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On the representation of a number as the sum of any number of squares, and in particular of twenty

by

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1. Introduction. We begin by considering the general problem of determining, for any non-negative integer n, the number $r_s(n)$ of representations of n as a sum of s squares, where s is any positive integer. Thus $r_s(n)$ is the number of solutions of the equation

$$x_1^2+x_2^2+\ldots+x_s^2 = n$$

in integers $x_1, x_2, \ldots, x_s$, positive, negative or zero.

There is a considerable literature on this problem (¹) and there exist different points of view as to what constitutes a solution of it. It is well known that it is possible to express $r_s(n)$ in the form

$$r_s(n) = \varrho_s(n)+R_s(n)\,, \tag{1}$$

where $\varrho_s(n)$ is a "divisor function" and $R_s(n)$ is a "remainder term" of smaller order. For example, when $s \geqslant 4$, $\varrho_s(n)$ is of the order of $n^{\frac{1}{2}s-1}$ for large n and $R_s(n) = O(n^{\frac{1}{4}(s-1)+\varepsilon})$ $(\varepsilon > 0)$. The function $\varrho_s(n)$ can be expressed in various explicit forms, for example as a "singular series" of the type derived by Hardy and Littlewood, or as a summation over divisors of n of different types. When $r_s(n)$ is only required to within an error of $O(n^{\frac{1}{2}s-1})$, the problem can be regarded as having been solved when $\varrho_s(n)$ has been found.

However, if an *exact* determination of $r_s(n)$ is required, more detailed consideration is necessary. Probably the most illuminating way of regarding the problem is in terms of the theory of modular forms, as was

(¹) For a very readable account and references to literature before 1940, see [4], Chapter IX and the notes at the end of this chapter. The most recent work on the problem is due to Lomadze [5], [6], who gives explicit formulae for all $s \leqslant 32$.

first done by Mordell [7], [8]. Write, in the usual notation for theta functions,

(2) $$\vartheta_2 = \vartheta_2(0\,|\,\tau) = 2\sum_{n=1}^{\infty} q^{(n+\frac{1}{2})^2},$$

(3) $$\vartheta_3 = \vartheta_3(0\,|\,\tau) = 1+2\sum_{n=1}^{\infty} q^{n^2},$$

(4) $$\vartheta_4 = \vartheta_4(0\,|\,\tau) = 1+2\sum_{n=1}^{\infty} (-1)^n q^{n^2},$$

where $q = e^{\pi i\tau}$ and $\operatorname{Im}\tau > 0$. Then, for any positive integer s,

(5) $$\vartheta_3^s = \sum_{n=0}^{\infty} r_s(n) q^n .$$

Put

$$I = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \quad V = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}, \quad U = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix},$$

$$A = \begin{bmatrix}0 & -1\\ 1 & 1\end{bmatrix}, \quad P = A^{-1}UA = -U^{-2}V = \begin{bmatrix}2 & 1\\ -1 & 0\end{bmatrix},$$

and let

$$T = \begin{bmatrix}a & b\\ c & d\end{bmatrix},$$

where a, b, c and d are integers satisfying $ad-bc = 1$. The set of all such matrices T forms the full modular group $\Gamma(1)$ and is generated by V and U. The subset Γ_3 of $\Gamma(1)$ consisting of all matrices $T \in \Gamma(1)$ for which

$$T \equiv I \quad \text{or} \quad T \equiv V \pmod 2$$

is a subgroup of index 3 in $\Gamma(1)$ and is generated by V and U^2. Further, the subset $\Gamma(2)$ of $\Gamma(1)$ and Γ_3 consisting of all $T \in \Gamma(1)$ for which $T \equiv I \pmod 2$, forms the principal congruence group of level 2, which is of index 2 in Γ_3.

The function ϑ_3 is an integral modular form for Γ_3 of dimension $-\frac{1}{2}$, possessing a multiplier system, which we denote by v. I.e. v is a function defined on Γ_3 with the property that

(6) $$\vartheta_3(0\,|\,T\tau) = (c\tau+d)^{1/2} v(T)\vartheta_3(0\,|\,\tau)$$

for all τ with $\operatorname{Im}\tau > 0$. Here

$$T\tau = \frac{a\tau+b}{c\tau+d}$$

and fractional powers of $c\tau+d$ are determined by taking

$$-\pi < \arg(c\tau+d) \leqslant \pi .$$

The values of $v(T)$ were determined by Hermite and may be found from Table XLII of [15] ($\nu = V = 0$, $\mu = 2$, $m'' = -1$). For each $T \in \Gamma_3$, $v(T)$ is an eighth root of unity and, in particular,

$$v(U^{2n}) = 1, \quad v(-U^{2n}) = -i \quad (n = 0, \pm 1, \pm 2, \ldots),$$

and

$$v(V) = e^{-\pi i/4}, \quad v(P) = e^{\pi i/4}.$$

More generally,

(7) $$v(T) = \begin{cases} \left(\dfrac{c}{d}\right) e^{\pi i(d-1)/4} & (d \text{ odd and positive}),\\ \left(\dfrac{d}{c}\right) e^{\pi i(c-2)/4} & (c \text{ odd and positive}).\end{cases}$$

In general, we write $\{\Gamma, k, w\}$ for the set of all integral modular forms of dimension $-k$ belonging to a subgroup Γ of $\Gamma(1)$ with multiplier system w, and put, in particular,

$$\mathfrak{B}_s = \{\Gamma_3, \tfrac{1}{2}s, v^s\}$$

for any positive integer s. Clearly $\vartheta_3^s \in \mathfrak{B}_s$.

If $s \equiv 0 \pmod 8$, $v^s(T) = 1$ for all $T \in \Gamma_3$, while, if $s \equiv 4 \pmod 8$,

(8) $$v^s(T) = \begin{cases} 1 & \text{if } T \equiv I \pmod 2,\\ -1 & \text{if } T \equiv V \pmod 2.\end{cases}$$

As a fundamental region for Γ_3 we may take the region D of the τ-plane for which $|\operatorname{Re}\tau| \leqslant 1$, $\operatorname{Im}\tau > 0$ and $|\tau| \geqslant 1$. The region D has three cusps. Two of these, namely $\tau = \pm 1$, belong to one parabolic cycle at which ϑ_3^s vanishes; the third, namely $\tau = \infty$, forms a second parabolic cycle at which ϑ_3^s takes the value 1.

More generally, if f is any member of $\mathfrak{B}_s$, then f has an absolutely convergent Fourier expansion

$$f(\tau) = \sum_{n=0}^{\infty} a_n q^n$$

and takes the value a_0 at ∞. Except when $s \equiv 0 \pmod 8$, however, f always vanishes at the cusps ± 1; for P is the "fundamental matrix" (see [11], p. 169) of the cusp -1 and $v^s(P) = 1$ if and only if $s \equiv 0 \pmod 8$.

The Eisenstein series in $\mathfrak{B}_s$ that is associated with the cusp ∞ is

(9) $$G_s(\tau) = \frac{1}{2}\sum_{T} \frac{1}{v^s(T)(c\tau+d)^{s/2}},$$

for $s \geqslant 5$; see [9]. Here the summation is over any maximal system of matrices T of Γ_3 having different second rows. Write

(10) $$G_s(\tau) = \sum_{n=0}^{\infty} \varrho_s(n) q^n .$$

Then $\varrho_s(0) = 1$ and, for $n \geqslant 1$,

$$\varrho_s(n) = \frac{\pi^{\frac{1}{2}s} n^{\frac{1}{2}s-1}}{\Gamma(\frac{1}{2}s)} \sum_{k=1}^{\infty} A_k(n), \tag{11}$$

where

$$A_k(n) = \sum_{\substack{h=1 \\ (h,k)=1}}^{k} \left(\frac{S_{h,k}}{k}\right)^s e^{-2\pi i n h/k}, \tag{12}$$

and $S_{h,k}$ is the Kloosterman sum

$$S_{h,k} = \sum_{j=1}^{k} e^{2\pi i h j^2/k}. \tag{13}$$

This is the "singular series" form of $\varrho_s(n)$. Its expression as a divisor function is, in general, much more complicated; see, for example, Suetuna [13], Hilfssatz 7 (s even), [14], Hilfssatz 4 (s odd) or Lomadze [5], [6]. However, when $s \equiv 0 \pmod 4$, we have the simpler expression

$$\varrho_s(n) = \frac{\pi^{\frac{1}{2}s} n^{\frac{1}{2}s-1}}{\Gamma(\frac{1}{2}s)\zeta(\frac{1}{2}s)(1-2^{-\frac{1}{2}s})} \Big\{ \sum_{\substack{d|n \\ d \text{ odd}}} d^{1-\frac{1}{2}s} + (-1)^{\frac{1}{4}s} \sum_{\substack{d|n \\ d \text{ even}}} (-1)^{n/d} d^{1-\frac{1}{2}s} \Big\}. \tag{14}$$

If we put

$$f_s = \vartheta_3^s - G_s,$$

then f_s is a cusp form; i.e. f_s vanishes at all the cusps of D. If $\mathfrak{C}_s$ denotes the vector subspace of $\mathfrak{B}_s$ consisting of all cusp forms, then $\mathfrak{C}_s$ has dimension $\varkappa_s = [\frac{1}{8}(s-1)]$; this can be deduced from the Riemann-Roch theorem (see [11], p. 188) or, in this special case, by more elementary methods.

In particular, $\varkappa_s = 0$ for $s \leqslant 8$, so that $f_s(\tau)$ is identically zero for $s = 5, 6, 7, 8$. When $s > 8$, the forms

$$\vartheta_3^{s-8r} \vartheta_2^{4r} \vartheta_4^{4r} \qquad (r = 1, 2, \ldots, \varkappa_s) \tag{15}$$

form a basis $\mathfrak{B}_s^{(1)}$ for $\mathfrak{C}_s$, as originally shown by Mordell [7], [8]. It follows that, for $9 \leqslant s \leqslant 16$,

$$\vartheta_3^s = G_s + C_s \vartheta_3^{s-8} \vartheta_2^4 \vartheta_3^4, \tag{16}$$

where [5]

$$C_9 = \frac{2}{17}, \quad C_{10} = \frac{2}{5}, \quad C_{11} = \frac{22}{31}, \quad C_{12} = 1,$$
$$C_{13} = \frac{871}{691}, \quad C_{14} = \frac{91}{61}, \quad C_{15} = \frac{73}{43}, \quad C_{16} = \frac{32}{17}.$$

However, $\varkappa_s > 1$ when $s > 16$ and the problem then is to find a suitable basis for $\mathfrak{C}_s$ and to express f_s as a linear combination of the cusp forms in this basis. For this purpose $\mathfrak{B}_s^{(1)}$ need not be the most convenient basis. Another possible basis, for example, would be an orthonormal basis formed by means of Petersson's scalar product

$$(f, g) = \iint_D f(\tau)\overline{g(\tau)} y^{\frac{1}{2}s-2} dx\, dy, \tag{17}$$

where f and g belong to $\mathfrak{C}_s$ and $\tau = x + iy$ (x, y real).

It is therefore relevant to ask what principles should guide one in the choice of a basis. To answer this question one must know the purpose for which $r_s(n)$ is being determined. If the approximation $\varrho_s(n)$ does not suffice, it would seem to be desirable that the remainder $R_s(n)$ should be expressible in a form that can be calculated without too much trouble.

One method, which has been used by Bulygin [2], Glaisher [3], Mordell [7], [8], Walfisz [16] and Lomadze [5], [6], is to express $R_s(n)$ as a sum of functions of the type

$$\varphi(k, r; n) = \tfrac{1}{8} \sum \{(x_1 + ix_2)^{4r} + (x_1 - ix_2)^{4r}\}, \tag{18}$$

where k and r are positive integers, and the summation is taken over all integral solutions $x_1, x_2, \ldots, x_k$ of the equation

$$x_1^2 + x_2^2 + \ldots + x_k^2 = n.$$

For example, when $s = 20$,

$$R_{20}(n) = \frac{1296}{31}\varphi(12, 1; n) - \frac{64}{31}\varphi(4, 2; n). \tag{19}$$

The identities

$$\vartheta_3^{s-8}\vartheta_2^4\vartheta_4^4 = 16\sum_{n=1}^{\infty}\varphi(s-8, 1; n)q^n \qquad (s \geqslant 10),$$

and

$$\vartheta_3^{s-16}\vartheta_2^4\vartheta_3^4(4\vartheta_3^8 + \vartheta_2^4\vartheta_4^4) = 16\sum_{n=1}^{\infty}\varphi(s-16, 2; n)q^n \qquad (s \geqslant 18),$$

make it clear why functions of the form (18) appear. In fact, if $s \not\equiv 1 \pmod 8$, the $\varkappa_s$ functions

$$f_{r,s}(\tau) = \sum_{m=0}^{2r} (-1)^{m-r}\binom{4r}{2m} \vartheta_3^{(2r-m)} \vartheta_3^{(m)} \vartheta_3^{s-2-8r} \tag{20}$$

($1 \leqslant r \leqslant \varkappa_s$) form a basis $\mathfrak{B}_s^{(2)}$ for $\mathfrak{C}_s$ and

$$f_{r,s}(\tau) = 4\pi^{2r}\sum_{n=1}^{\infty}\varphi(s-8r, r; n)q^n. \tag{21}$$

In (20) the derivatives of ϑ_3 are with respect to τ. For $s \equiv 1 \pmod 8$, the function $\vartheta_3\vartheta_2^{(s-1)/2}\vartheta_4^{(s-1)/2}$, whose coefficients can be expressed in terms of summations over $\frac{1}{2}(s-1)$ squares and $\frac{1}{2}(s+1)$ half-odd squares must be added to the functions $f_{r,s}$ $(1 \leqslant r < \varkappa_s)$ to form the basis $\mathfrak{B}_s^{(2)}$. Formulae for $R_s(n)$ of the type (19) have been given by the four authors just mentioned, and have been extended as far as ([2]) $s = 32$ by Lomadze [5], [6].

Such formulae are satisfactory in the sense that they are perfectly explicit and can be used to give a numerical value for $r_s(n)$ in any given case. On the other hand, if used for several values of n, the provision of tables of the functions $\varphi(k, r; n)$ becomes necessary. If this has to be done, it is not clear that these functions are the most suitable for the purpose. In this connexion, it would seem that Glaisher had the right idea. In this choice of cusp forms for even $s \leqslant 18$, he always preferred those whose coefficients have multiplicative properties. It is rather remarkable that, with the methods at his disposal, he was able to do this, although he could not always prove the multiplicative properties suggested by the tables he constructed.

For subgroups of the modular groups, our knowledge of these multiplicative cusp forms is still incomplete. In particular, for $\mathfrak{C}_s$ and $s > 16$, these multiplicative cusp forms seem to be known only for $s = 24$. In the case of 20 squares, however, I recently noticed that two of the three multiplicative cusp forms of dimension -10, that I constructed by elementary methods in 1946 [12], form a basis for $\mathfrak{C}_{20}$, and the remainder of the paper is devoted to the determination of $R_{20}(n)$ in terms of these cusp forms and to related questions.

2. Twenty squares. We write 1 and -1 to denote the multiplier systems v^8 and v^4 $(= v^{20})$, respectively; see (8) for the latter. Suppose that $F \in \mathfrak{C}_{20} = \{\Gamma_3, 10, -1\}$. Then clearly $F \in \mathfrak{C}'_{20} = \{\Gamma(2), 10, 1\}$. Let $\varkappa_{20}$, $\varkappa'_{20}$ and $\varkappa''_{20}$ denote the dimensions of the vector spaces $\mathfrak{C}_{20}$, $\mathfrak{C}'_{20}$ and $\mathfrak{C}''_{20} = \{\Gamma_3, 10, 1\}$, respectively. Then

$$\varkappa_{20} = 2, \quad \varkappa'_{20} = 3, \quad \varkappa''_{20} = 1.$$

As a basis for $\mathfrak{C}''_{20}$ we take single form

$$\Psi_0(\tau) = \sum_{n=1}^{\infty} \psi_0(n) q^n = \tfrac{1}{16}\vartheta_2^4\vartheta_3^8\vartheta_4^4(\vartheta_4^4 - \vartheta_2^4) \tag{22}$$

$$= q - 16q^2 - 156q^3 - 256q^4 + \dots \tag{23}$$

As a basis for $\mathfrak{C}_{20}$ we may take the forms

$$\Psi_1(\tau) = \sum_{n=1}^{\infty} \psi_1(n) q^n = \tfrac{1}{16}\vartheta_2^4\vartheta_3^4\vartheta_4^4(\vartheta_3^8 - \vartheta_2^4\vartheta_4^4) \tag{24}$$

$$= q + 228q^3 + \dots, \tag{25}$$

and

$$\Psi_1^*(\tau) = \sum_{n=1}^{\infty} \psi_1^*(n) q^n = \tfrac{1}{16}\vartheta_2^4\vartheta_3^4\vartheta_4^4(\vartheta_3^8 + 2\vartheta_2^4\vartheta_4^4) \tag{26}$$

$$= q + 48q^2 - 156q^3 + 768q^4 + \dots \tag{27}$$

That Ψ_0, Ψ_1 and Ψ_1^* are cusp forms is clear; it is easily verified that they belong to the vector spaces stated, and that Ψ_1 and Ψ_1^* are linearly independent. The three functions form a basis for $\mathfrak{C}'_{20}$. In [12] the functions Ψ_0, Ψ_1 and Ψ_1^* were denoted by F_6, G_8 and G'_8, respectively, their coefficients $\psi_0(n)$, $\psi_1(n)$ being denoted by $f_6(n)$, $g_8(n)$ and $g'_8(n)$, respectively.

It was shown in [12] that the coefficients $\psi_0(n)$, $\psi_1(n)$ and $\psi_1^*(n)$ have the multiplicative property

$$\psi(mn) = \psi(m)\psi(n) \tag{28}$$

whenever $(m, n) = 1$. Also, for any odd prime p and positive integer l, the equations

$$\psi(p^{l+1}) = \psi(p)\psi(p^l) - p^9\psi(p^{l-1}) \tag{29}$$

holds for each of the three functions. Further

$$\psi_1(2^l) = 0, \quad \psi_1^*(2^l) = 3\cdot 16^l \quad (l > 0). \tag{30}$$

In [12] no explicit formula for $\psi_0(2^l)$ was given. However, by using the easily verified fact that

$$\Psi_1^*(\tau) = \Psi_0(\tau) - 64\Psi_0(2\tau + 1), \tag{31}$$

we deduce that

$$\psi_0(2^l) = -16^l \quad (l > 0). \tag{32}$$

Also, by (14),

$$\varrho_{20}(n) = \tfrac{8}{31}\Big\{\sum_{\substack{d|n\\ n/d\ \text{odd}}} d^9 - \sum_{\substack{d|n\\ n/d\ \text{even}}} (-1)^d d^9\Big\}. \tag{33}$$

We have

$$\vartheta_3^{20}(0|\tau) = G_{20}(\tau) + A\Psi_1(\tau) + A^*\Psi_1^*(\tau), \tag{34}$$

where A and A^* are constants to be determined. Hence

$$r_{20}(n) = \varrho_{20}(n) + A\psi_1(n) + A^*\psi_1^*(n).$$

By using the values $r_{20}(1) = 40$, $r_{20}(2) = 760$, $\varrho_{20}(1) = 8/31$, $\varrho_{20}(2) = 4104/31$, $\psi_1(1) = 1$, $\psi_1(2) = 0$, $\psi_1^*(1) = 1$, $\psi_1^*(2) = 48$, we deduce that

$$A = \frac{2480}{93}, \quad A^* = \frac{1216}{93},$$

so that, for $n \geqslant 1$,

$$r_{20}(n) = \varrho_{20}(n) + \tfrac{16}{93}\{155\psi_1(n) + 76\psi_1^*(n)\}. \tag{35}$$

([2]) When $s \equiv 1 \pmod 8$, formulae of a slightly different type are required.

It can also be shown that Bulygin's functions $\varphi(12,1;n)$ and $\varphi(4,2;n)$ (see (18), (19)) are related to $\psi_1(n)$ and $\psi_1^*(n)$ by the formulae

$$\varphi(12,1;n) = \tfrac{2}{3}\psi_1(n)+\tfrac{1}{3}\psi_1^*(n)\,,$$
$$\varphi(4,2;n) = \tfrac{7}{12}\psi_1(n)+\tfrac{5}{12}\psi_1^*(n)\,.$$

The functions $\varphi(12,1;n)$ and $\varphi(4,2;n)$ are not multiplicative.

We note, in conclusion that the functions Ψ_1 and Ψ_1^* are orthogonal in the metric on $\mathfrak{C}_{20}$; i.e. (see (17) with $s=20$) $(\Psi_1,\Psi_1^*)=0$. Further, in the metric on $\mathfrak{C}_{20}'$, the three functions Ψ_0, Ψ_1 and Ψ_1^* are mutually orthogonal in pairs.

To prove these facts we observe that $D' \doteq D\cup VD$ is a fundamental region for $\Gamma(2)$ and write

$$(f,g)' = \iint_{D'} f(\tau)\overline{g(\tau)}y^8\,dx\,dy$$

for any two forms of $\mathfrak{C}_{20}'$. Write $Vz = Z = X+iY$ (X, Y real). Then, since

$$\Psi_0(V\tau) = \tau^{10}\Psi_0(\tau)\,,\qquad \Psi_1(V\tau) = -\tau^{10}\Psi_1(\tau)\,,$$
$$(\Psi_0,\Psi_1)' = (\Psi_0,\Psi_1)+\iint_{VD}\Psi_0(\tau)\overline{\Psi_1(\tau)}y^8\,dx\,dy$$
$$= (\Psi_0,\Psi_1)+\iint_{D}\Psi_0(V\tau)\overline{\Psi_1(V\tau)}\,Y^8\,dX\,dY\,.$$
$$= (\Psi_0,\Psi_1)-(\Psi_0,\Psi_1) = 0$$

and $(\Psi_0,\Psi_1^*)'=0$, similarly. Further,

$$2(\Psi_1,\Psi_1^*) = (\Psi_1,\Psi_1^*)'\,.$$

The double integral for $(\Psi_1,\Psi_1^*)'$ we split into two parts corresponding to the regions D_1 $(x>0)$, D_2 $(x<0)$ to which we apply the transformations U^{-1} and U respectively. Since $\Psi_1(\tau\pm1) = -\Psi_1(\tau)$ and

$$\Psi_1^*(\tau\pm1) = \tfrac{1}{2}\Psi_1^*(\tau)-\tfrac{3}{2}\Psi_0(\tau)\,,$$

we have

$$(\Psi_1,\Psi_1^*)' = (\Psi_1,\tfrac{3}{2}\Psi_0-\tfrac{1}{2}\Psi_1^*)' = \tfrac{3}{2}(\Psi_1,\Psi_0)'-\tfrac{1}{2}(\Psi_1,\Psi_1^*)' = -\tfrac{1}{2}(\Psi_1,\Psi_1^*)\,,$$

from which it follows that

$$(\Psi_1,\Psi_1^*) = (\Psi_1,\Psi_1^*)' = 0\,.$$

These results on orthogonality with respect to $\Gamma(2)$ can also be deduced from Petersson's results on Euler products for Dirichlet series associated with cusp forms in $\mathfrak{C}_{20}'$ [10] (Sätze 5, 6; note that $N=2$, $t=1$, $\varepsilon(n)=1$).

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