ACTA ARITHMETICA
LXX.2 (1995)

Odometers and systems of numeration

by

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1. Introduction. Our starting point is the study of systems of numeration with respect to a general base from an arithmetical and a dynamical point of view. Let $G = (G_n)_{n\ge 0}$ be a strictly increasing sequence of positive integers with $G_0 = 1$. In the following such a sequence is called *G-scale*. Any positive integer n can be represented in G-scale as follows:

Let L be the unique integer satisfying $G_L \le n < G_{L+1}$. Then there exist integers $\varepsilon_L(n)$ and n_L with $n = \varepsilon_L(n)G_L + n_L$ and $0 \le n_L < G_L$. This is the greedy algorithm (see for example [Fr]) and by iteration we finally get the G-expansion of n

$$n = \varepsilon_0(n)G_0 + \ldots + \varepsilon_L(n)G_L, \tag{1.1}$$

where the digits $\varepsilon_j(n)$ satisfy $0 \le \varepsilon_j(n) < G_{j+1}/G_j$. It is well-known that the expansion (1.1) is uniquely determined provided that

$$\varepsilon_0(n)G_0 + \ldots + \varepsilon_j(n)G_j < G_{j+1}, \quad 0 \le j \le L. \tag{1.2}$$

A lot of special examples of such expansions have been studied in the literature. The classical case is the q-ary number system with respect to an integral base $q \ge 2$. A well-known extension is Cantor's number system, where G_n is given as the product $q_0 \ldots q_n$ of positive integers; see [HW] and [KT]. Another important number system was introduced by Ostrowski [Os]. In that case G_n is the denominator of the nth convergent in the continued fraction expansion of an irrational real number θ. Later this expansion played an important rôle in proving precise estimates and exact formulas for the discrepancy of the sequence $(n\theta)$; cf. [Dup], [Sg], and [S/os]. A particular case is the golden ratio $\theta = (\sqrt{5}-1)/2$ which leads to the Fibonacci sequence. This example can also be viewed as a special linear recurring sequence for G. This class of expansions has been studied extensively from a number-

The second author is partially supported under contract 901636/A000/DRET/DS/SR. The other authors are supported by Austrian Science Foundation project P8274-PHY.

theoretic point of view. The investigations in this direction were initiated by J. Coquet (see for example [Co] and the survey [Li1] on his work). One motivation is the study of discrepancy estimates for sequences of the type $(s(n)\theta)$ where $s(n) = s_G(n)$ denotes the sum of digits of n to base G, i.e.

$$s_G(n) = \sum_{j \le L} \varepsilon_j(n).$$

Another motivation is the problem to find the asymptotic behaviour of the moments of the sum of digits function. The most recent papers in this direction are [GT1], [GT2], [FGKPT], [GKPT] and [Be]. In connection with substitution automata we refer to the very recent article [DT1], [DT2] by Dumont and Thomas.

The most popular sequence in this area is the Thue–Morse sequence t, which can be defined as the sum of digits to base 2 viewed mod 2. From a dynamical point of view this sequence was first investigated by M. Keane. The function $f(n) = e^{2\pi i \theta s(n)}$ ($\theta \in \mathbb{R}$) is G-multiplicative, which means in general

$$f(n) = \prod_{j=0}^{L} f(\varepsilon_j(n) G_j), \qquad f(0) = 1. \tag{1.3}$$

Let us consider a sequence $u = (u_n)$ in a compact metric space X. Let Ω be the infinite product space X^N and let σ be the usual shift transformation on Ω and for $u \in \Omega$ define the orbit closure $\mathcal{O}_u = \overline{\{\sigma^n u : n \ge 0\}}$. Then the dynamical system

$$\mathcal{K}(u) = (\mathcal{O}_u, \sigma|_{\mathcal{O}})$$

can be associated with the sequence u. One of the most natural questions is to ask for the properties of such dynamical systems. For example, in the case of the Thue–Morse sequence $\mathcal{K}(t)$ is metrically isomorphic to a two-point group extension of the classical 2-adic machine. For Cantor expansions a similar result can be proved replacing the two-point group by a closed subgroup of the circle (see [Li2]). In this paper we introduce the general concept of a G-adic machine (we will also call it a G-*odometer*) which serves us to understand the dynamics of G-multiplicative sequences. There is a strong connection to the Bratteli diagram introduced by A. M. Vershik [Ve]. Recently B. Solomyak [So1]–[So3] used this approach to give a full description of G-odometers in the case where G comes from a special expansion based on a linear recurring sequence.

In Section 2 we will define the general G-odometer and we will establish a necessary and sufficient condition for the continuity. In Section 3 we will investigate in detail expansions with respect to linear recurrences. In particular, for second-order linear recurring sequences we identify the G-

odometer to a rotation on the circle. In the final Section 4 we study the discrepancy of some related sequences.

2. General G-odometers. Let $E_j = \{k \in \mathbb{N} : 0 \le k < G_{j+1}/G_j\}$ be endowed with the discrete topology. The G-expansion of positive integers leads to a natural injective mapping $n \mapsto \overline{n}$ from $\mathbb{N}$ to the infinite product space $\prod_{j\ge 0} E_j$ given by

$$\overline{n} = \varepsilon_0(n)\ldots\varepsilon_L(n)0^\infty \tag{2.1}$$

according to the G-expansion (1.1) of n (here the string $\overline{n}$ ends with an infinite sequence of digits 0). Now we consider the closure of the image $\overline{\mathbb{N}}$ in E which is the set

$$\mathcal{K}_G = \{x = (x_0x_1x_2\ldots) \in E : \forall\, j \ge 0,\ x_0G_0 + \ldots + x_jG_j < G_{j+1}\}. \tag{2.2}$$

The infinite strings in $\mathcal{K}_G$ will be called G-*admissible* and we extend this definition to any finite string $X = x_0\ldots x_n$, if $X0^\infty$ is G-admissible.

Obviously, $\mathcal{K}_G$ is compact and it will be called the G-*compactification* of $\mathbb{N}$. In the following we use the notation $x(j) = x_0G_0 + \ldots + x_jG_j$. Now we want to extend the translation $\overline{n} \mapsto \overline{n+1}$ on $\overline{\mathbb{N}}$ to $\mathcal{K}_G$. For this purpose we introduce the set

$$\mathcal{K}_G^0 = \{x \in \mathcal{K}_G : \exists\, M_x, \forall\, j \ge M_x\ x(j) < G_{j+1} - 1\}. \tag{2.3}$$

For $x \in \mathcal{K}_G^0$ and $j \ge M_x$ let us set

$$\tau(x) = (\varepsilon_0(x(j)+1)\ldots\varepsilon_j(x(j)+1))x_{j+1}x_{j+2}\ldots \tag{2.4}$$

A straightforward computation shows that this definition does not depend on the choice of $j \ge M_x$. In fact, let l be the greatest integer such that $x(l-1)+1 = G_l$ provided that such an l exists; otherwise there is no carry and we just add one to the first digit. Then for all $j \ge l$ we have

$$\begin{aligned}(x(j)+1) &= (\varepsilon_0(x(l)+1)\ldots\varepsilon_l(x(l)+1))x_{l+1}\ldots x_j\\ &= 0^l(x_l+1)x_{l+1}\ldots x_j.\end{aligned} \tag{2.5}$$

We extend the definition of τ by $\tau(x) = 0$ $(= 000\ldots)$ for $x \in \mathcal{K}_G \setminus \mathcal{K}_G^0$. Now the transformation τ is well defined on the space $\mathcal{K}_G$ and it is called the G-*odometer*. We need to give a precise description of the G-expansion. Let $x \in \mathcal{K}_G$ and let $D(x) = (d_n)_{n\ge 0}$ denote the increasing sequence of all integers d such that $x(d) = G_{d+1} - 1$. Note that $D(x)$ may be empty, finite or infinite. The number of elements in the sequence $D(x)$ will be called its *length*. From the definition we easily obtain

$$x \in \mathcal{K}_G^0 \Leftrightarrow D(x) \text{ is finite or empty.} \tag{2.6}$$

Proposition 1. (i) *If $D(x) = (d_0,\ldots,d_s)$ is finite, then $x = B_0B_1\ldots$ $\ldots B_sX^{(d_s+1)}$ with the notation $X^{(m)} = x_mx_{m+1}\ldots$, where the strings B_j*

are given by:

$$(2.7)\qquad \begin{gathered} B_0 0^\infty = \overline{G_{d_0+1}-1}, \qquad B_0\ldots B_j 0^\infty = \overline{G_{d_j+1}-1}, \\ 0^{d_{j-1}+1} B_j 0^\infty = \overline{G_{d_j+1}-G_{d_{j-1}+1}} \end{gathered}$$

for $0<j\le s$. *Moreover*,

$$(2.8)\qquad \tau(x) = 0^{(d_s+1)}(x_{d_s+1}+1)X^{(d_s+2)}.$$

(ii) *If* $D(x)=(d_0,d_1,\ldots)$ *is infinite then* $x = B_0B_1\ldots$, *where the* B_j *satisfy* (2.7) *for all* $j\ge 0$ *and* $\tau(x)=0$.

(iii) *The map* τ *is injective on* $\mathcal{K}_G^0$.

(iv) *The map* τ *is surjective if and only if* $\tau^{-1}(0)\neq\emptyset$.

Proof. (i) If $0\le n<d_0$ then clearly, $\overline{x(n)} = x_0\ldots x_n$ and $\overline{x(n)+1} = (x_0+1)x_1\ldots x_n$. If $d_m\le n<d_{m+1}$, then

$$x(n) = x_0G_0+\ldots+x_{d_m}G_{d_m}+\ldots+x_nG_n = (G_{d_m+1}-1)+\sum_{d_m<j\le n} x_jG_j.$$

Therefore

$$(2.9)\qquad x(n)+1 = (x_{d_m+1}+1)G_{d_m+1}+\sum_{d_m+1<j\le n} x_jG_j.$$

We claim that (2.9) is a G-expansion. If this were not the case there would exist an integer k with $d_m+1\le k\le n$ such that

$$x(k)+1 = (x_{d_m+1}+1)G_{d_m+1}+\sum_{d_m+1<j\le k} x_jG_j \ge G_{k+1}.$$

But in fact $x(k)+1<G_{k+1}$, a contradiction, and so (2.9) is a G-expansion. It follows that

$$(2.10)\qquad G_{d_j+1}-G_{d_{j-1}+1} = \sum_{d_{j-1}<l\le d_j} x_lG_l.$$

This expansion is a G-expansion and corresponds to the string $0^{(d_{j-1}+1)}B_j0^\infty$ which serves us to define the string $B_j = x_{d_{j-1}+1}\ldots x_{d_j}$ and we get

$$\overline{x(n)} = B_0\ldots B_{d_m}x_{d_m+1}x_{d_m+2}\ldots x_n0^\infty.$$

In particular, $B_0\ldots B_j0^\infty = \overline{G_{d_j+1}-1}$. The same computation works for $n\ge d_s$ and in that case we also obtain

$$\overline{\tau(x)(n)} = 0^{(d_s+1)}(x_{d_s+1}+1)x_{d_s+2}\ldots x_n0^\infty.$$

From this (2.8) follows immediately.

(ii) follows by the same arguments as (i) by taking into account that $D(x)$ is an infinite string.

For (iii) let $x = B_0B_1\ldots B_sX^{(d_s+1)}$ and $x' = B_0'B_1'\ldots B_{s'}'X'^{(d_{s'}'+1)}$ be given such that $\tau(x)=\tau(x')$. From (2.8) we have $d_s = d_{s'}'$, $X^{(d_s+1)} =$

$X'^{(d'_{s'}+1)}$, and since $B_0B_1\ldots B_s0^\infty$ corresponds to the G-expansion of $G_{d_s+1} - 1$ we obtain

$$B_0B_1\ldots B_s = B'_0B'_1\ldots B'_{s'}.$$

Thus injectivity is proved.

To show (iv) we only have to prove sufficiency. Let $x \in \mathcal{K}_G \setminus \mathcal{K}_G^0$. Then $x = B_0B_1\ldots$ where B_00^∞ corresponds to the G-expansion of $G_{d_0+1} - 1$. Let B'_00^∞ be the G-expansion of $G_{d_0+1} - 2$ and take $x' = B'_0B_1B_2\ldots \in \mathcal{K}_G$ (clearly this is a G-expansion). Thus by construction $\tau(x') = x$.

Now let $x \in \mathcal{K}_G^0$. If $D(x)$ is not empty then according to (i) we can write $x = B_0B_1\ldots B_sX^{(d_s+1)}$. Define B'_0 as above and take $x' = B'_0B_1\ldots \ldots B_sX^{(d_s+1)}$ and we get $\tau(x') = x$. It remains to consider the case where $D(x)$ is empty. For $x = 0$ just notice that $\tau^{-1}(0)$ is not empty by assumption. If $x \neq 0$ then x has the form

$$x = 0^{(k)}x_kx_{k+1}\ldots, \qquad x_k > 0.$$

Now take $x = X'(x_k - 1)x_{k+1}\ldots$ where $\overline{G_k - 1} = X'0^\infty$. Then again $\tau(x') = x$, which yields surjectivity and the proof of the proposition is complete. ∎

Example 1. In the classical case of the q-adic number system $G_n = q^n$ it is easy to see that $\mathcal{K}_G$ corresponds to the group of q-adic integers and τ is just addition of 1 in this group. Note that $D(-1) = \{0, 1, 2, \ldots\}$ since the G-expansion of -1 is $(q-1)^\infty$.

Example 2. Take $G_n = 2^{n+1}-1$. Then $G_{n+1}-1 = 2G_n$ and $\overline{G_{n+1} - 1} = 0^{(n)}2$. Then the cardinality of $D(x)$ is not greater than 1. If $D(x) = d$ then

$$x = 0^{(d)}2x_{d+1}x_{d+2}\ldots$$

with $x_j \neq 2$ for all $j \geq d+1$. If $D(x) = \emptyset$ then $x = x_0x_1\ldots$ with $x_j \neq 2$ for all j.

Thus, for this example we have $\mathcal{K}_G^0 = \mathcal{K}_G$ and $\tau^{-1}(0) = \emptyset$.

Example 3. In the Cantor expansions $G_n = q_0\ldots q_n$ the set $\mathcal{K}_G$ corresponds to the group of general G-adic integers (cf. for example [HR]). In that case we have

$$\tau^{-1}(0) = \{(q_1 - 1)(q_2 - 1)\ldots\}.$$

Example 4. In Ostrowski's number system with respect to an irrational θ given in continued fraction expansion $\theta = [0; a_1, a_2, \ldots]$ we have

$$G_1 = a_1, \qquad G_{n+1} = a_{n+1}G_n + G_{n-1}.$$

This yields $G_{2n} - 1 = (G_2 - G_0) + (G_4 - G_2) + \ldots + (G_{2n} - G_{2n-2})$ and we obtain

$$\overline{G_{2n} - 1} = 0a_20\ldots a_{2n-2}0a_{2n}.$$

Similarly we get

$$\overline{G_{2n+1}-1} = (a_1-1)0a_30\ldots a_{2n-1}0a_{2n+1}.$$

This odometer is extensively studied in [Li3]. In particular, it is proved that $\tau^{-1}(0)$ contains two points θ_1 and θ_2 satisfying $D(\theta_i)=\{2n+i : n\geq 0\}$ for $i=1,2$.

We compare the G-odometer with the adding shift introduced by Vershik [Ve]. These notions are both similar and quite identical in some special case. In particular, Example 4 could also be understood via the adding shift. We define the partial ordering $\prec$ on $\mathcal{K}_G$ as follows. Let $x=x_0x_1\ldots$ and $y=y_0y_1\ldots$ be elements in $\mathcal{K}_G$. Then $x\prec y$ if and only if $x=y$ or there exists an integer $k\geq 0$ such that $x_k<y_k$ and $x_j=y_j$ for all $j>k$.

Lemma 1. *For positive integers m, n one has*:

$$m\leq n \Leftrightarrow \overline{m}\prec\overline{n}.$$

P r o o f. Let m and n be two different positive integers. Let k be the integer defined by $\varepsilon_k(m)\neq\varepsilon_k(n)$ and $\varepsilon_j(m)=\varepsilon_j(n)$ for all $j>k$. By the greedy algorithm one has $\varepsilon_0(m)G_0+\ldots+\varepsilon_k(m)G_k<(\varepsilon_k(m)+1)G_k$. Then it easily follows that $m<n$ if and only if $\varepsilon_k(m)<\varepsilon_k(n)$. This completes the proof. ∎

Now we may replace (2.2) by

$$x=x_0x_1\ldots\in\mathcal{K}_G \Leftrightarrow \forall\, k\in\mathbb{N},\ x_0\ldots x_k0^\infty\prec\overline{G_{k+1}-1}. \tag{2.2a}$$

Lemma 2. *Let $x=x_0x_1\ldots$ and $y=y_0y_1\ldots$ be in $\mathcal{K}_G$ such that $x\neq y$ but $x\prec y$. Let k be the integer defined by $x_k<y_k$ and $x_j=y_j$ for all $j>k$. Then the interval $[x,y]=\{z\in\mathcal{K}_G : x\prec z\prec y\}$ contains $y(k)-x(k)+1$ points given by all infinite G-admissible strings z defined by $x(k)\leq z(k)\leq y(k)$ and $z_j=x_j$ for all $j>k$.*

P r o o f. Essentially, we have to prove that if z is an infinite string such that $z_0\ldots z_k$ is G-admissible, $x(k)\leq z(k)\leq y(k)$ and $z_j=x_j$ for all $j>k$ then z is G-admissible. But $z(j)\leq y(j)<G_{j+1}$ for all $j>k$. Therefore $z(j)<G_{j+1}$ for all $j\geq 0$, as expected. ∎

By the above lemma, if x is not maximal in $\mathcal{K}_G$ then the interval $[x,\rightarrow)$ $=\{z\in\mathcal{K}_G : x\prec zx\neq z\}$ is not empty, totally ordered and we can define the successor x^+ of x, namely $x^+=\min\{z\in[x,\rightarrow) : x\neq z\}$.

Proposition 2. *$\tau^{-1}(0)$ is the set of maximal points of $(\mathcal{K}_G,\prec)$ and each $x\in\mathcal{K}_G^0$ has a successor given by $x^+=\tau(x)$.*

P r o o f. Clearly if $x\prec y$ in $\mathcal{K}_G$ with $x\neq y$ then $x(j)<y(j)$ for all j large enough. This implies that $D(x)$ is finite, $x^+=\tau(x)$ and elements in $\tau^{-1}(0)$ are maximal. It remains to prove that if x is a maximal element, then $D(x)$

is infinite. Assume otherwise; then directly from the above lemma we have $x \prec \tau(x)$, a contradiction. ∎

Our main results in this section concern a criterion for the continuity of the odometer and its minimality, the latter meaning that the only closed subsets F of $\mathcal{K}_G$ such that $\tau(F) \subset F$ are the empty set and the full space $\mathcal{K}_G$. Let Δ be the set of finite (or empty) sequences δ such that there exists $x \in \mathcal{K}_G^0$ satisfying $D(x) = \delta$.

THEOREM 1. *The G-odometer τ is continuous if and only if for all $(d_0, d_1, \ldots, d_k) \in \Delta$ the set $\{d > d_k : (d_0, d_1, \ldots, d_k, d) \in \Delta\}$ is finite.*

Proof. First we prove the sufficiency of the above condition. Using the notation of Proposition 1 let $x = B_0B_1\ldots$ be in $\tau^{-1}(0)$ and let y be close to x. Then $y = B_0B_1\ldots B_sY_s$ with large s and $\tau(y) = 0^{(j)}\ldots$ with $j \geq d_s$. Therefore τ is continuous at the points of $\tau^{-1}(0)$.

Now we take $x \in \mathcal{K}_G^0$. Using again the notation of Proposition 1, $D(x) = (d_0, \ldots, d_s)$ and $x = B_0B_1\ldots B_sX^{(d_s+1)}$ or $D(x)$ is empty. If y is close enough to x and $D(x) \neq \emptyset$ then the sequence $D(y)$ starts with $d_0, \ldots, d_s, d(y)$, where $d(y)$ can be omitted. If $D(x) = \emptyset$ then either $D(y) = \emptyset$ or $D(y) = (d(y), \ldots)$. In the latter case $d(y)$ is bounded by assumption, uniformly in y. Therefore we can choose y close enough to x such that $D(x) = D(y)$, and the continuity at x follows easily.

The necessity is proved by contradiction. Assume that there exists $(d_0, \ldots, d_k)$ in Δ (this sequence is possibly empty) and an infinity of integers d such that $(d_0, \ldots, d_k, d) \in \Delta$. Thus we can choose a sequence of elements $y^{(n)}$ in $\mathcal{K}_G$ such that

$$D(y^{(n)}) = (d_0, \ldots, d_s, \delta_n)$$

with $\delta_n < \delta_{n+1}$ for all n and $y^{(n)}$ converges to an element y. By construction $D(y) = D(x)$, so that $\tau(y) \neq 0$ but $\lim_{n\to\infty} \tau(y^{(n)}) = 0$. This is a contradiction and Theorem 1 is proved. ∎

Remark 1. In our examples above the odometer is always continuous except in Example 2 where it is also not surjective. In the next section we give another example of a non-continuous odometer which is surjective (its set $\tau^{-1}(0)$ is not empty).

THEOREM 2. *Assume that the G-odometer is continuous. Then it is also surjective and minimal.*

Proof. Let $\sigma : \Delta \to \mathbb{N}^{\mathbb{N}}$ be defined by $\sigma(\delta) = (0, 0, \ldots)$ if the length of $\delta \in \Delta$ is ≤ 1, and $\sigma(d_0, d_1, \ldots, d_k) = (d_1 - d_0, \ldots, d_k - d_{k-1}, 0, 0, \ldots)$ if $k \geq 2$. Assume that the G-odometer τ is continuous. The criterion of continuity in Theorem 1 implies the following: For all integers $k \geq 0$, there exists an integer $m_k \geq 0$ such that

$$\forall \delta \in \varDelta, \quad \sigma(\delta)_k \leq m_k.$$

This means that $\sigma(\varDelta) \subset \varOmega = \prod_{k=0}^{\infty}\{0,1,\ldots,m_k\}$. By a compactness argument there exists a convergent sequence $y^{(n)}$ in $\mathcal{K}_G$ such that the length of $D(y^{(n)})$ tends to infinity, all sequences $D(y^{(n)})$ begin with the same two consecutive values d_0, d_1 and the sequence $n \mapsto \sigma(D(y^{(n)}))$ converges in $\varOmega$, say to $s = (s_k)_{k\geq 0}$, with $s_k \geq 1$ for all $k \geq 0$. This implies that if $y = \lim_{n\to\infty} y^{(n)}$ then $D(y) = (d_0, d_0 + s_0, \ldots, d_0 + s_0 + \ldots + s_k, \ldots)$. In other words, $y \in \tau^{-1}(0)$ and the surjectivity follows from Proposition 1(iv).

It remains to prove the minimality. To this end we show that for all $x \in \mathcal{K}_G$, the orbit $\{x, \tau(x), \tau^2(x), \ldots\}$ is dense. Let the interval $[x, \rightarrow)$ be finite. By Lemma 2 there exists $y \in \mathcal{K}_G$ such that $[x, \rightarrow) = [x, y]$. But y must be maximal and by Proposition 2, $\tau^m(x) = y$ for an integer $m \geq 1$. Therefore $\tau^{m+1}(x) = 0$ and the orbit of x under τ contains the dense set $\overline{\mathbb{N}}$. Now assume that $[x, \rightarrow)$ is infinite. Then for any $L \geq 0$ there exists $z \in [x, \rightarrow)$ such that the integer $l(z) = l$ defined by $x_l < z_l$ and $x_j = z_j$ for all $j > l$ satisfies $l \geq L$. It is clear that we can construct a sequence $y^{(n)}$ in $[x, \rightarrow)$ with $l(y^{(n)}) < l(y^{(n+1)})$. Put $l_n = l(y^{(n)})$ for short. Then the infinite string

$$z^{(n)} = \overline{G_{l_n} - 1}\, x_{l_n} x_{1+l_n} \ldots x_{k+l_n} \ldots$$

is G-admissible and in fact $x \prec z^{(n)} \prec y^{(n)}$. Taking a subsequence if necessary we may assume, by the same argument as above, that both sequences $\sigma(D(z^{(n)}))$ and $z^{(n)}$ converge and the limit of $z^{(n)}$ belongs to $\tau^{-1}(0)$. From this fact and the continuity of τ we derive the existence of a non-decreasing sequence of integers k_n such that

$$\lim_{n\to\infty} \tau^{k_n}(x) = 0.$$

Again by continuity, for any given positive integer m the sequence $\tau^{m+k_n}(x)$ converges to $\overline{m}$ $(= \tau^m(0))$. This proves that the orbit closure of x contains $\overline{\mathbb{N}}$ and finally the orbit is dense. ■

3. Systems of numeration with respect to linear recurrences.

Let $\alpha > 1$ be a real number. Then Parry's α-expansion (cf. [Pa]) of an arbitrary real number x is given by

$$x = \xi_0 + \frac{\xi_1}{\alpha} + \frac{\xi_2}{\alpha^2} + \ldots, \tag{3.1}$$

where $\xi_0 = [x]$, the greatest integer $\leq x$, and the other digits $\xi_1, \xi_2, \ldots$ can be computed in the usual way with the help of the transformation $Tx = \{\alpha x\}$ ($\{x\} = x - [x]$). The uniqueness of the representation (3.1) is guaranteed by the following requirements on the digits ξ_j:

$$(\xi_n, \xi_{n+1}, \dots) < (a_0, a_1, \dots) \quad \text{for } n \ge 1, \tag{3.2}$$

where the a_j are the digits of α in α-expansion, i.e.

$$\alpha = a_0 + \frac{a_1}{\alpha} + \frac{a_2}{\alpha^2} + \dots$$

(in the case of ambiguity we take the infinite representation of α) and "$<$" denotes the lexicographical order (cf. [Pa]). A more general version of digit expansions was already studied by A. Rényi [Ré]. Furthermore, we note that digital properties such as periodicity and finiteness of expansions were extensively studied in the literature (cf. [Be], [F1], [F2], [F3], [FS], [Sch]).

We consider the digit expansion of integers with respect to the linear recurrence

$$G_{n+1} = \sum_{k=0}^{n} a_{n-k} G_k + 1, \qquad G_0 = 1. \tag{3.3}$$

As in [GT2] we introduce the generating functions

$$A(z) = \sum_{n=0}^{\infty} a_n z^n, \quad G(z) = \sum_{n=0}^{\infty} G_n z^n$$

satisfying

$$G(z) = \frac{1}{(1-z)(1-zA(z))}.$$

Note that by the theorem of Pólya–Carlson $A(z)$ is either rational and therefore the string $(a_0, a_1, \dots)$ is finally periodic or has the unit circle as its natural boundary. We obtain the asymptotic formula

$$G_n \sim C\alpha^n,$$

where C can be easily computed by residue calculus. Hence the sequence of integers G_n is strongly connected with the α-expansion. Let

$$n = \sum_{j=0}^{L} \varepsilon_j G_j$$

be the G-expansion. Then the digits $\varepsilon_j = \varepsilon_j(n)$ satisfy

$$(\varepsilon_k, \varepsilon_{k-1}, \dots, \varepsilon_0, 0, 0, \dots) < (a_0, a_1, \dots) \quad \text{for } k = 0, \dots, L, \tag{3.4}$$

where $L = L(n)$ is chosen such that $G_L \le n < G_{L+1}$. These are just the finite admissible blocks as defined in the previous section. From (3.4) it immediately follows that the inequalities (1.2) are satisfied in this case. Thus the blocks of digits given by the (uniquely determined) G-expansion correspond to the admissible blocks. In this paper we will show that for our purposes only the case of periodic sequences $(a_0, a_1, \dots)$ is interesting. In this case the sequence G_n is generated by a finite linear recurrence of order $d+1$, where $d+1$ is the period length:

$$G_{n+d+1} = a_0 G_{n+d} + a_1 G_{n+d-1} + \ldots + (a_d + 1) G_n \quad \text{for } n \geq 0 \tag{3.5}$$

and the initial values are given by (3.3). This follows immediately from the generating functions since finite recurrences correspond to rational functions; of course it also follows from the recurrence (3.3). For a detailed discussion concerning the initial values of such finite recurrences we refer to our earlier paper [GT1].

The main aim of this section is the investigation of this G-odometer. Let us recall that the set of all infinite admissible sequences is given by

$$\mathcal{K} = \mathcal{K}_G = \{(x_0, x_1, \ldots) \in E : x_0 G_0 + \ldots + x_j G_j < G_{j+1} \ \forall j \geq 0\},$$

and for short we put $\mathcal{K}^0 = \mathcal{K}^0_G$.

Now we present a special G-expansion with non-continuous odometer.

Example 5. Let $(a_0, a_1, \ldots) = (2, 1, 1, \ldots)$ define the sequence G_n and

$$\xi_n = (\underbrace{1, \ldots, 1}_{n}, 2, 0, 0, \ldots) \to (1, 1, \ldots),$$

$$\tau(\xi_n) = (\underbrace{0, \ldots, 0}_{n+1}, 1, 0, 0, \ldots) \to (0, 0, \ldots),$$

$$\tau((1, 1, \ldots)) = (2, 1, 1, \ldots).$$

Thus τ is not continuous.

The above example shows that it is necessary to establish conditions that ensure continuity. In the following we establish a continuity criterion for this special type of expansions which restates Theorem 1 in a different form.

Theorem 3. *Let $\xi_n = a_n a_{n-1} \ldots a_0 0^\infty$. Then τ is continuous if and only if all accumulation points of ξ_n are in $\mathcal{K} \setminus \mathcal{K}_0$.*

Proof. Let τ be continuous. Then obviously all accumulation points of ξ_n are contained in $\mathcal{K} \setminus \mathcal{K}_0$ (see the above example). For the converse direction we assume that all accumulation points of ξ_n are in $\mathcal{K} \setminus \mathcal{K}_0$. First we prove the continuity of τ in $\mathcal{K}_0$. Let $x \in \mathcal{K}_0$, $\tau(x) = z = (z_0 z_1 \ldots)$ and consider

$$\tau^{-1}(\{(z_0, z_1, \ldots, z_n, y_{n+1}, y_{n+2}, \ldots) \in \mathcal{K}\}),$$

where $n > M_x$ and $y_{n+1}, y_{n+2}, \ldots$ are arbitrary. This set is open since the digits y_j are not affected by τ^{-1} and there is no accumulation point in $\mathcal{K}_0$. Thus τ is continuous on $\mathcal{K}_0$. To prove continuity on $\mathcal{K} \setminus \mathcal{K}_0$ we only have to consider neighbourhoods of 0. As

$$\tau^{-1}(\{(0^{k+1}, y_{k+1}, \ldots)\}) = \bigcup_{l=k}^{\infty} \{(a_l, \ldots, a_0, y_{l+1}, \ldots) \in \mathcal{K}\}$$

is open, τ is continuous on $\mathcal{K} \setminus \mathcal{K}_0$. ∎

THEOREM 4. *τ is continuous if and only if $(a_0, a_1, \ldots)$ is periodic (i.e. the sequence G_n is a finite recurrence).*

Proof. Applying Theorem 3 we have to show that in the case of an aperiodic string $(a_0, a_1, \ldots)$ at least one accumulation point of $\xi_n = a_n a_{n-1} \ldots \ldots a_0 0^\infty$ is in $\mathcal{K}_0$. Define k_n to be the minimal integer such that $a_{n-k_n} = a_0$, $a_{n-k_n+1} = a_1, \ldots, a_n = a_{k_n}$. If k_n takes arbitrarily large values we consider a subsequence n_j such that $k_{n_j} \to \infty$. Then ξ_{n_j} has an accumulation point in $\mathcal{K}_0$.

Suppose now that k_n is bounded. Then we have subsequences $n_j^{(i)}$ with $i = 0, \ldots, K$ such that $k_{n_j^{(i)}} = i$. At least one of these sequences has to have an infinity of terms. Let l be the maximal index of a sequence with an infinity of terms. We write $n_j = n_j^{(l)}$ for short. Then we have $a_{n_j-l} = a_0, \ldots, a_{n_j} = a_l$. By our hypothesis we have $k_{n_j+1} \leq l$ for j sufficiently large. Thus we have $a_{n_j+1} < a_{l+1}$ $(k_{n_j+1} > 0)$ or $a_{n_j+1} = a_0$ $(k_{n_j+1} = 0)$. The first case would imply that $a_{n_j-k_{n_j+1}+1} = a_0, \ldots, a_{n_j+1} = a_{k_{n_j+1}} < a_{l+1}$ which is a contradiction to the lexicographic condition. Repeating this procedure yields the periodicity of the sequence $(a_0, a_1, \ldots)$. ∎

Remark 2. If $d+1$ is the period length, then ξ_n has exactly $d+1$ accumulation points:

$$a_l \ldots a_0 (a_d, \ldots, a_0)^\infty \quad \text{for } l = 0, \ldots, d.$$

These are the elements of $\tau^{-1}(0)$.

Remark 3. The proof of Theorem 4 shows that all accumulation points of ξ_n are contained in $\mathcal{K}^0$ provided that one is contained in $\mathcal{K}^0$. To give an example, we construct such a string: start with some string $a_0 a_1 \ldots a_k$, add one digit 0 and repeat the whole string, then add two digits 0 and repeat the whole string, add three digits 0 and so on.

From now on we only consider recurrences G_n leading to continuous τ, i.e. the case of finite recurrences.

PROPOSITION 3. *The sequence $D(x)$ is non-empty iff*

$$x = a_l a_{l-1} \ldots a_0 (a_d \ldots a_0)^k B,$$

where $l = 0, \ldots, d$, $k \in \mathbb{N}_0$ and B is a block not starting with $a_d \ldots a_0$. In this case we have

$$D(x) = \{l, l+d+1, \ldots, l+k(d+1)\}.$$

In the case of the accumulation points we have

$$D(a_l \ldots a_0(a_d \ldots a_0)^\infty) = \{l + k(d+1) : k = 0, 1, \ldots\}.$$

Proof. Simple computations. ∎

Next we will prove that the odometer is a uniquely ergodic transformation. For this purpose we need the following lemma, the proof of which is given later in a slightly different context (cf. Section 4, Proposition 4).

LEMMA 3. *Let $f_l : \mathbb{N} \to \mathbb{C}$ be arbitrary number-theoretic functions and let $f : \mathcal{K} \to \mathbb{C}$ be a function satisfying*

$$f\Big(\sum_{l=0}^{L} \varepsilon_l G_l\Big) = \prod_{l=0}^{K} f_l(\varepsilon_l),$$

where (*in the case $L < K$*) *leading* 0*'s are considered in the evaluation of f. Then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=m}^{N+m-1} f(n) = C_f \tag{3.6}$$

uniformly in m.

Sketch proof. Let

$$F_k = \sum_{n<G_k} f(n).$$

Then it is easy to see that

$$F_{k+d+1} = a_0 F_{k+d} + \ldots + (a_d + 1) F_k \tag{3.7}$$

for $k > K$, which is the recurrence of G_k. Therefore the limit $\lim_{k\to\infty} F_k/G_k$ exists, since by the positivity of coefficients and simple estimates there exists one positive dominating root. By an argument used in Proposition 4 of Section 4 this implies the existence of

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n<N} f(n)$$

and also the uniformity in (3.6). ∎

THEOREM 5. *The odometer τ is a uniquely ergodic transformation*, *i.e. there is a unique invariant measure μ given by*

$$\mu(Z) = \frac{F_{K+1}\alpha^d + (F_{K+2} - a_0F_{K+1})\alpha^{d-1} + \ldots + (F_{K+d+1} - a_0F_{K+d} - \ldots - a_{d-1}F_{K+1})}{\alpha^K(\alpha^d + \alpha^{d-1} + \ldots + 1)},$$

where

$$F_k = \sum_{n<G_k} \chi_Z(n),$$

Z is a cylinder with fixed digits $\varepsilon_0 \ldots \varepsilon_K$ and χ_Z denotes the characteristic function of Z.

Proof. Using the fact that the functions f considered in Lemma 3 are dense in the space of continuous functions on $\mathcal{K}$ and a criterion for unique ergodicity (cf. [Wa, Theorem 6.19]) yields the first part of the theorem. Inserting the function χ_Z, where Z is a cylinder set, in Lemma 3 yields the same recurrence for F_k as for G_k (compare (3.7)). An easy application of generating functions can be used to compute $\lim_{k\to\infty} F_k/G_k$ which is $\mu(Z)$ by Lemma 3. ■

In the following we prove that under a certain hypothesis the odometer has purely discrete spectrum. For linear recurrences with decreasing coefficients Solomyak [So2] proved that result. Our approach is related but somewhat different and we use the following two lemmata.

LEMMA 4. *Let*

$$B = \begin{pmatrix} 1 & 1 \\ 1 & C \end{pmatrix}$$

be a matrix with positive entries such that all entries of the first row and column are equal to 1. *Let β be the maximun eigenvalue of B and γ the maximum eigenvalue of C. Assume that there exists $n > 0$ with $B^n > 0$* (*componentwise*). *Then $\gamma < \beta$.*

Proof. This is a standard application of the Perron–Frobenius theorem. ■

LEMMA 5. *For integers b, k and l with $0 \le k < l$, $b > 0$ define the set*

$$E_k(l,b) := \{x \in \mathcal{K}_G : \exists s \text{ with } k \le s \le l \text{ and } x_{s(b+1)} \ldots x_{(s+1)(b+1)-1} = 0^{(b+1)}\}.$$

Then there exist absolute constants c and ϱ with $0 < \varrho < 1$ such that for all k and l $(0 \le k < l)$ we have

$$\mu(E_k(l,b)) \ge 1 - c\varrho^l,$$

where μ denotes the measure given in Theorem 5.

Proof. It follows from the periodicity of the sequence (a_n) and (2.2a) that a point $x = (x_0x_1\ldots)$ belongs to $\mathcal{K}_G$ if and only if for all $k \ge 0$ the strings $x_k \ldots x_{k+b}$ are G-admissible (see [Pa] for details). Let $\mathcal{W}$ be the set of G-admissible strings $W = w_0 \ldots w_b$ of length $b+1$. Let $\mathcal{W}$ be ordered by the lexicographic order and let B be the matrix whose entries are

$$B_{W_iW_j} = \begin{cases} 1 & \text{if } W_iW_j0^\infty \in \mathcal{K}_G, \\ 0 & \text{otherwise.} \end{cases}$$

Thus any string x in $\mathcal{K}_G$ can be written as an infinite string $x=(X_0X_1X_2\ldots)$ over the alphabet $\mathcal{W}$ such that $B_{X_jX_{j+1}} = 1$ for $j = 0, 1, 2, \ldots$ The matrix

B has the form $\left(\begin{smallmatrix}1 & 1\\ 1 & C\end{smallmatrix}\right)$. Let $\mathcal{L}_l(C)$ be the set of finite strings $X = X_0 \dots X_l$ over the alphabet $\mathcal{W}$ such that $C_{W_j W_{j+1}} = 1$ for $j = 0, \dots, l-1$. By the unique ergodicity of the odometer the μ-measure of $\mathcal{K}_G \setminus E_0(l,b)$ is given by

$$\lim_{N\to\infty} \frac{\#\{n : 0 \le n < G_{(N+1)(b+1)} \text{ and } e_0(n) \dots e_{(l+1)(b+1)-1}(n) \in \mathcal{L}_l(C)\}}{G_{(N+1)(b+1)}}.$$

Let $|A| = \sum_{i,j} |a_{ij}|$ denote the 1-norm of a matrix A. Then with the notations of Lemma 4 we have $G_{(N+1)(b+1)} = |B^N|$ and for $N \ge l$,

$$\#\{n : 0 \le n < G_{N(b+1)} \text{ and } e_0(n) \dots e_{(l+1)(b+1)-1}(n) \in \mathcal{L}_l(C)\} \le |C^l| \cdot |B^{N-l}|.$$

The matrix B satisfies the assumptions of Lemma 4. Let β and γ be the maximal eigenvalues of B and C respectively and let $K \ge 1$, $L > 0$ be constants such that $K^{-1}\beta^m \le |B^m| \le K\beta^m$ and $|C^m| \le L\gamma^m$. Then

$$\mu(\mathcal{K}_G \setminus E_0(l,b)) \le LK^2(\gamma/\beta)^l.$$

But $\mu(E_k(l,b)) \ge \mu(E_0(l,b)) = 1 - \mu(\mathcal{K}_G \setminus E_0(l,b)) \ge 1 - LK^2(\gamma/\beta)^l$. This proves the lemma with $\varrho = \gamma/\beta$ and $c = LK^2$. ∎

We now state a combinatorial assumption concerning the backward carries in digit expansions with respect to linear recurrences.

Hypothesis B. *There is an integer* $b > 0$ *such that for all* k *and*

$$\widetilde{N} = (\varepsilon_0, \dots, \varepsilon_k, 0^{(b+1)}, \varepsilon_{k+b+2}, \dots),$$

addition of G_m *to* N (*with* $m \ge k + b + 2$) *does not change the digits* $\varepsilon_0, \dots, \varepsilon_k$, *i.e.*

$$\widetilde{N + G_m} = (\varepsilon_0, \dots, \varepsilon_k, \dots).$$

A simple consideration shows that for instance the Multinacci sequence defined by $M_{k+d} = M_{k+d-1} + \dots + M_k$ has this property. It seems to be quite clear that this hypothesis is closely related to the finiteness of α-expansions. In a recent paper [FS] it is shown that for recurrences with decreasing coefficients all positive integers have finite α-expansions with respect to the dominating characteristic root α. An immediate consequence of the hypothesis is the following

Lemma 6. *Let* $x = (x_0 x_1 \dots)$ *be in* $E_k(l,b)$ (*for some block-length* b, *as above*). *Then for all* $m \ge (l+1)(b+1)$ *we have* $x_j = (\tau^{G_m}(x))_j$ *for* $j = 0, \dots, k(b+1) - 1$.

Theorem 6. $\mathcal{K}_G$ *is* (*measure-theoretically*) *isomorphic to a group rotation with purely discrete spectrum given by the countable group*

$$\Gamma := \{z \in \mathbb{C} : \lim_{n\to\infty} z^{G_n} = 1\}$$

provided that Hypothesis B *is satisfied.*

Remark 4. Notice that for $\alpha = \frac{1}{2}(a+\sqrt{a^2+4})$ the corresponding sequence $(a_0, a_1, \ldots)$ is $(a, 0, a, 0, \ldots)$. In this case the odometer is metrically isomorphic to the translation $x \mapsto x+\alpha \bmod 1$. This is well-known for $a=1$; the general case follows from [Li3].

Proof of Theorem 6. First step. Let $z \in \Gamma$ and write $z = e^{2i\pi\lambda}$. By definition $\lim_{n\to\infty} \|\lambda G_n\| = 0$ ($\|\cdot\|$ denoting the distance to the nearest integer) and in the following we give a short argument showing the (well-known) fact that the convergence is indeed geometric.

We can write $\lambda G_n = u_n + \eta_n$, where u_n is an integer and $\lim_{n\to\infty} \eta_n = 0$. This last condition implies the sequence (u_n) satisfies the same linear recurrence as (G_n) (for sufficiently large n) so that the same is true for (η_n). Thus $\eta_n = A_1(n)\alpha_1^n + \ldots + A_s(n)\alpha_s^n$ with $\alpha_1 = \alpha$ and the other α_j are less than α in absolute values. Moreover, the $A_j(n)$ denote polynomials. Taking into account the growth of each term it essentially remains to consider the case of sequences $\eta_n = B_1\beta_1^n + \ldots + B_r\beta_r^n$ with $|\beta_j| = 1$ for all j. At this moment we have to distinguish two cases. First assume that all β_j are roots of unity. Therefore there exists an integer K such that $\eta_{n+mK} = \eta_n$ for all integers m and this implies $\eta_n = 0$. In the other case the closed subgroup generated by the $(\beta_1, \ldots, \beta_r)$ contains a torus. From this it follows that η_n is identically 0.

Let $\zeta \in \Gamma$. Let $x \in \mathcal{K}_G$. Then by the above, the series $\sum_k |\zeta^{x_k G_k} - 1|$ converges. Thus the limit

$$\zeta^x := \lim_{k\to\infty} \zeta^{x(k)}$$

exists. Note that the map $\widehat{\zeta} : x \to \zeta^x$ is continuous on $\mathcal{K}_G$. Now we easily get $\widehat{\zeta}\cdot(\tau(x)) = \zeta\widehat{\zeta}(x)$. This means that each element ζ of Γ is an eigenvalue of τ with continuous eigenfunction $\widehat{\zeta}$, thus $\Gamma \subset \mathrm{Spec}(\tau)$.

Second step. We claim that for all maps f as considered in Lemma 3 we have

$$\sum_n \int_{\mathcal{K}_G} |f\circ\tau^{G_n} - f|^2\, d\mu < \infty, \tag{3.8}$$

where μ is the τ-invariant measure given explicitly in Theorem 5. From this, by standard arguments (see e.g. [So1]–[So3]) τ has a purely discrete spectrum which is contained in Γ. Concluding the proof, we use the notations of Lemma 5 and split the integral $\int_{\mathcal{K}_G} |f\circ\tau^{G_n} - f|^2\, d\mu$ into two integrals assuming $(l+1)(b+1) \le n < (l+2)(b+1)$. First $\int_{E_n(l,b)} |f\circ\tau^{G_n} - f|^2\, d\mu = 0$ because of Lemma 6 and the remaining integral can be estimated by $2\|f\|_\infty^2 \mu(\mathcal{K}_G \setminus E_k(l,b))$. Thus by Lemma 5, the series (3.8) is convergent.

Third step. It remains to show the countability of Γ. This is a well-known fact due to Pisot [Pi] which can be proved using the same argument as in the first step. In fact, using the generating functions $U(z)$ and $Y(z)$ of u_n and η_n, respectively, we get the relation

$$\frac{\lambda}{(1-z)(1-zA(z))}=U(z)+Y(z),$$

where $G(z)$ and $A(z)$ are given at the beginning of Section 3. Notice that (a_n) is purely periodic, thus $A(z)=Q(z)/(1-z^{d+1})$, where $Q(z)=a_0+a_1z+\ldots$ $\ldots+a_dz^d$. But (u_n) satisfies the same linear recurrence (for sufficiently large n) as G_n; hence $U(z)=P(z)/((1-z)(1-zA(z)))$, where $P(z)$ is a polynomial with integral coefficients. Moreover, the series Y has no pole in the closed unit disk. Therefore

$$\lambda=\frac{P(\theta^{-1})(1-\theta^{-1})}{1-\theta^{-(d+1)}},$$

where θ is any of the characteristic roots of (G_n) with modulus ≤ 1. This shows the countability of Γ and the proof is complete. ■

Remark 5. Notice that for all $\zeta\in\Gamma$ the map $\widehat{\zeta}$ is continuous and we can define the following factor:

$$F:\mathcal{K}_\alpha\to\Pi=\prod_{\zeta\in\Gamma}U_\zeta,\qquad x\mapsto(\zeta^x)_{\zeta\in\Gamma},$$

where U_ζ is the closed group generated by ζ. The transformation T on Π corresponding to τ is $T:(u_\zeta)\mapsto(\zeta U_\zeta)$ and the image $F(\mathcal{K}_\alpha)$ is the closed subgroup $\Pi(\Gamma)$ of Π generated by $(\zeta)_{\zeta\in\Gamma}$.

Remark 6. The above theorem was also proved in [So2] for recurrences with decreasing coefficients. In that paper also a more general theorem is proved under the assumption that all numbers in $\mathbb{Z}_+[\alpha^{-1}]$ have finite α-expansions. Iti is an open problem whether this is equivalent to Hypothesis B.

4. Exponential sums and applications. In [GT2] we considered sequences of the type $s_G(n)x$, where x is a given irrational number and $s_G(n)$ denotes the G-ary sum-of-digits function. A basic tool for investigating the distribution behaviour of such sequences is the following proposition.

PROPOSITION 4. *Let f be G-multiplicative*, $|f(n)|\le 1$ *and*

$$\Bigl|\frac{1}{G_k}\sum_{n<G_k}f(n)\Bigr|\le\frac{1}{g(G_k)}$$

for an increasing function g with $g(x)\le x$. Then for some constant D only depending on G we have

$$\Bigl|\frac{1}{N}\sum_{n=m}^{N+m-1}f(n)\Bigr|\le\frac{D}{g(\sqrt{N})}$$

uniformly in m.

Proof. It has been proved in [GT2] that under the hypotheses of the proposition

$$\left|\frac{1}{N}\sum_{n=0}^{N-1} f(n)\right| \le \frac{D_1}{g(\sqrt{N})}.$$

We now have to consider some cases:

(a) $m < G_{L(N)+1}$. Here we have

$$\left|\frac{1}{N}\sum_{n=m}^{N+m-1} f(n)\right| = \frac{1}{N}\left|\sum_{n=0}^{N+m-1} f(n) - \sum_{n=0}^{m-1} f(n)\right| \le \frac{D_2}{g(\sqrt{N})}.$$

(b) $m \ge G_{L(N)+1}$. In this case we write

$$m = \sum_{k=0}^{L(m)} \delta_k G_k = \underbrace{\sum_{k=0}^{L(N)} \delta_k G_k}_{m_1} + \underbrace{\sum_{k=L(N)+1}^{L(m)} \delta_k G_k}_{m_2}.$$

Now there are two subcases.

(b1) $L(N+m_1) = L(N)$. In this case we have

$$\sum_{n=m}^{N+m-1} f(n) = f(m_2) \sum_{n=m_1}^{N+m_1-1} f(n)$$

and this last sum was estimated in case (a).

(b2) $L(m_1+N) = L(m_1)+1$. In this case we split the range of summation into two parts

$$\sum_{n=m}^{m_2+G_{L(N)}-1} f(n) + \sum_{n=m_2+G_{L(N)}}^{N+m-1} f(n)$$

and both of these parts are of the type considered in case (b1). ∎

Definition. A sequence x_n is called *pseudo-random* if the following three conditions are satisfied for all integers $k \ne 0$:

(i) $\lim_{N\to\infty} \frac{1}{N}\sum_{n<N} e^{2k\pi i x_n} = 0$,

(ii) $\gamma(h) = \lim_{N\to\infty} \frac{1}{N}\sum_{n<N} e^{2k\pi i(x_{n+h}-x_n)}$ exists,

(iii) $\lim_{H\to\infty} \frac{1}{H}\sum_{h=0}^{H} |\gamma(h)|^2 = 0.$

A sequence x_n is called *well-distributed* if for all integers $k \neq 0$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=l+1}^{N+l} e^{2k\pi i x_n} = 0$$

uniformly in l.

THEOREM 7. *The sequence $xs_G(n) + yn$ is well-distributed if at least one of the numbers x and y is irrational.*

Proof. Upon setting $f(n) = e^{2\pi i k(xs_G(n)+yn)}$ and observing that this is a G-multiplicative function, Proposition 4 immediately yields the result. ∎

Remark 7. It is an open conjecture if the sequence considered above is pseudo-random. For the dyadic expansion this was proved by K. Mahler [Ma] using a general approach by N. Wiener [Wi]. The main problem is to establish (iii) since (i) follows from the above proposition and (ii) can be managed by Wiener's approach. M. Mendès France [Me1] has extended this (elaborate) method to q-ary expansions. Finally, we note that in [Li2] a dynamic approach is developed which may lead to a proof of this conjecture.

Remark 8. In the theory of pseudo-random sequences a stronger version of condition (iii) is known:

(iii′) $\lim_{h\to\infty} \gamma(h) = 0$

(cf. [Ba1, Ba2]). By arguments as in [Ma] it follows easily that in our case this stronger condition is not satisfied.

Remark 9. The sequence $xs_G(n)$ has empty spectrum in the sense of Mendès France (cf. [Me3]). We note here that pseudo-randomness and spectrum of sequences were studied in a sequence of papers by different authors, e.g. Bass [Ba1, Ba2], Bertrandias [Ber] and Mendès France [Me1, Me2].

A quantitative measure for distribution behaviour of a sequence x_n of real numbers is the discrepancy

$$D_N(x_n) = \sup_J \left| \frac{1}{N} \sum_{n=0}^{N-1} \chi_J(\{x_n\}) - \lambda(J) \right|, \tag{4.1}$$

where the supremum is taken over all intervals J of length $\lambda(J)$ and $\{\cdot\}$ denotes the fractional part. A measure for the well-distribution of a sequence is the uniform discrepancy

$$T_N(x_n) = \sup_{h\in\mathbb{N}_0} D_N(x_{n+h}). \tag{4.2}$$

In a series of papers [TT1], [GT2], [KLTT], [TT2] estimates for the discrepancy and uniform discrepancy of the sequence $xs_G(n)$, where x denotes an irrational number, were established. A major tool for establishing such estimates is the Erdős–Turán inequality

$$D_N(x_n) \leq 6\biggl(\frac{1}{H} + \sum_{h=1}^{H} \frac{1}{h}\biggl|\frac{1}{N}\sum_{n=0}^{N-1} e^{2h\pi i x_n}\biggr|\biggr) \tag{4.3}$$

for every positive integer H.

Remark 10. By using Proposition 4 and choosing a suitable H in (4.3), estimates for the uniform discrepancy of $xn + ys_G(n)$ can be established. However, the estimates seem to be quite weak, so that we do not work them out in detail. The different method of [GT2] cannot be applied directly to that type of sequences. It remains an open problem to find sharp bounds for the discrepancy and the uniform discrepancy of $xn + ys_G(n)$.

Remark 11. In [KLTT] the discrepancy of the sequence $xs_G(n)$ was estimated in the case of Ostrowski expansions with respect to the continued fraction expansion of a given real number. We note here that in the recent PhD thesis [Ko] some further results concerning such expansions are proved under special assumptions on the growth of the partial quotients.

Acknowledgements. The third author is indebted to Y. Lacroix for valuable discussions concerning the spectral properties of the odometer.

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Received on 2.7.1993
and in revised form on 7.12.1993 (2455)