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On some theorems of Littlewood and Selberg, IV

by

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*Dedicated to the eighty-first birthday of
Professor Paul Erdős*

1. Introduction and notation. As usual we write $s = \sigma + it$, and

$$\zeta(s) \equiv \sum_{n=1}^{\infty}\Big(n^{-s} - \int_{n}^{n+1} u^{-s}\,du\Big) + \frac{1}{s-1}, \qquad \sigma > 0,$$
$$= \sum_{n=1}^{\infty} n^{-s} = \prod_{p}(1-p^{-s})^{-1}, \qquad \sigma > 1.$$

In the last product p runs over all primes $2, 3, 5, 7, 11, \ldots$ The object of this note is to prove the following theorem.

Theorem 1. *There exist effective absolute positive constants C and C^* with the following property. Let $T \ge 20$, $H = C\log\log\log T$ and $\zeta(s) \ne 0$ in $(1/2 + (10\log\log T)^{-1} < \sigma \le 1,\ T - H \le t \le T + H)$. Then there is at least one zero of $\zeta(s)$ in the disc of radius $C^*(\log\log T)^{-1}$ with centre $1/2 + iT$.*

Remark 1. The proof of this theorem depends on Theorem 1 of our earlier paper [3], and a significant use of Ramachandra's kernel function of the third order, namely $R_3(z) \equiv \mathrm{Exp}(e - \mathrm{Exp}(\cos z))$. These kernels were known to Ramachandra for a long time. Ramachandra's kernel function of the second order, namely $R_2(z) \equiv \mathrm{Exp}((\sin z)^2)$, was used by him and his collaborators in various papers. Besides these the proof uses Borel–Carathéodory theorem and Hadamard's three circles theorem (the application of these last two theorems is similar to that explained in [4], pp. 210, 211).

Remark 2. A more complicated application of Borel–Carathéodory theorem and Hadamard's three circles theorem was employed by E. C. Titchmarsh to give an alternative (simpler) proof of a theorem of J. E. Littlewood (see [4], Theorem 9.12, p. 224). Littlewood's theorem asserts that given any $t > 0$ there exists at least one zero $\varrho = \beta + i\gamma$ with $|t-\gamma| \le D_1(\log\log\log(|t|+$

$10^{100}))^{-1}$. Our theorem asserts that $|t-\gamma| \le D_2(\log\log(|t|+10^{100}))^{-1}$ if we assume Riemann hypothesis. Of course the result just mentioned is not difficult to prove. But Theorem 1 is not so easy to prove and is also of sufficient interest in itself.

Remark 3. It is possible to generalize Theorem 1 to functions $F(s)$ which have an Euler product and a functional equation. But the fact that there is at least one zero-free region for $F(s)$ required by Theorem 1 is not known for any such $F(s)$. Hence we have to restrict $F(s)$ to ordinary L-functions and L-functions of quadratic fields. In these cases we can prove that

$$\frac{1}{U}\int\limits_{U}^{2U} |F(1/2+it)|^2\,dt \le (\log U)^A \qquad (U \ge 3)$$

where $A>0$ is some constant. This last condition on the mean-value ensures such a zero-free region for $F(s)$. In fact under this condition the number of zeros of $F(s)$ in $(\operatorname{Re} s \ge \sigma_0,\ |\operatorname{Im} s| \le U)$ is $\le U^{4(3-2\sigma_0)^{-1}}(\log U)^B$ where $B>0$ is a constant which depends on A.

Remark 4. While dealing with applications of Theorem 1 for $U \le T \le 2U$ it is convenient to prove it with $(10\log\log U)^{-1}$ in place of $(10\log\log T)^{-1}$. This follows from the method of our proof. Also deeper results like

$$\frac{1}{U'}\int\limits_{U}^{U+U'} |\zeta(1/2+it)|^2\,dt \le (\log U)^A \qquad (U \ge 3)$$

with $U' = U^{1/3}$ due to R. Balasubramanian [1] and $U' = U^{7/22+\varepsilon}$ due to D. R. Heath-Brown and M. N. Huxley [2] are known. These results imply the existence of intervals of the type $T \le t \le T+H$ contained in $[U, U+U']$ for which $\sigma > 1/2 + (10\log\log U)^{-1}$ are zero free. Theorem 1 has applications to such cases also.

Remark 5. The following result may be of some interest. Let D_3 be any positive constant. Let $\zeta(s) \ne 0$ in the region $\sigma \ge 1/2 + D_3(\log\log(|t| + 100))^{-1}$. Then given any real number $t>0$ there exists a zero ϱ of $\zeta(s)$ such that

$$|1/2+it-\varrho| \le D_4(\log\log(|\varrho|+100))^{-1}$$

where D_4 is a constant which depends only on D_3. This is a hypothetical result, the hypothesis being weaker than Riemann's hypothesis. It is possible to obtain this result (by our method) by employing Ramachandra's kernel of the second order, namely $\mathrm{Exp}((\sin z)^2)$.

Remark 6. In Section 2 we prove an estimate for Ramachandra's kernel of the third order, namely $R_3(z)$ mentioned in Remark 1. In Section 3 we state a special case of Theorem 1 of our previous paper [3]. In Section 4 we complete the proof of Theorem 1 by using the Borel–Carathéodory theorem and Hadamard's three circles theorem.

Notation. We use $A, B, D_1, D_2, D_3, D_4, C, C^*, C_1, \ldots, C_9$ to denote effective positive numerical constants. The letter $\underline{a}$ will denote an effective large positive constant to be chosen at the end. We write $\alpha = 1/2 + (10 \log\log T)^{-1}$.

2. An estimate for Ramachandra's kernel of the third order. The kernel in question is $R_3(z) = \mathrm{Exp}(e - \mathrm{Exp}(\cos z))$ where $z = x + iy$. We prove the following theorem.

Theorem 2. *Let $R = |R_3(z)|$ and $|x| \le 2e^{-|y|}$. Then for all y we have $R \le e^e$. Also if $|x| \le 2e^{-|y|}$ and $|y| \ge 1$, we have*

$$R \le \mathrm{Exp}\big(e - \tfrac{1}{2}\,\mathrm{Exp}\,\big(\tfrac{1}{2}e^{|y|} - e^{-|y|}\big)\big).$$

In particular, if $|y| = \log\log\log T + 2$ and $|x| \le 2e^{-2}(\log\log T)^{-1}$ we have $R \le e^e T^{-(\log T)/4}$.

Remark. Note that $2e^{-2} \ge 1/5$.

Proof. We have

$$\begin{aligned} R &= \mathrm{Exp}\{e - \mathrm{Re}\, e^{\cosh y \cos x - i \sinh y \sin x}\} \\ &= \mathrm{Exp}\{e - e^{\cosh y \cos x} \cos(\sinh y \sin x)\}; \end{aligned}$$

since this expression depends only on $|x|$ and $|y|$ we may suppose that $x > 0$ and $y > 0$. Put $\theta = (\sinh y \sin x)$ and let $x \le 2e^{-y}$. Then $0 \le \theta \le \frac{1}{2}xe^y \le 1$ and so

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots \ge 1 - \frac{\theta^2}{2} \ge \frac{1}{2}.$$

Thus

$$R \le \mathrm{Exp}\big(e - \tfrac{1}{2}e^{\cosh y \cos x}\big).$$

Next

$$\begin{aligned} \cosh y \cos x &= (\cosh y)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right) \\ &= (\cosh y)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\left(1 - \frac{x^2}{5.6}\right) + \frac{x^8}{8!}\left(1 - \frac{x^2}{9.10}\right) + \ldots\right) \\ &> (\cosh y)\left(1 - \frac{x^2}{2}\right) \ge \frac{1}{2}e^y\left(1 - \frac{1}{2}(2e^{-y})^2\right) \ge \frac{1}{2}e^y - e^{-y}. \end{aligned}$$

This proves the first two assertions. To prove the last assertion let $|y| = \log\log\log T + 2$ and $|x| \le 2e^{-|y|} = 2e^{-2}\log\log T$. Then by the second assertion of Theorem 2 we have

$$\begin{aligned}R &\le \operatorname{Exp}\big(e - \tfrac{1}{2}\operatorname{Exp}\big(\tfrac{1}{2}e^2\log\log T - e^{-2}(\log\log T)^{-1}\big)\big)\\ &= \operatorname{Exp}\big(e - \tfrac{1}{2}(\log T)^{e^2/2}\operatorname{Exp}(-e^{-2}(\log\log T)^{-1})\big)\\ &\le \operatorname{Exp}\big(e - \tfrac{1}{4}(\log T)^{e^2/2}\big) = e^e T^{-(\log T)/4}.\end{aligned}$$

This proves Theorem 2 completely.

Remark. To prove Theorem 1 we may assume that T exceeds any large positive constant since we may increase C^* to cover smaller values of T.

3. An application of our previous results to $\zeta(s)$. In this section we record a special case of Theorem 1 of [3], as Theorem 3.

Theorem 3. *Suppose $\zeta(s) \ne 0$ in ($\sigma > \alpha = \frac{1}{2} + (10\log\log T)^{-1}$, $T - H \le t \le T + H$). Then for ($\sigma \ge \alpha$, $T - H/2 \le t \le T + H/2$) we have*

$$|\zeta(\sigma + it)| \le \operatorname{Exp}(C_1(\log T)(\log\log T)^{-1})$$

and for ($\alpha + C_2(\log\log T)^{-1} \le \sigma \le 3/4$, $T - H/2 \le t \le T + H/2$) we have

$$|\log\zeta(\sigma + it)| \le C_3(\log T)^{2-2\sigma}(\log\log T)^{-1}.$$

Also for ($\sigma \ge 3/4$, $T - H/2 \le t \le T + H/2$) we have

$$|\log\zeta(\sigma + it)| \le C_4(\log T)^{1/4}(\log\log T)^{-1}.$$

Corollary. *For ($\sigma \ge 1/2$, $T - H/4 \le t \le T + H/4$) we have*

$$|\zeta(\sigma + it)| \le \operatorname{Exp}(C_5(\log T)(\log\log T)^{-1}).$$

Proof of the Corollary. To get the inequality of the Corollary in ($1 - \alpha - 10(\log\log T)^{-1} \le \sigma \le 1 - \alpha$, $T - H/2 \le t \le T + H/2$) we can apply the first assertion of Theorem 3 and the functional equation. After this we have simply to apply the maximum modulus principle to

$$\zeta(s + z)R_3(z)$$

where $s = \sigma + it$ ($1 - \alpha \le \sigma \le \alpha$, $T - H/4 \le t \le T + H/4$) is any point in question and $z = x + iy$ is on the boundary of the rectangle defined by $|x| \le (5\log\log T)^{-1}$ and $|y| = \log\log\log T + 2$. We have only to apply Theorem 2.

4. Completion of the proof. We borrow the Borel–Carathéodory theorem from page 174 of Titchmarsh's book [5].

Theorem 4. *Let $f(z)$ be an analytic function regular for $|z| \le R$ and let $M(r)$ and $A(r)$ denote as usual the maximum of $|f(z)|$ and $R\{f(z)\}$ on $|z| = r$. Then for $0 < r < R$,*

$$M(r) \le \frac{2r}{R - r}A(R) + \frac{R - r}{R + r}|f(0)|.$$

R e m a r k. We have stated the theorem in the notation of Titchmarsh's book. The letter R should not be confused with our notation in Section 2. In Theorem 4, $R\{f(z)\}$ denotes the real part of $f(z)$.

We also borrow Hadamard's three circles theorem from the same book (see [5], p. 172).

THEOREM 5. *Let $f(z)$ be an analytic function regular for $r_1 \le |z| \le r_3$. Let $r_1 < r_2 < r_3$ and let M_1, M_2, M_3 be the maximum of $|f(z)|$ on the three circles $|z| = r_1, r_2, r_3$ respectively. Then*

$$M_2^{\log(r_3/r_1)} \le M_1^{\log(r_3/r_2)} M_3^{\log(r_2/r_1)}.$$

From now on we write $L_1 = \log T$, $L_2 = \log\log T$, and we assume that $\zeta(s) \neq 0$ in $(\sigma \ge 1/2 - 10aL_2^{-1}, \||t-T| \le 10aL_2^{-1})$. We obtain a contradiction by a suitable application of Theorems 4 and 5. We put $z_0 = \alpha + (C_2+a)L_2^{-1} + iT$.

LEMMA 1. *Let $f(z) = \log\zeta(z+z_0)$. Then with $R = 6aL_2^{-1}$ and $r = 4aL_2^{-1}$ we have*

$$M(r) \le C_6 aL_1L_2^{-1}.$$

P r o o f. By using the asymptotic properties of the conversion factor in the functional equation for $\zeta(s)$, we see that $A(R)$ does not exceed a constant times $aL_1L_2^{-1}$. Certainly $|f(0)| \le C_7e^{-a}L_1L_2^{-1}$. This completes the proof.

It is to be noted here that we have used Theorem 3 and its corollary to get the bounds for $A(R)$ and $|f(0)|$ required for the application of Theorem 4.

LEMMA 2. *As before let $f(z) = \log\zeta(z+z_0)$, $r_1 = L_2^{-1}$, $r_2 = 2aL_2^{-1}$ and $r_3 = 4aL_2^{-1}$. We have*

$$(M(2aL_2^{-1}))^{\log(4a)} \le (C_8e^{-a}L_1L_2^{-1})^{\log 2}(C_6aL_1L_2^{-1})^{\log(2a)}.$$

P r o o f. We have $M_2 = M(2aL_2^{-1})$ and $M_1 \le C_8e^{-a}L_1L_2^{-1}$ by Theorem 3 and $M_3 \le C_6aL_1L_2^{-1}$ by Lemma 1. Hence the lemma follows from Theorem 5.

LEMMA 3. *We have* $\log|\zeta(z_0 - 2aL_2^{-1})| \ge C_9aL_1L_2^{-1}$ *and so*

$$M(2aL_2^{-1}) \ge C_9aL_1L_2^{-1}.$$

P r o o f. The proof follows by the functional equation and the lower bound for $\log|\zeta(s)|$ in $(\alpha + C_2L_2^{-1} \le \sigma \le 3/4,\ T - H/2 \le t \le T + H/2)$ provided by Theorem 3. This proves the lemma.

The proof of Theorem 1 is now complete since the inequalities asserted by Lemmas 2 and 3 contradict each other, if we choose for $\underline{a}$ a large constant.

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