminus \{0\}$ *with* $\deg R \le D$, $H(R) \le H$,

$$|R(\underline{\omega})| \ge \exp(-U).$$

Proof. Lemma 4 can be derived from Jabbouri's criterion [J] analogous to the proof of the proposition in [T2]. ∎

LEMMA 5. *Let* $f_1, \ldots, f_m \in \mathbb{C}[[z]]$ *be formal power series which satisfy*

$$A_0(z, \underline{f}(z))\underline{f}(T(z)) = \underline{A}(z, \underline{f}(z)),$$

where $\underline{f}(z) = (f_1(z), \ldots, f_m(z))$, $T(z) = T_1(z)/T_2(z)$ *is a rational function with* $T_1, T_2 \in \mathbb{C}[z]$, $d = \max\{\deg T_1, \deg T_2\}$, $\delta = \operatorname{ord}_0 T \ge 2$, $\underline{A}(z, \underline{y}) = (A_1(z, \underline{y}), \ldots, A_m(z, \underline{y}))$, *and* $A_i(z, \underline{y}) \in \mathbb{C}[z, y_1, \ldots, y_m] \setminus \{0\}$ $(0 \le i \le m)$ *are polynomials with* $\deg_z A_i \le s$ *and* $\deg_{y_1, \ldots, y_m} A_i \le t$. *Suppose that* $t^m < \delta$ *and* $Q \in \mathbb{C}[z, y_1, \ldots, y_m]$ *with* $\deg_z Q \le M$, $\deg_{y_1, \ldots, y_m} Q \le N$ *and* $M \ge N \ge 1$. *If* $Q(z, \underline{f}(z)) \ne 0$, *then*

$$\operatorname{ord}_0 Q(z, \underline{f}(z)) \le c_7 M N^{m \log d/(\log \delta - m \log t)}$$

with a constant $c_7 \in \mathbb{R}_+$ *depending on* $\underline{f}$.

Proof. See Theorem 1 in [T3]. ∎

The following result of Kubota is often useful to verify the algebraic independence of the functions $f_1, \ldots, f_m$.

LEMMA 6. *Suppose* $f_{i,j} \in \mathbb{C}[[z]]$ $(1 \le i \le m, 1 \le j \le n(i))$ *are formal power series satisfying the functional equations*

$$f_{i,j}(z) = a_i(z) f_{i,j}(T(z)) + b_{i,j}(z) \quad (1 \le i \le m, 1 \le j \le n(i))$$

with $a_i, b_{i,j} \in \mathbb{C}(z)$, $T \in \mathbb{C}(z)$ *is not constant*, $a_i \ne 0$, *and* a_{i_1}/a_{i_2} *is not of the form* $g(T(z))/g(z)$ *with* $g \in \mathbb{C}(z)$ *for* $i_1 \ne i_2$. *If* $f_{1,1}, \ldots, f_{m,n(m)}$ *are algebraically dependent, then there exist indices* $1 \le i_1 < \ldots < i_R \le m$, *complex numbers* $c_{i_r,j}$ *for* $1 \le r \le R$ *and* $1 \le j \le n(i_r)$, *not all zero*, *and functions* $g_1, \ldots, g_R \in \mathbb{C}(z)$ *with the following properties*:

(i) $g_r(z) = a_{i_r}(z) g_r(T(z)) + \sum_{j=1}^{n(i_r)} c_{i_r,j} b_{i_r,j}(z)$ *for* $1 \le r \le R$,

(ii) *there exist* $m_1, \ldots, m_R \in \mathbb{Z}$, *not all zero*, *such that*

$$\prod_{r=1}^{R} \Big(\sum_{j=1}^{n(i_r)} c_{i_r,j} f_{i_r,j}(z) - g_r(z) \Big)^{m_r} \in \mathbb{C}(z).$$

Proof. See Theorem 2 in [K2]. ∎

4. Proof of Theorem 1. The first step in the proof of the theorems is the reduction to the case $\omega = 0$, as shown in [B3]. This is done by means of a suitable Möbius transformation $\Phi(z)$, which is defined as

$$\Phi(z) = \begin{cases} z - \omega & \text{for } \omega \in \mathbb{C}, \\ \dfrac{1}{z - \beta} & \text{for } \omega{=}\infty \text{ with an algebraic number } \beta{\ne}T^k(\alpha) \text{ for } k \in \mathbb{N}_0. \end{cases}$$

Then we consider the functions $f_i^*(z) = f_i(\Phi^{-1}(z))$ and the transformation $T^*(z) = \Phi(T(\Phi^{-1}(z)))$ (notice that $\deg T^* = \deg T$ and $\operatorname{ord}_0 T^* = \operatorname{ord}_\omega T$). Since the functional equations

$$a^*(z) \underline{f}^*(z) = A^*(z) \underline{f}^*(T^*(z)) + \underline{B}^*(z)$$

with $a^*(z) = a(\Phi^{-1}(z))$, $A^*(z) = A(\Phi^{-1}(z))$, $\underline{B}^*(z) = \underline{B}(\Phi^{-1}(z))$ hold, the assumptions of Theorem 1 are fulfilled for $\underline{f}^*$, $d(z)a^*(z)$, $d(z)A^*(z)$, $d(z)\underline{B}^*(z)$, where $d(z) \in \mathbb{Q}[z]$ is a common denominator for the rational functions in A^*, $\underline{B}^*$, a^*, and further $\omega = 0$.

The next step in the proof of Theorem 1 is the estimate of the power series coefficients of the functions f_i and the construction of an auxiliary function with high vanishing order at $z = 0$. This yields a sequence of auxiliary polynomials in $f_1(\alpha), \ldots, f_m(\alpha)$. Application of Lemmas 3 and 5 and a suitable choice of the parameters completes the proof.

For the proof of Lemmas 7–9 we suppose that $T(z) = T_1(z)/T_2(z)$ with $T_1, T_2 \in \overline{\mathbb{Q}}[z]$, $\omega = 0$, $d = \deg T \ge \delta = \operatorname{ord}_0 T \ge 2$. Further we define for $f_i(z) = \sum_{h=0}^{\infty} f_{i,h} z^h$ the power series coefficients of the jth power $f_i^j(z)$ by

$$f_i^j(z) = \sum_{h=0}^{\infty} \Big(\sum_{h_1+\ldots+h_j=h} f_{i,h_1} \ldots f_{i,h_j} \Big) z^h = \sum_{h=0}^{\infty} f_{i,h}^{(j)} z^h \tag{9}$$

and for $\underline{j} = (j_1, \ldots, j_m) \in \mathbb{N}_0^m$,

$$\underline{f}(z)^{\underline{j}} = f_1^{j_1}(z)\ldots f_m^{j_m}(z) = \sum_{h=0}^{\infty}\Big(\sum_{h_1+\ldots+h_m=h} f_{1,h_1}^{(j_1)}\ldots f_{m,h_m}^{(j_m)}\Big)z^h = \sum_{h=0}^{\infty} f_h^{(\underline{j})}z^h. \tag{10}$$

Lemma 7. *Suppose the above mentioned assumptions are fulfilled, and $\underline{f}$ satisfies* (3). *Then for all $h\in\mathbb{N}_0$ and $j\in\mathbb{N}$, $\underline{j}\in\mathbb{N}_0^m$ with $|\underline{j}| = j_1+\ldots+j_m$,*

(i) $f_{i,h}\in K$,

(ii) $\overline{|f_{i,h}|} \le \exp(c_8(1+h))$, $D^{[c_8(1+h)]}f_{i,h}\in O_K$,

(iii) $\overline{|f_{i,h}^{(j)}|} \le \exp(c_9(j+h))$, $D^{[c_9(j+h)]}f_{i,h}^{(j)}\in O_K$,

(iv) $\overline{|f_h^{(\underline{j})}|} \le \exp(c_{10}(|\underline{j}|+h))$, $D^{[c_{10}(|\underline{j}|+h)]}f_h^{(\underline{j})}\in O_K$,

where $D\in\mathbb{N}$, $c_8, c_9, c_{10}\in\mathbb{R}_+$, and the algebraic number field K depend on $f_1,\ldots,f_m$.

Proof. Without loss of generality we may assume that $f_i(0)=0$ for all i (otherwise we consider $f_i(z)-f_i(0)$), and the entries of $a(z)^{-1}A(z)$ (hence of $a(z)^{-1}\underline{B}(z)$) are regular in $z=0$. If there exist entries of $a(z)^{-1}A(z)$ which are not regular in $z=0$, and the pole order is at most s, we put

$$R_i(z) = \sum_{h=0}^{s-1} f_{i,h}z^h \quad (1\le i\le m), \qquad \underline{R}(z) = (R_1(z),\ldots,R_m(z)),$$

and consider the functions $g_i(z) = (f_i(z)-R_i(z))z^{-s}$, which satisfy the functional equation

$$\underline{g}(z) = T(z)^s z^{-s} a(z)^{-1}A(z)\underline{g}(T(z)) - z^{-s}(\underline{R}(z) - a(z)^{-1}(A(z)\underline{R}(T(z)) + \underline{B}(z))),$$

and then $T(z)^s z^{-s}a(z)^{-1}A(z)$ is regular in $z=0$ because of $\delta\ge 2$. Now let K denote the algebraic number field which is generated by the coefficients of the power series expansion of the entries of $a(z)^{-1}A(z)$ and $a(z)^{-1}\underline{B}(z)$, the fixed point ω (remember the Möbius transformation Φ), the coefficients of T, finitely many power series coefficients of $f_1,\ldots,f_m$ (if necessary, see above), and the point β from the beginning of this section (if necessary). With $a(z)^{-1}A(z) = (a_{i,j}(z))_{1\le i,j\le m}$, $a(z)^{-1}\underline{B}(z) = (b_i(z))_{1\le i\le m}$ and

$$a_{i,j}(z) = \sum_{h=0}^{\infty} a_{i,j,h}z^h, \qquad b_i(z) = \sum_{h=0}^{\infty} b_{i,h}z^h,$$

$$T(z) = \sum_{h=\delta}^{\infty} p_h z^h, \qquad (T(z))^l = \sum_{h=\delta^l}^{\infty} p_h^{(l)} z^h,$$

the functional equation implies

$$\sum_{h=1}^{\infty} f_{i,h} z^h = \sum_{j=1}^{m} \Big(\sum_{h=0}^{\infty} a_{i,j,h} z^h \Big) \Big(\sum_{l=1}^{\infty} f_{j,l} \Big(\sum_{h=\delta^l}^{\infty} p_h^{(l)} z^h \Big) \Big) + \sum_{h=0}^{\infty} b_{i,h} z^h$$
$$= \sum_{h=\delta}^{\infty} \Big(\sum_{j=1}^{m} \sum_{k=\delta}^{h} a_{i,j,h-k} \Big(\sum_{l=1}^{[\log k/\log \delta]} f_{j,l}\, p_k^{(l)} \Big) \Big) z^h + \sum_{h=0}^{\infty} b_{i,h} z^h,$$

and we get the identity

$$f_{i,h} = \sum_{k=\delta}^{h} \sum_{j=1}^{m} a_{i,j,h-k} \Big(\sum_{l=1}^{[\log k/\log \delta]} f_{j,l}\, p_k^{(l)} \Big) + b_{i,h}. \tag{11}$$

Now assertion (i) is obvious. According to Lemma 1(ii) the power series coefficients p_h of T are bounded by $\overline{|p_h|} \le \exp(\gamma_0(h+1))$ with $\gamma_0 \in \mathbb{R}_+$, and then

$$\overline{\big|p_h^{(l)}\big|} \le \sum_{h_1+\ldots+h_l=h} \overline{|p_{h_1}|} \ldots \overline{|p_{h_l}|} \le \exp(\gamma_1(l+h)).$$

Together with (11) and the bounds of Lemma 1(ii) for the power series coefficients of the $a_{i,j}(z)$ and $b_i(z)$ this yields the first part of (ii) by induction, and with suitable $D \in \mathbb{N}$ the second part of (ii) follows from Lemma 1(iii).

Assertions (iii) and (iv) are consequences of (ii) and the identities (9), (10) (notice that the number of $\underline{h} \in \mathbb{N}_0^j$ with $|\underline{h}| = h$ is bounded by $\binom{h+j-1}{j-1} \le 2^{h+j}$). ■

LEMMA 8. *For $N \in \mathbb{N}$ there exists a polynomial $R_N(z, \underline{y}) \in O_K[z, y_1, \ldots \ldots, y_m] \setminus \{0\}$ with the following properties*:

(i) $\deg_z R_N \le N$, $\deg_{\underline{y}} R_N \le N$,
(ii) $H(R_N) \le \exp(c_{11} N^{1+m})$,
(iii) $c_{12} N^{1+m} \le \nu(N) = \operatorname{ord}_0 R_N(z, \underline{f}(z)) \le c_{13} N^{1+m \log d/\log \delta}$.

Proof. Put

$$R_N(z, \underline{y}) = \sum_{\nu=0}^{N} \sum_{|\underline{\mu}| \le N} r_{\nu,\underline{\mu}} z^{\nu} \underline{y}^{\underline{\mu}}$$

with unknown coefficients $r_{\nu,\underline{\mu}}$. Then

$$R_N(z, \underline{f}(z)) = \sum_{\nu=0}^{N} \sum_{|\underline{\mu}| \le N} r_{\nu,\underline{\mu}} z^{\nu} \underline{f}(z)^{\underline{\mu}} = \sum_{h=0}^{\infty} \beta_h z^h$$

with

$$\beta_h = \sum_{\nu=0}^{\min\{h,N\}} \sum_{|\underline{\mu}| \le N} r_{\nu,\underline{\mu}} f_{h-\nu}^{(\underline{\mu})}. \tag{12}$$

The left-hand inequality of assertion (iii) is equivalent to the condition $\beta_h = 0$ for $0 \le h < c_{12}N^{1+m}$. This yields at most $[c_{12}N^{1+m}]+1$ linear equations in the $(N+1)\binom{N+m}{m}$ unknowns $r_{\nu,\underline{\mu}}$. After multiplication with $D^{[c_{12}N^{1+m}]}$ (see Lemma 7) the coefficients of the linear equations are algebraic integers, and the houses are bounded by $\exp(\gamma_0 N^{1+m})$. Since $(N+1)\binom{N+m}{m} \ge \frac{1}{m!}N^{1+m} > 2c_{12}N^{1+m}+1$ for suitable $c_{12} \in \mathbb{R}_+$, Siegel's lemma yields the assertion of Lemma 8 apart from the upper bound for the zero order $\nu(N)$ in (iii), but this is a consequence of Lemma 5. ∎

LEMMA 9. *For $k \in \mathbb{N}$ with $\delta^k \ge c_{14}\nu(N)$,*

$$\exp(-c_{15}\nu(N)\delta^k) \le |R_N(T^k(\alpha), \underline{f}(T^k(\alpha)))| \le \exp(-c_{16}\nu(N)\delta^k),$$

where the constants $c_{14}, c_{15}, c_{16} \in \mathbb{R}_+$ depend only on $\underline{f}$ and α.

Proof. From Lemma 7 and (12) we get (notice that $h \ge c_{12}N^{1+m}$)

$$|\beta_h| \le \overline{|\beta_h|} \le \exp(\gamma_0 h), \qquad D^{[\gamma_0 h]}\beta_h \in O_K. \tag{13}$$

Then we consider

$$R_N(T^k(\alpha), \underline{f}(T^k(\alpha))) = \beta_{\nu(N)}(T^k(\alpha))^{\nu(N)}\biggl(1 + \sum_{h=1}^{\infty}\frac{\beta_{h+\nu(N)}}{\beta_{\nu(N)}}(T^k(\alpha))^h\biggr).$$

Since

$$|\beta_{\nu(N)}| \ge (D^{[\gamma_0\nu(N)]}\overline{|\beta_{\nu(N)}|})^{-[K:\mathbb{Q}]} \tag{14}$$

and

$$\left|\frac{\beta_{h+\nu(N)}}{\beta_{\nu(N)}}\right| \le \exp(\gamma_1(h+\nu(N)))$$

for $h \in \mathbb{N}$, Lemma 2 implies for $k \in \mathbb{N}$ with $\delta^k \ge \gamma_2\nu(N)$,

$$\left|\sum_{h=1}^{\infty}\frac{\beta_{h+\nu(N)}}{\beta_{\nu(N)}}(T^k(\alpha))^h\right| \le \sum_{h=1}^{\infty}\exp(\gamma_1(h+\nu(N)) - \gamma_3 h\delta^k) < \frac{1}{2},$$

hence

$$\frac{1}{2}|\beta_{\nu(N)}||T^k(\alpha)|^{\nu(N)} \le |R_N(T^k(\alpha), \underline{f}(T^k(\alpha)))| \le \frac{3}{2}|\beta_{\nu(N)}||T^k(\alpha)|^{\nu(N)}.$$

Now (13), (14) together with Lemma 2 complete the proof. ∎

From now on we suppose in addition that $\delta = \operatorname{ord}_0 T = \deg T = d$, i.e. the assumptions of Theorem 1 are fulfilled with $\omega = 0$. For the application of Lemma 4 we define polynomials $R_{k,N} \in K[z, \underline{y}]$ for $k, N \in \mathbb{N}$ with $\delta^k \ge c_{14}\nu(N)$ by

$$R_{0,N}(z,\underline{y}) = R_N(z,\underline{y}),$$
$$R_{k+1,N}(z,\underline{y}) = (\det A(z))^N T_2(z)^{d_k N} R_{k,N}(T(z), A(z)^{-1}(a(z)\underline{y} - \underline{B}(z))),$$

where the degree of the entries of $A(z)$ and $\underline{B}(z)$ is at most $s \in \mathbb{N}$, and $d_k = c_{17}(d^k - 1)/(d-1) + d^k$ with $c_{17} = ms$.

LEMMA 10. *Suppose $k, N \in \mathbb{N}$. Then*

(i) $R_{k,N} \in K[z, \underline{y}]$,

(ii) $\deg_z R_{k,N} \le d_k N \le 2c_{17} d^k N$, $\deg_{\underline{y}} R_{k,N} \le N$,

(iii) $H(R_{k,N}) \le \exp(c_{18} N(d^k + N^m))$,

and if $d^k \ge c_{19}\nu(N)$, *then*

(iv) $\exp(-c_{20}\nu(N)d^k) \le |R_{k,N}(\alpha, \underline{f}(\alpha))| \le \exp(-c_{21}\nu(N)d^k)$.

Proof. (i), (ii) are proved by induction; (i) follows from the fact that the matrix $\det A(z) A(z)^{-1}$ has entries in $K[z]$, and (ii) is a consequence of $\deg T = d$ and the definition of c_{17}. Suppose that L is an upper bound for the length of $a(z)$ and the entries of $A(z)$ and $\underline{B}(z)$. Then assertion (iii) follows from

$$\begin{aligned} H(R_{k+1,N}) &\le L(R_{k+1,N}) \\ &\le L(R_{k,N}) \max\{1, L\}^{mN} \max\{1, L(T_1), L(T_2)\}^{d_k N} \\ &\le L(R_N) \exp\Big(\gamma_0 \sum_{l=0}^{k} d_l N\Big) \le \exp(\gamma_1 d^{k+1} N + \gamma_2 N^{1+m}). \end{aligned}$$

The last assertion is a consequence of $d = \delta$, Lemma 8, and

$$\begin{aligned} &R_{k,N}(\alpha, \underline{f}(\alpha)) \\ &\quad = \prod_{j=0}^{k-1} (\det A(T^j(\alpha)))^N \prod_{j=0}^{k-1} (T_2(T^j(\alpha)))^{d_{k-1-j}N} R_N(T^k(\alpha), \underline{f}(T^k(\alpha))), \end{aligned}$$

since

$$\exp(-\gamma_3 d^k N) \le \prod_{j=0}^{k-1} |\det A(T^j(\alpha))|^N \le \exp(\gamma_4 d^k N) \tag{15}$$

and

$$\exp(-\gamma_5 d^k N) \le \prod_{j=0}^{k-1} |T_2(T^j(\alpha))|^{d_{k-1-j}N} \le \exp(\gamma_6 d^k N). \ \blacksquare \tag{16}$$

Suppose that D_1 is a denominator of α, D_2 is a common denominator of the coefficients of $T(z)$, and D_3 is a common denominator of the coefficients of $a(z)$ and the entries of $A(z)$ and $\underline{B}(z)$. Then we put

$$Q_{k,N}(\underline{y}) = (D_1 D_2)^{[2c_{17} d^k N]+1} D_3^{mkN} R_{k,N}(\alpha, \underline{y}). \tag{17}$$

Thus for $N \ge N_0$ and $k \in \mathbb{N}$ with $d^k \ge c_{22} N^{1+m}$ (cf. Lemma 8(iii)),

$$Q_{k,N} \in O_K[\underline{y}], \quad \deg Q_{k,N} \le N, \quad H(Q_{k,N}) \le \exp(c_{23} d^k N),$$
$$\exp(-c_{24} d^k N^{1+m}) \le |Q_{k,N}(\underline{f}(\alpha))| \le \exp(-c_{25} d^k N^{1+m}).$$

With sufficiently large constants $\gamma_0, \gamma_1 \in \mathbb{R}_+$, which depend only on $\underline{f}$, α, N_0, and the constant c_6 of Lemma 4, we choose $N_1 = [\gamma_0 D]$ and the parameters $k_0(N)$, $k_1(N)$ for $N \in \{N_0, \ldots, N_1\}$ such that

$$d^{k_0(N)-1} < c_{22} N^{1+m} \le d^{k_0(N)},$$
$$k_1 = k_1(N) = \left[\frac{1}{\log d} \log\left(D^{m+1} + \frac{\log H}{D}\right) + \gamma_1\right],$$

D and H as in the assumptions of Theorem 1. Hence $k_0(N) \le k_1$, and for the application of Lemma 4 we define

$$\Phi_1 = N_1, \quad \Phi_2 = c_{23} N_1 d^{k_1},$$
$$\Psi_1(k, N) = c_{24} d^k N^{1+m}, \quad \Psi_2(k, N) = c_{25} d^k N^{1+m}.$$

Then obviously (i), (ii), (iii) of Lemma 4 are fulfilled with $\Lambda = dc_{24}/c_{25}$ and

$$U = c_{24} d^{k_1} N_1^{1+m}, \quad \tau = c_{24} d^{k_0(N_0)} N_0^{1+m}.$$

Furthermore, we see that

$$\begin{aligned} U &\ge \gamma_2 N_1^m \max\{\log H + d^{k_1} D, \tau D/N_1\} \\ &\ge c_6 \Lambda^{m-1} \Phi_1^{m-1} \max\{\tau D, \Lambda(\Phi_1 \log H + \Phi_2 D)\}, \end{aligned}$$

and Lemma 4 implies

$$\begin{aligned} |Q(\underline{f}(\alpha))| &> \exp(-U) \\ &\ge \exp(-\gamma_3 d^{k_1} N_1^{1+m}) \\ &\ge \exp\left(-\gamma_4 D^{m+1}\left(D^{m+1} + \frac{\log H}{D}\right)\right). \ \blacksquare \end{aligned}$$

5. Proof of Theorem 2. The first part of the proof up to Lemma 9 and the definition of the polynomials $R_{k,N}$ in the paragraph after Lemma 9 is identical with the proof of Theorem 1. Since $2 \le \delta \le d$, Lemma 10 must be slightly modified.

LEMMA 11. *Suppose* $k, N \in \mathbb{N}$. *Then*

(i) $R_{k,N} \in K[z, \underline{y}]$,

(ii) $\deg_z R_{k,N} \le d_k N \le 2c_{17} d^k N$, $\deg_{\underline{y}} R_{k,N} \le N$,

(iii) $H(R_{k,N}) \le \exp(c_{18} N(d^k + N^m))$,

and if $\delta^k \ge c_{26} \nu(N)$ *and* $N d^k \le c_{27} \nu(N) \delta^k$, *then*

(iv) $\exp(-c_{28} \nu(N) \delta^k) \le |R_{k,N}(\alpha, \underline{f}(\alpha))| \le \exp(-c_{29} \nu(N) \delta^k)$.

P r o o f. The additional assumption in (iv) is necessary to compensate the bounds of Lemma 9 and (15), (16). ■

With denominators D_1, D_2, D_3 as in (17) we define polynomials $Q_{k,N}$ by

$$Q_{k,N}(\underline{y}) = (D_1 D_2)^{[2c_{17}d^k N]+1} D_3^{mkN} R_{k,N}(\alpha, \underline{y}).$$

Thus for $k \in \mathbb{N}$ with $Nd^k \le c_{30}\nu(N)\delta^k$ and $\delta^k \ge c_{31}\nu(N)$ we have

$$Q_{k,N} \in O_K[\underline{y}], \quad \deg Q_{k,N} \le N, \quad H(Q_{k,N}) \le \exp(c_{32}d^k N),$$
$$\exp(-c_{33}\delta^k \nu(N)) \le |Q_{k,N}(\underline{f}(\alpha))| \le \exp(-c_{34}\delta^k \nu(N)).$$

With sufficiently large $\gamma_0, \gamma_1 \in \mathbb{R}_+$, which depend on $\underline{f}$ and α, we define

$$k_0 = \left[\frac{\log \nu(N)}{\log \delta} + \gamma_0\right], \quad k_1 = \left[\frac{\log \nu(N) - m_0 \log N}{\log d - \log \delta} - \gamma_1\right]$$

(notice that $c_{30} \in \mathbb{R}_+$ may be very small). Then obviously $Nd^k \le c_{30}\nu(N)\delta^k$ and $\delta^k \ge c_{31}\nu(N)$ for $k_0 \le k \le k_1$ (without loss of generality $m_0 \ge 1$), and $k_0 \le k_1$ is shown in (19). Furthermore,

$$\nu(N)\delta^{k_1} \ge \gamma_2 N^{m_0} d^{k_1}, \tag{18}$$

and the definition of m_0, k_0, k_1 together with $\nu(N) \ge c_{12}N^{1+m}$ yields

$$\delta^{k_1} \ge \gamma_3 N^{m_0 - 1}\delta^{k_0} \tag{19}$$

with $\gamma_2, \gamma_3 \in \mathbb{R}_+$ for $N \ge N_0(\gamma_0, \ldots, \gamma_3)$. Thus we define

$$\Phi_1 = N, \quad \Phi_2 = c_{32}d^{k_1}N,$$
$$\Psi_1(k) = c_{33}\delta^k \nu(N), \quad \Psi_2(k) = c_{34}\delta^k \nu(N), \quad \Lambda = \delta c_{33}/c_{34},$$

and if we now fix $N \in \mathbb{N}$ sufficiently large with respect to $\gamma_0, \ldots, \gamma_3, \delta, \underline{f}, \alpha$, and c_5, we put $Q_k = Q_{k,N}$ for $k_0 \le k \le k_1$ and this value of N. Then (18), (19) imply

$$\Psi_2(k) \ge c_5 \Lambda^{m_0-1}\Phi_1^{m_0-1}\max\{\Psi_1(k_0), \Phi_2\},$$

and the other assumptions of Lemma 3 are also fulfilled for this choice of parameters. The application of Lemma 3 completes the proof of Theorem 2. ■

6. Proof of Theorem 3. Under the assumptions of Theorem 3 we can give sharper bounds for the power series coefficients of $f_1, \ldots, f_m$ in the expansion at ω. This yields a weaker condition for k_0, hence a better bound for m_0.

Analogously to Section 4 we apply the Möbius transformation Φ to get $\omega = 0$. Then the sharper estimates for the power series coefficients depend on the fact that $a(z) = 1$, and $T(z)$ and the entries of $A(z)$ and $\underline{B}(z)$ are polynomials. For the sake of simplicity the case $\omega = \infty$ is excluded, because then Φ transforms the functional equation into another system, where in general $a(z)$ is not constant, and $T(z)$ is rational.

Since the proof of Theorem 3 is analogous to the proof of Theorem 2 apart from the estimates for the power series coefficients, most proofs are shortened or omitted.

Lemma 12. *Suppose that the assumptions of Theorem* 3 *are fulfilled with* $\omega = 0$. *Then for all* $h \in \mathbb{N}_0$ *and* $j \in \mathbb{N}$, $\underline{j} \in \mathbb{N}_0^m$,

(i) $f_{i,h} \in K$,

(ii) $\overline{|f_{i,h}|} \le \exp(c_{34}\log(h+2))$, $D^{[c_{34}\log(h+2)]} f_{i,h} \in O_K$,

(iii) $\overline{|f_{i,h}^{(j)}|} \le \exp(c_{35} j \log(h+2))$, $D^{[c_{35} j\log(h+2)]} f_{i,h}^{(j)} \in O_K$,

(iv) $\overline{|f_h^{(\underline{j})}|} \le \exp(c_{36}|\underline{j}|\log(h+2))$, $D^{[c_{36}|\underline{j}|\log(h+2)]} f_h^{(\underline{j})} \in O_K$,

where $D \in \mathbb{N}$, $c_{34}, c_{35}, c_{36} \in \mathbb{R}_+$, *and the algebraic number field* K *depend on* $\underline{f}$.

Proof. Without loss of generality $f_i(0) = 0$ for all i (since $f_1(0), \ldots, f_m(0) \in \overline{\mathbb{Q}}$, the functions $f_i(z) - f_i(0)$, $1 \le i \le m$, satisfy functional equations of the required form). Then with $A(z) = (a_{i,j}(z))_{1\le i,j\le m}$, $\underline{B}(z) = (B_i(z))_{1\le i\le m}$ and

$$a_{i,j}(z) = \sum_{h=0}^{s} a_{i,j,h} z^h, \qquad B_i(z) = \sum_{h=0}^{s} b_{i,h} z^h,$$
$$T(z) = \sum_{h=\delta}^{d} p_h z^h, \qquad (T(z))^l = \sum_{h=\delta^l}^{d^l} p_h^{(l)} z^h,$$

the functional equation implies

$$\begin{aligned}\sum_{h=1}^{\infty} f_{i,h} z^h &= \sum_{j=1}^{m}\Big(\sum_{h=0}^{s} a_{i,j,h} z^h\Big)\Big(\sum_{l=1}^{\infty} f_{j,l}\Big(\sum_{h=\delta^l}^{d^l} p_h^{(l)} z^h\Big)\Big) + \sum_{h=0}^{s} b_{i,h} z^h \\ &= \sum_{h=\delta}^{\infty}\Big(\sum_{j=1}^{m}\sum_{k=\max\{\delta,h-s\}}^{h} a_{i,j,h-k}\Big(\sum_{\log k/\log d \le l \le \log k/\log\delta} f_{j,l}\, p_k^{(l)}\Big)\Big) z^h \\ &\quad + \sum_{h=0}^{s} b_{i,h} z^h,\end{aligned}$$

and from the identity

$$f_{i,h} = \sum_{k=\max\{\delta,h-s\}}^{h}\sum_{j=1}^{m} a_{i,j,h-k}\Big(\sum_{\log k/\log d \le l \le \log k/\log\delta} f_{j,l}\, p_k^{(l)}\Big) + b_{i,h} \tag{20}$$

(with $b_{i,h} = 0$ for $h > s$) assertion (i) follows immediately. Since

$$\overline{|p_h^{(l)}|} \le \sum_{h_1+\ldots+h_l=h} \overline{|p_{h_1}|}\ldots\overline{|p_{h_l}|} \le \exp(\gamma_0 l)$$

(notice that $\delta \le h_i \le d$ for $i = 1, \ldots, l$), the first part of (ii) follows from (20), if we choose $D \in \mathbb{N}$ as a suitable denominator for the coefficients of $T(z)$ and the entries of $A(z)$ and $\underline{B}(z)$. Then (iii), (iv) can be derived from (9), (10) respectively (notice that the number of $\underline{h} \in \mathbb{N}_0^j$ with $|\underline{h}| = h$ is bounded by $\binom{h+j-1}{j-1} \le \exp(j \log(h+1))$. ∎

LEMMA 13. *For $N \in \mathbb{N}$ there exists a polynomial $R_N(z, \underline{y}) \in O_K[z, y_1, \ldots \ldots, y_m] \setminus \{0\}$ with the following properties*:

(i) $\deg_z R_N \le N$, $\deg_{\underline{y}} R_N \le N$,

(ii) $H(R_N) \le \exp(c_{37} N \log(N+1))$,

(iii) $c_{38} N^{1+m} \le \nu(N) = \operatorname{ord}_0 R_N(z, \underline{f}(z))$.

Proof. Analogous to Lemma 8. ∎

LEMMA 14. *For $k \in \mathbb{N}$ with $\delta^k \ge c_{39} N \log \nu(N)$,*

$$\exp(-c_{40}\nu(N)\delta^k) \le |R_N(T^k(\alpha), \underline{f}(T^k(\alpha)))| \le \exp(-c_{41}\nu(N)\delta^k),$$

where $c_{39}, c_{40}, c_{41} \in \mathbb{R}_+$ depend only on $\underline{f}$ and α.

Proof. Analogous to Lemma 9. Notice that

$$|\beta_h| \le \overline{|\beta_h|} \le \exp(\gamma_0 N \log h), \qquad D^{[\gamma_0 N \log h]}\beta_h \in O_K$$

and $h \ge \nu(N)$. ∎

Now we define polynomials $R_{k,N}$ by

$$R_{0,N}(z, \underline{y}) = R_N(z, \underline{y}),$$
$$R_{k+1,N}(z, \underline{y}) = (\det A(z))^N R_{k,N}(T(z), A(z)^{-1}(\underline{y} - \underline{B}(z))),$$

where the degree of the entries of $A(z)$ and $\underline{B}(z)$ is at most s.

LEMMA 15. *Suppose $k, N \in \mathbb{N}$. Then*

(i) $R_{k,N} \in K[z, \underline{y}]$,

(ii) $\deg_z R_{k,N} \le c_{42}(d^k - 1)/(d-1) + d^k \le 2c_{42} d^k$, $\deg_{\underline{y}} R_{k,N} \le N$,

(iii) $H(R_{k,N}) \le \exp(c_{43} N(\log(N+1) + d^k))$

with $c_{42} = sm$, $c_{43} \in \mathbb{R}_+$.

If $\delta^k \ge c_{44} N \log \nu(N)$ and $N d^k \le c_{45}\nu(N)\delta^k$, then

(iv) $\exp(-c_{46}\nu(N)\delta^k) \le |R_{k,N}(\alpha, \underline{f}(\alpha))| \le \exp(-c_{47}\nu(N)\delta^k)$.

Proof. Analogous to Lemma 10 resp. Lemma 11. ∎

Suppose that D_1 is a denominator of α, D_2 is a common denominator of the coefficients of $T(z)$, and D_3 is a common denominator of the coefficients of the entries of $A(z)$ and $\underline{B}(z)$. Then we define

$$Q_{k,N}(\underline{y}) = (D_1 D_2)^{[2c_{42} d^k N]+1} D_3^{mkN} R_{k,N}(\alpha, \underline{y}).$$

Thus for $N \geq N_0$ and $\delta^k \geq c_{48} N \log \nu(N)$ and $N d^k \leq c_{49} \nu(N) \delta^k$ we have

$$Q_{k,N} \in O_K[\underline{y}], \quad \deg Q_{k,N} \leq N, \quad H(Q_{k,N}) \leq \exp(c_{50} d^k N),$$
$$\exp(-c_{51} \delta^k \nu(N)) \leq |Q_{k,N}(\underline{f}(\alpha))| \leq \exp(-c_{52} \delta^k \nu(N)).$$

With sufficiently large $\gamma_0, \gamma_1 \in \mathbb{R}_+$, which depend on $\underline{f}$ and α, we choose

$$k_0 = \left[\frac{\log(N \log \nu(N))}{\log \delta} + \gamma_0\right], \quad k_1 = \left[\frac{\log \nu(N) - m_0 \log N}{\log d - \log \delta} - \gamma_1\right].$$

This implies $\delta^k \geq c_{48} N \log \nu(N)$ and $N d^k \leq c_{49} \nu(N) \delta^k$. Furthermore,

$$\nu(N) \delta^{k_1} \geq \gamma_2 N^{m_0} d^{k_1}$$

for $N \geq N_0(\gamma_2)$. Since $m_0 \log d < (1 - \varepsilon)(m + 1) \log \delta$ for some $\varepsilon \in \mathbb{R}_+$ and $\nu(N) \geq c_{38} N^{1+m}$, we have for all $N \geq N_0(\gamma_0, \ldots, \gamma_3, \varepsilon)$,

$$\delta^{k_1} \geq \gamma_3 N^{m_0 - 1} \delta^{k_0}.$$

Thus let

$$\Phi_1 = N, \quad \Phi_2 = c_{50} N d^{k_1},$$
$$\Psi_1(k) = c_{51} \delta^k \nu(N), \quad \Psi_2(k) = c_{52} \delta^k \nu(N), \quad \Lambda = \delta c_{51}/c_{52},$$

where N is fixed sufficiently large with respect to $\gamma_0, \ldots, \gamma_3, \varepsilon, \delta, \underline{f}, \alpha$, and c_5, and put

$$Q_k(\underline{y}) = Q_{k,N}(\underline{y})$$

for $k_0 \leq k \leq k_1$ and this value of N. Then

$$\Psi_2(k_1) \geq c_5 \Lambda^{m_0 - 1} \Phi_1^{m_0 - 1} \max\{\Psi_1(k_0), \Phi_2\},$$

and since all other assumptions of Lemma 3 are fulfilled, the assertion of Theorem 3 now follows from Lemma 3. ∎

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