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Finally, if $a_{[(y+1)/2]} > n$, we have by (34) for $c_6 = c_6(\varepsilon)$

$$\sum_{1 \leqslant i \leqslant (y+1)/2} a_i > c_6 y^{1+1/(\alpha+\varepsilon)}. \tag{37}$$

Thus we have to show that

$$c_6 y^{1+1/(\alpha+\varepsilon)} > 3T\left(\frac{2y}{C}\right)^{1/\alpha},$$

or, for sufficiently large C,

$$y^{1-\varepsilon/\alpha^2} > T. \tag{38}$$

Indeed (38) is trivial for sufficiently large C and n, since from

$a_{[(y+1)/2]} > n$, $y > A(n) > Cn^{\alpha}$ and by (19) we have $T < 80n^{1-\alpha}/C$.

In the third case (7) holds. By (7) we have for $t > t_0$, $a_t > \frac{1}{2}(t/D)^{1/\alpha}$. Thus

$$\sum_{1 \leqslant i \leqslant (y+1)/2} a_i > \frac{c_7}{D^{1/\alpha}} y^{1+1/\alpha} - c_8,$$

where c_7 and c_8 are absolute constants. Thus we only have to show that for $y > L^{\alpha}$

$$\frac{c_7}{D^{1/\alpha}} y^{1+1/\alpha} - c_8 > 3T\left(\frac{2y}{C}\right)^{1/\alpha}. \tag{39}$$

As before, it suffices to prove (39) for $y = L^{\alpha}$. By (8), (11) and (19), (39) follows from $\alpha^2 + \alpha = 1$ by a simple computation for sufficiently large C and n (if n is large $y = L^{\alpha}$ is also large).

Thus the proof of our Theorem is complete.

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Reçu par la Rédaction le 6. 1. 1962

Uniform distribution mod 1 (II)

by

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1. Introduction. Let $[a, b]$ be an interval properly contained in $[0, 1]$ and define

$$f(\xi) = \begin{cases} 1 & \text{if} \quad a \leqslant \xi \leqslant b, \\ 0 & \text{if} \quad 0 \leqslant \xi < a \quad \text{or} \quad b < \xi < 1, \end{cases} \tag{1.1}$$
$$f(\xi+1) = f(\xi).$$

$f(\xi)$ is the characteristic function of $[a, b]$ extended periodically. The present paper is concerned with the distribution of the sums

$$\sum_{k=1}^{N} f(y+kx)$$

which equal the number of terms among $y+x, y+2x, \ldots, y+Nx$ with fractional part in $[a, b]$. We assume that x and y are independent random variables each with a uniform distribution on $[0, 1]$ and show that

$$(\log N)^{-1} \sum_{k=1}^{N} \big(f(y+kx) - (b-a)\big)$$

has asymptotically a Cauchy distribution. This is expressed in the following

Theorem. *If $|E|$ denotes the Lebesgue measure of the set E, then, for every real α,*

$$\begin{aligned} &\lim_{N\to\infty} P\Big\{(\log N)^{-1} \sum_{k=1}^{N} \big(f(y+kx) - (b-a)\big) \leqslant \alpha\Big\} \\ &= \lim_{N\to\infty} \Big|\Big\{x, y \mid (\log N)^{-1} \sum_{k=1}^{N} \big(f(y+kx) - (b-a)\big) \leqslant \alpha,\ 0 \leqslant x,\ y \leqslant 1\Big\}\Big| \\ &= \frac{1}{\pi} \int_{-\infty}^{\alpha} \frac{dt}{1+t^2}. \end{aligned} \tag{1.2}$$

(1) Research supported by the National Science Foundation under project NSF-G 18837.

ϱ is a positive constant depending on $b-a$ and given by (4.2) *and* (4.3) *below. It has the same value for all irrational values of $b-a$.*

In the first member of (1.2) we used $P\{\ \}$ for the probability of the event between braces. This corresponds to the point of view that x and y are random variables. The proof is based on probabilistic techniques and written in probabilistic terminology even though the final result can be stated without any probabilistic concepts as in the second member of (1.2).

The present paper is a sequel to [4] where the theorem was proved for $b-a$ rational. We use most of the proof of [4] and only replace one or two lemmas which were based on the rationality of $b-a$. In the next section we introduce Lemma 1 and show that the proof of [4] can be taken over with minor modifications if Lemma 1 is used to replace Lemma 2.6 of [4]. Lemma 1 itself is proved in Section 3. In the last section we give an expression for ϱ as a function of $b-a$. Some minor misprints of [4] are corrected at the end of Section 4.

2. Reduction of the proof of the theorem. Let us first introduce some notation, some already used in [4]. As in [4] we use probabilistic terminology (2) and base the proof on properties of continued fractions. Any number ξ, $0 \leqslant \xi \leqslant 1$ can be written as a continued fraction ([3])

$$\xi = [a_1(\xi), a_2(\xi), \ldots] = \cfrac{1}{a_1(\xi)+\cfrac{1}{a_2(\xi)+\ldots}}. \tag{2.1}$$

Except for the zero set of rational ξ (which can be ignored) this expansion does not break off and we can define for irrational ξ the nth convergent $p_n(\xi)/q_n(\xi)$ by

$$\begin{aligned} &q_0(\xi)=1\,, \quad q_1(\xi)=a_1(\xi)\,, \quad q_{n+1}(\xi)=a_{n+1}(\xi)q_n(\xi)+q_{n-1}(\xi)\,,\\ &p_0(\xi)=0\,, \quad p_1(\xi)=1\,, \quad p_{n+1}(\xi)=a_{n+1}(\xi)p_n(\xi)+p_{n-1}(\xi)\,. \end{aligned} \tag{2.2}$$

$\delta_n(\xi)$ is defined by (cf. (3.9) in [4])

$$\delta_n(\xi) = \xi q_n(\xi) - p_n(\xi)\,, \tag{2.3}$$

and the functions U and V by

$$U(z,q,y) = \frac{2}{\pi^2}\sum_{k=1}^{\infty}\frac{\sin 2\pi kz \sin \pi k q(b-a)\cos 2\pi ky}{k^2}\,, \tag{2.4}$$

$$V(z,u,y) = \frac{2}{\pi^2}\sum_{k=1}^{\infty}\frac{\sin 2\pi kz \sin \pi ku\cos 2\pi ky}{k^2}\,. \tag{2.5}$$

(2) $P\{E_1\}$ and $P\{E_1|E_2\}$ are respectively the probability of the event E_1 and the conditional probability of E_1, given E_2. For any function g $Eg(x)$ and $E\{g(x)|E_1\}$ are respectively the expectation of $g(x)$ and the conditional expectation of $g(x)$ given E_1. Similar definitions hold for functions of other variables.

For rational $b-a$ this use of U agrees with [4] and it was proved there without assumptions about the rationality of $(b-a)$ (cf. p. 464) that (1.2) follows if one proves

$$\lim_{N\to\infty} P\Big\{\frac{1}{\log N}\sum_{m=0}^{(\tau-\varepsilon)\log N}(-1)^m a_{m+1}(x)\,U\big(\tfrac{1}{2}N\delta_m(x)\,,\,q_m(x)\,,\,y_m\big)\leqslant a\Big\} = \frac{1}{\pi}\int_{-\infty}^{\varrho(\varepsilon)a}\frac{dt}{1+t^2} \tag{2.6}$$

for sufficiently small $\varepsilon>0$ and

$$\lim_{\varepsilon\downarrow 0}\varrho(\varepsilon) = \varrho\,. \tag{2.7}$$

Here τ is defined by means of (2.10) in [4] and turns out to equal $\pi^{-2}\cdot 12\log 2$ (cf. proposition 2 below). The probability in (2.6) is computed under the assumption that $x, y_0, y_1, \ldots$ are (completely) independent random variables, each with a uniform distribution on $[0,1]$. By the Lévy continuity theorem ([7], pp. 48, 49), (2.6) is equivalent to

$$\lim_{N\to\infty} E\exp\Big\{\frac{it}{\log N}\sum_{m=0}^{(\tau-\varepsilon)\log N}(-1)^m a_{m+1}(x)\,U\big(\tfrac{1}{2}N\delta_m(x)\,,\,q_m(x)\,,\,y_m\big)\Big\} = \exp\Big(-\frac{|t|}{\varrho(\varepsilon)}\Big) \tag{2.8}$$

for each fixed t and this is the relation we intend to prove. Lemma 3.3 and most of the proof of Lemma 3.4 of [4] still hold even if $b-a$ is irrational. The difficulty starts when Lemma 2.6 is used on p. 469 of [4]. Lemma 2.6 will be replaced by

LEMMA 1. *If $b-a$ is irrational, then there exists a sequence $K=\{k_1<k_2<\ldots\}$ of non-negative integers of density zero (i.e. such that*

$$\lim_{n\to\infty} n^{-1}\{\textit{number of } k_i\leqslant n\} = 0) \tag{2.9}$$

such that

$$\begin{aligned} \lim_{p\to\infty}\varlimsup_{\substack{k\to\infty\\ k\notin K\\ N\to\infty}}\sum_{b_1,\ldots,b_k} P\{a_i(x)=b_i\,,\ i=1,\ldots,k\}\cdot\log N\times\\ \times\Big|E\exp\Big\{\frac{it}{\log N}(-1)^{k+p+1}a_{k+p+2}(x)\,U\big(z,q_{k+p+1}(x),y\big)\Big|\,a_i(x)=b_i\,,\\ i=1,\ldots,k,\ a_{k+j}(x)\leqslant(\log N)^3\,,\ j=1,\ldots,p+2\Big\}-\Big(1-\frac{C|t|}{\log N}\Big)\Big| = 0 \end{aligned} \tag{2.10}$$

where

$$C = \frac{\pi}{2\log 2}\int_0^1\int_0^1\int_0^1 dz\,du\,dy\,|V(z,u,y)| \tag{2.11}$$

and the expectation in the right hand side of (2.10) *is computed under the assumption that* x, y, z *are independent random variables, each uniformly distributed on* $[0,1]$.

Before we prove Lemma 1 we shall show that it can take the place of Lemma 2.6 in [4]. More precisely we prove the following

PROPOSITION 1. *If Lemma* 1 *holds then the theorem follows. For irrational* $b-a$ ϱ *is given by* $\tau^{-1}C^{-1}$ *with* τ *defined in* (2.10) *of* [4] *and* C *by* (2.11) *above.*

Proof. Since the theorem has been proved in [4] for $b-a$ rational (except for the computation of ϱ given in Proposition 2), we may, and shall, assume $b-a$ to be a fixed irrational number.

$C_{12}, C_{13}, \ldots$ will be finite positive constants in the proof below. We start with C_{12} to avoid confusion with the constants in [4]. Let t be fixed and $\varepsilon_1 > 0$. Choose p, k_0, N_0 such that for $k \geqslant k_0$, $k \notin K$, $N \geqslant N_0$

$$\frac{1}{\log N}\{\text{number of } k_i \leqslant \tau\log N\} \leqslant \varepsilon_1 \tag{2.12}$$

and

$$\sum_{b_1,\ldots,b_k} P\{a_i(x) = b_i,\ i=1,\ldots,k\}\cdot\log N \times \tag{2.13}$$

$$\times\left|E\exp\left\{\frac{it}{\log N}(-1)^{k+p+1}a_{k+p+2}(x)\,U\big(z, q_{k+p+1}(x), y\big)\right| a_i(x) = b_i,\right.$$

$$\left. i=1,\ldots,k,\ a_{k+j}(x) \leqslant (\log N)^3,\ j=1,\ldots,p+2\right\} - \left(1-\frac{C|t|}{\log N}\right)\Bigg| \leqslant \varepsilon_1 .$$

We start with proving the analogue of Lemma 3.4 of [4]. Till the top of p. 469 no real changes are needed. Except for the fact that p is not a function of N now, we put as in [4]

$$\eta = \tfrac{1}{2}\varepsilon\tau^{-2},$$

$$G_m = \left\{x \,\Big|\, \sup_{1\leqslant k\leqslant m+2} a_k(x) \leqslant (\log N)^3 \ \&\ q_{m-p}(x) \leqslant \exp\left(\left(\frac{1}{\tau}+\eta\right)(m-p)\right)\right\},$$

$$f_m(x,y) = \exp\left\{\frac{it}{\log N}\sum_{s=0}^{m}(-1)^s a_{s+1}(x)\,U\left(\tfrac{1}{2}N\delta_s, q_s(x), y_s\right)\right\} - 1 .$$

Then we define $r_1(x), r_2(x)$ and $f'_m(x,y)$ as in [4] and without change of the proof one obtains for $p \leqslant m \leqslant (\tau-\varepsilon)\log N$ (dropping for shortness the argument x of several functions)

$$\left|Ef_m(x,y)\left(\exp\left\{\frac{it}{\log N}(-1)^{m+1}a_{m+2}\,U\left(\tfrac{1}{2}N\delta_{m+1}, q_{m+1}, y_{m+1}\right)\right\}-1\right)\right. \tag{2.14}$$

$$-\int_0^1 dy\int_{G_m} dx\left[(f'_{r_1-1}(x,y)+1)\int_0^1 \exp\left\{\frac{it}{\log N}\sum_{s=r_1}^{m-p}(-1)^s a_{s+1}\,U(z_s,q_s,y_s)\right\}dz-1\right]$$

$$\left.\times\int_0^1\left(\exp\left\{\frac{it}{\log N}(-1)^{m+1}a_{m+2}\,U(z_{m+1},q_{m+1},y_{m+1})\right\}-1\right)dz_{m+1}\right|$$

$$\leqslant \frac{C_{12}p(\log\log N)^2}{(\log N)^2} + C_{12}A(\eta)\exp\big(-(m-p)B(\eta)\big)$$

$$+ C_{12}(\log N)^{3p+9}N^{-1}\exp\left(\left(\frac{1}{\tau}+\eta\right)(m-p)\right) + C_{12}\frac{\log\log N}{(\log N)^2}\int_{G_m} dx\sum_{s=r_2(x)}^{r_1(x)-1} a_{s+1}(x) .$$

The expectation in (2.14) is computed for $x, y_0, \ldots, y_{m+1}$ independent random variables, each uniformly distributed on $[0,1]$. $\int_0^1 dy$ stands for $\int_0^1\ldots\int_0^1 dy_1\ldots dy_{m+1}$ and $\int_0^1 dz$ stands for $\int_0^1\ldots\int_0^1 dz_{r_1}\ldots dz_{m-p}$. $A(\eta), B(\eta)$ are positive constants introduced in Lemma (2.1) of [4]. It is important to realize that $r_1(x)$ and $f'_{r_1-1}(x,y)$ depend on x only through the values of $a_1(x), \ldots, a_{m-p}(x)$. Let now $m-p \notin K$, $m-p \geqslant k_0$, $N \geqslant N_0$. We then evaluate in (2.14) first the integral over $0 \leqslant y_{m+1}, z_{m+1} \leqslant 1$ and $x \in G_m$ with $a_i(x) = b_i$, $i = 1, \ldots, m-p$.

$$\left[(f'_{r_1-1}(x,y)+1)\int_0^1 \exp\left\{\frac{it}{\log N}\sum_{s=r_1}^{m-p}(-1)^s a_{s+1}\,U(z_s,q_s,y_s)\right\}dz-1\right]$$

is a constant in this region and the integral of

$$\exp\left\{\frac{it}{\log N}(-1)^{m+1}a_{m+2}\,U(z_{m+1},q_{m+1},y_{m+1})\right\}-1$$

becomes

$$\cdot P\{a_i(x) = b_i,\ i=1,\ldots,m-p,\ a_{m-p+j}(x) \leqslant (\log N)^3,\ j=1,\ldots,p+2\}\times$$

$$\times\left(E\left\{\exp\left(\frac{it}{\log N}(-1)^{m+1}a_{m+2}\,U(z,q_{m+1},y)\right)\right| a_i(x) = b_i,\ i=1,\ldots,m-p,$$

$$\left. a_{m-p+j}(x) \leqslant (\log N)^3,\ j=1,\ldots,p+2\right\}-1\right).$$

Summing over the possible choices of $b_1, \dots, b_{m-p}$ for $x \in G_m$ gives then, by (2.13)

(2.15)
$$\Big|\int_0^1 dy \int_{G_m} dx \Big[(f'_{r_1-1}(x,y)+1)\int_0^1 \exp\Big\{\frac{it}{\log N}\sum_{s=r_1}^{m-p}(-1)^s a_{s+1}U(z_s,q_s,y_s)\Big\}dz - 1\Big]\times$$
$$\times \int_0^1 \Big(\exp\Big\{\frac{it}{\log N}(-1)^{m+1}a_{m+2}U(z_{m+1},q_{m+1},y_{m+1})\Big\}-1\Big)dz_{m+1}$$
$$-\int_0^1 dy \int_{G_m} dx \Big[(f'_{r_1-1}(x,y)+1)\int_0^1 \exp\Big\{\frac{it}{\log N}\sum_{s=r_1}^{m-p}(-1)^s a_{s+1}U(z_s,q_s,y_s)\Big\}dz - 1\Big]\times$$
$$\times\Big(-C\frac{|t|}{\log N}\Big)\Big| \leqslant \frac{2\varepsilon_1}{\log N}.$$

Finally, as in [4] we can reverse the steps to show

(2.16)
$$\Big|\int_0^1 dy \int_{G_m} dx \Big[(f'_{r_1-1}(x,y)+1)\int_0^1 \exp\Big\{\frac{it}{\log N}\sum_{s=r_1}^{m-p}(-1)^s a_{s+1}U(z_s,q_s,y_s)\Big\}dz - 1\Big]\times$$
$$\times C\frac{|t|}{\log N} - Ef_m(x,y)\frac{C|t|}{\log N}\Big|$$
$$\leqslant C_{12}\frac{p\log\log N}{(\log N)^2} + C_{12}A(\eta)\exp\big(-(m-p)B(\eta)\big) + \frac{C_{12}}{(\log N)^2}\int_{G_m} dx \sum_{s=r_2(x)}^{r_1(x)-1} a_{s+1}(x).$$

Combining (2.14)-(2.16) one has for $m-p \notin K$, $m-p \geqslant k_0$, $N \geqslant N_0$, $p \leqslant m \leqslant (\tau-\varepsilon)\log N$,

(2.17)
$$\Big|Ef_m(x,y)\Big(\exp\Big\{\frac{it}{\log N}(-1)^{m+1}a_{m+2}U(\tfrac{1}{2}N\delta_{m+1},q_{m+1},y_{m+1})\Big\}-1\Big)$$
$$+Ef_m(x,y)\,C\frac{|t|}{\log N}\Big|$$
$$\leqslant 2C_{12}\frac{p(\log\log N)^2}{(\log N)^2} + 2C_{12}A(\eta)\exp\big(-(m-p)B(\eta)\big)$$
$$+C_{12}(\log N)^{3p+9}N^{-1}\exp\Big(\Big(\frac{1}{\tau}+\eta\Big)(m-p)\Big)$$
$$+2C_{12}\frac{\log\log N}{(\log N)^2}\int_{G_m} dx \sum_{s=r_2(x)}^{r_1(x)-1} a_{s+1}(x) + \frac{2\varepsilon_1}{\log N}.$$

Moreover, for any $m, b_1, \dots, b_{m+1}$

(2.18)
$$\Big|\int_0^1\int_0^1 dy\,dz\,E\Big\{\exp\Big(\frac{it}{\log N}(-1)^{m+1}a_{m+2}U(z,q_{m+1},y)\Big)\,|\,a_i(x)=b_i,$$
$$i=1,\dots,m+1\Big\}-1\Big| \leqslant \frac{C_{13}}{\log N}$$

(cf. (3.5), (3.41), (3.50) in the next section) so that, replacing $\frac{2\varepsilon_1}{\log N}$ by $\frac{C_{13}}{\log N}$ in the right hand side of (2.17) makes (2.17) valid for all m, for which $p \leqslant m \leqslant (\tau-\varepsilon)\log N$.

Notice that (2.13) also implies

(2.19)
$$\Big|E\exp\Big\{\frac{it}{\log N}(-1)^{m+1}a_{m+2}U(\tfrac{1}{2}N\delta_{m+1},q_{m+1},y_{m+1})\Big\}-1+\frac{C|t|}{\log N}\Big|$$
$$\leqslant \frac{C_{12}}{(\log N)^2}+C_{12}A(\eta)\exp\big(-(m-p)B(\eta)\big)$$
$$+C_{12}(\log N)^{3p+9}N^{-1}\exp\Big(\Big(\frac{1}{\tau}+\eta\Big)(m-p)\Big)+\frac{2\varepsilon_1}{\log N}$$

if $m-p \notin K$, $m-p \geqslant k_0$, $N \geqslant N_0$. For other m (2.19) again holds with $\frac{2\varepsilon_1}{\log N}$ replaced by $\frac{C_{13}}{\log N}$.

We now complete the proof as in [4] by Lemma 2.7 in [4]. More precisely, from the proof of Lemma 2.7 in [4] and (2.17)-(2.19) one deduces

(2.20)
$$\Big|E\exp\Big\{\frac{it}{\log N}\sum_{m=\log\log N}^{(\tau-\varepsilon)\log N}(-1)^m a_{m+1}U(\tfrac{1}{2}N\delta_m(x),q_m(x),y_m)\Big\}-\prod_{m=\log\log N}^{(\tau-\varepsilon)\log N}\Big(1-\frac{C|t|}{\log N}\Big)\Big|$$
$$\leqslant 2\sum_{m=\log\log N}^{(\tau-\varepsilon)\log N}\Big|E\exp\Big\{(-1)^m a_{m+1}U(\tfrac{1}{2}N\delta_m(x),q_m(x),y_m)\Big\}-\Big(1-\frac{C|t|}{\log N}\Big)\Big|$$
$$+\sum_{m=\log\log N}^{(\tau-\varepsilon)\log N}\Big|Ef_m(x,y)\Big(\exp\Big\{\frac{it}{\log N}(-1)^{m+1}a_{m+2}U(\tfrac{1}{2}N\delta_{m+1},q_{m+1},y_{m+1})\Big\}-1\Big)$$
$$+Ef_m(x,y)\frac{C|t|}{\log N}\Big|$$
$$\leqslant C_{14}\frac{p(\log\log N)^2}{\log N}+C_{14}\sum_{m=\log\log N}^{(\tau-\varepsilon)\log N}A(\eta)\exp\big(-(m-p)B(\eta)\big)$$
$$+C_{14}\sum_{m=\log\log N}^{(\tau-\varepsilon)\log N}(\log N)^{3p+9}N^{-1}\exp\Big(\Big(\frac{1}{\tau}+\eta\Big)(m-p)\Big)$$
$$+C_{14}\sum_{m=\log\log N}^{(\tau-\varepsilon)\log N}\frac{\log\log N}{(\log N)^2}\int_{G_m} dx\sum_{s=r_2(x)}^{r_1(x)-1}a_{s+1}(x)+C_{14}\varepsilon_1$$
$$+C_{14}\frac{1}{\log N}\{\text{number of } k_i \leqslant \tau\log N,\ k_i \in K\}.$$

It was shown in [4], pp. 469, 470 that the first four terms in the right hand side of (2.20) tend to zero as $N\to\infty$ and the last one is at most $C_{14}\varepsilon_1$ by (2.12). Finally

$$\frac{1}{\log N}\sum_{m=0}^{\log\log N}(-1)^m a_{m+1}U\left(\tfrac{1}{2}N\delta_m(x),\, q_m(x),\, y_m(x)\right)\to 0$$

in probability (e.g. by [7], bottom of p. 322).

$$\lim_{N\to\infty}\prod_{m=0}^{(\tau-\varepsilon)\log N}\left(1-\frac{C|t|}{\log N}\right)=\exp C(\tau-\varepsilon)|t|$$

so that

$$\overline{\lim_{N\to\infty}}\left|E\exp\frac{it}{\log N}\sum_{m=0}^{(\tau-\varepsilon)\log N}(-1)^m a_{m+1}U\left(\tfrac{1}{2}N\delta_m(x),\, q_m(x),\, y_m(x)\right)-\exp C(\tau-\varepsilon)|t|\right|\leqslant 2C_{14}\varepsilon_1\,.$$

Since $\varepsilon_1>0$ was arbitrary this proves (2.8) with $\varrho(\varepsilon)=C^{-1}(\tau-\varepsilon)^{-1}$ and the theorem follows.

The proof of the theorem has therefore reduced to the proof of Lemma 1 which will be given in the next section.

3. Proof of Lemma 1. Unfortunately, we have not been able to find a short proof of Lemma 1. The proof will therefore be broken down into several lemmas. The present proof shows resemblance to the proof of Theorem 4 in [2].

First some new notation. For any $0\leqslant u,\, v\leqslant 1$, put

$$\begin{aligned}&w_{-1}(u,v)=v\,,\\ &w_0(u,v)=u\,,\\ &w_k(u,v)=a_k(x)w_{k-1}(u,v)+w_{k-2}(u,v)\,,\qquad k=1,2,\ldots\end{aligned}\tag{3.1}$$

Actually w_k depends on x also and is therefore a random variable whenever x is. We usually don't exhibit this dependence. Using (2.2) one shows by induction

$$w_k(u,v)=q_k(x)u+p_k(x)v\,,\qquad k=0,1,\ldots\tag{3.2}$$

If x is again uniformly distributed on $[0,1]$ and A is some condition on x we put

$$\nu_k(r,s;u,v\,|\,A)=E\left\{\exp\left(2\pi ir\,w_k(u,v)+2\pi is\,w_{k-1}(u,v)\right)|\,A\right\}.\tag{3.3}$$

Before proving the next lemmas we recall some facts about the distribution of continued fractions (most of which appear in [7] and have been used in [4]; also [5] contains some of the material). C_i again denotes a finite positive constant.

For any measurable set $E\subseteq[0,1]$ put

$$\mu(E)=\frac{1}{\log 2}\int_E\frac{dt}{1+t}\,.$$

This measure is stationary i.e. $\mu\{x\colon [a_1(x),a_2(x),\ldots]\in E\}=\mu\{x\colon [a_k(x),a_{k+1}(x),\ldots]\in E\}$. For any set E_k of the form $E_k=\{x\colon a_{k+j}(x)=b_j,\ j=1,2,\ldots,n\}$ one has uniformly in q,n,a_j,b_j

$$\left|P\{x\in E_k\,|\,a_j(x)=a_j\,,\ j=1,\ldots,q\}-\mu(E_k)\right|\leqslant C_{15}\lambda_3^{k-q}\mu(E_k)\tag{3.4}$$

for some $0<\lambda_3<1$ (cf. (2.6) in [4], [7] or [8]). Also, for all $a_1,\ldots,a_k$

$$C_{15}^{-1}\mu(E_k)\leqslant P\{x\in E_k\,|\,a_j(x)=a_j\,,\ j=1,\ldots,k\}\leqslant C_{15}\mu(E_k)\,.\tag{3.5}$$

Moreover, $P\{x\in E_k\}$ depends mainly on the digits just before k, i.e. for $s\geqslant -k+q$

$$\begin{aligned}&\left|P\{x\in E_k\,|\,a_j(x)=a_j\,,\ j=1,\ldots,k\}\right.\\ &\qquad\left.-P\{x\in E_{k+s}\,|\,a_{j+s}(x)=a_j\,,\ j=k-q+1,\ldots,k\}\right|\leqslant C_{15}\lambda_3^q\mu(E_k)\,.\end{aligned}\tag{3.6}$$

This follows from [7], p. 292, formula 8 and has been derived more explicity in [1], pp. 355, 356, footnote 6. Since $\frac{1}{\log 2}\cdot\frac{1}{1+t}$, the density of μ, is continuously differentiable and since with $a_j(x)=a_j$, $j=1,\ldots,k$ fixed, x can vary at most over an interval of length $2\lambda_1^{-2k}$, $\lambda_1>1$ (cf. (2.3) in [4]) one has also

$$\begin{aligned}&\mu(E_k\,|\,a_j(x)=a_j\,,\ j=1,\ldots,k)\\ &=\left\{\frac{1}{\log 2}\int_{\substack{a_j(t)=a_j\\ j=1,\ldots,k}}\frac{dt}{1+t}\right\}^{-1}\frac{1}{\log 2}\int_{\substack{a_j(t)=a_j\\ j=1,\ldots,k\\ t\in E_k}}\frac{dt}{1+t}\\ &=\left\{\int_{\substack{a_j(t)=a_j\\ j=1,\ldots,k}}dt\right\}^{-1}\int_{\substack{a_j(t)=a_j\\ j=1,\ldots,k\\ t\in E_k}}dt(1+2\,\theta\lambda_1^{-2k})\\ &=P\{E_k\,|\,a_j(x)=a_j,\ j=1,\ldots,k\}(1+2\,\theta\lambda_1^{-2k})\qquad\text{with }|\theta|\leqslant 1\,.\end{aligned}\tag{3.7}$$

Since $\mu(E_k\,|\,a_j(x)=a_j,\ j=k-q+1,\ldots,k)$ is an average of expressions $\mu(E_k\,|\,a_j(x)=a_j,\ j=1,\ldots,k)$, (3.6)-(3.7) combined give

$$\begin{aligned}&\left|P\{E_k\,|\,a_j(x)=a_j\,,\ j=k-q+1,\ldots,k\}\right.\\ &\qquad\left.-\mu(E_k\,|\,a_j(x)=a_j\,,\ j=k-q+1,\ldots,k)\right|\leqslant C_{16}\lambda^q\mu(E_k)\end{aligned}\tag{3.8}$$

for some $0<\lambda<1$.

LEMMA 2. *Let A_k be arbitrary conditions defined in terms of $a_{k+1}(x)$, $a_{k+2}(x)$, ... and let B stand for the condition $a_j(x) = a_j$, $j = 1, \dots, q$. Then for $r, s \neq 0, 0$*

$$\int_0^1\int_0^1 du\,dv \left| \nu_k(r, s; u, v \mid A_k, B)\right|^2 \leqslant C_{17}\lambda^k \tag{3.9}$$

where C_{17} depends on $r, s, a_1, \dots, a_q$ and λ is a constant with $0 < \lambda < 1$.

Proof. Write

$$P\{m_1, m_2, m_3, m_4\} = P\{q_k(x) = m_1, q_{k-1}(x) = m_2, p_k(x) = m_3, p_{k-1}(x) = m_4 \mid A_k, B\}.$$

Then using (3.2)

$$\int_0^1\int_0^1 du\,dv \left|\nu_k(r, s; u, v \mid A_k, B)\right|^2 = \int_0^1\int_0^1 du\,dv \sum_{\substack{1\leqslant m_j\leqslant\infty \\ j=1,\dots,8}} P\{m_1, m_2, m_3, m_4\}\times$$

$$\times P\{m_5, m_6, m_7, m_8\}\times$$

$$\times \exp\{2\pi i u(r(m_1-m_5)+s(m_2-m_6)) + 2\pi i v(r(m_3-m_7)+s(m_4-m_8))\}$$

$$= \sum_{m_5,\dots,m_8} P\{m_5, m_6, m_7, m_8\} \sum_{\substack{rm_1+sm_2=rm_5+sm_6 \\ rm_3+sm_4=rm_7+sm_8}} P\{m_1, m_2, m_3, m_4\}$$

$$\leqslant \sup_{m,n} P\{rq_k(x)+sq_{k-1}(x) = m,\ rp_k(x)+sp_{k-1}(x) = n \mid A_k, B\}.$$

But ([3]) for any ξ

$$\left|\frac{p_k(\xi)}{q_k(\xi)} - \xi\right| \leqslant \frac{1}{q_k^2(\xi)}, \qquad \left|\frac{p_{k-1}(\xi)}{q_{k-1}(\xi)} - \xi\right| \leqslant \frac{1}{q_k(\xi)q_{k-1}(\xi)}$$

and for some $\lambda_1 > 1$ (cf. (2.2) in [4])

$$q_k(\xi) \geqslant \lambda_1^k$$

so that for $m \neq 0$

$$P\{rq_k+sq_{k-1} = m,\ rp_k+sp_{k-1} = n \mid A_k, B\} \tag{3.10}$$

$$\leqslant P\left\{rq_k+sq_{k-1} = m,\ |x(rq_k+sq_{k-1})-n| \leqslant \frac{|r|+|s|}{\lambda_1^k},\ B\right\}\times$$

$$\times \frac{P\{A_k \mid rq_k+sq_{k-1} = m,\ rp_k+sp_{k-1} = n, B\}}{P\{B\}\cdot P\{A_k \mid B\}}$$

$$\leqslant P\left\{\left|x-\frac{n}{m}\right| \leqslant \frac{|r|+|s|}{m\lambda_1^k}\right\} \frac{P\{A_k \mid rq_k+sq_{k-1} = m,\ rp_k+sp_{k-1} = n, B\}}{P\{B\}\cdot P\{A_k \mid B\}}.$$

By (3.5), for any $a_1, \dots, a_k$, $b_1, \dots, b_k$

$$\frac{P\{A_k \mid a_j(x) = a_j,\ j = 1, \dots, k\}}{P\{A_k \mid a_j(x) = b_j,\ j = 1, \dots, k\}} \leqslant C_{18} \tag{3.11}$$

and of course

$$P\left\{\left|x-\frac{n}{m}\right| \leqslant \frac{|r|+|s|}{m\lambda_1^k}\right\} \leqslant \frac{2|r|+2|s|}{m\lambda_1^k}$$

so that the right hand side of (3.10) is at most

$$\frac{2C_{18}(|r|+|s|)}{P\{B\}\lambda_1^k}.$$

If $m = 0$, then by the second member of (3.10) we can only take $n = 0$ also, as soon as $\lambda_1^k > |r|+|s|$. But then the probability in (3.10) equals zero since $\frac{p_k}{q_k} \neq \frac{p_{k-1}}{q_{k-1}}$ with probability one. This shows (3.9) for sufficiently large k and by proper choice of C_{17} for all $k \geqslant 1$.

LEMMA 3. *Let A stand for $a_j(x) = a_j$, $j = 1, \dots, q$ and let I_1, I_2 be arbitrary intervals contained in $[0, 1]$. Then for any $c_1, \dots, c_t$* [3]

$$\lim_{p\to\infty} P\{w_p(u, v) \in I_1 \,(\mathrm{mod}1),\ w_{p-1}(u, v) \in I_2 \,(\mathrm{mod}1),\ a_{p-t+j} = c_j,$$

$$j = 1, \dots, t \mid A\} = |I_1|\,|I_2|\,\mu(a_j(x) = c_j,\ j = 1, \dots, t) \tag{3.12}$$

for almost all u, v. ($w_p(u, v) \in I_1 \,(\mathrm{mod}1)$ of course means that the fractional part of $w_p(u, v)$ lies in I_1.)

Proof. Let us write B_p for the condition

$$a_{p-t+j} = c_j, \qquad j = 1, \dots, t$$

and let us drop for shortness the expressions "mod1". Since we know (cf. (3.4))

$$\lim_{p\to\infty} P\{B_p \mid a_j(x) = a_j,\ j = 1, \dots, q\} = \mu(a_j(x) = c_j,\ j = 1, \dots, t)$$

one only has to show

$$\lim_{p\to\infty} P\{w_p(u, v) \in I_1,\ w_{p-1}(u, v) \in I_2 \mid A, B_p\} = |I_1|\,|I_2| \quad \text{a.e. in } u, v.$$

The convergence of $\sum_{k=1}^{\infty} C_{17}\lambda^k$ and Lemma 2 imply (cf. [9], p. 345) for $r, s \neq 0, 0$

$$\lim_{p\to\infty} \nu_{p-t}(r, s; u, v \mid A, B_p) = 0 \quad \text{a.e. in } u, v. \tag{3.13}$$

Of course, for all $u, v, p \geqslant t$

$$\nu_{p-t}(0, 0; u, v \mid A, B_p) = 1. \tag{3.14}$$

[3] $|I|$ denotes the length of the interval I.

(3.13) and (3.14) show that the Fourier coefficients of the distribution function

$$P\{w_{p-t}(u,v) \leqslant \alpha \,(\mathrm{mod}1)\,,\ w_{p-t-1}(u,v) \leqslant \beta \,(\mathrm{mod}1) \,|\, A, B_p\}$$

converge to the Fourier coefficients of the distribution function

$$G(\alpha,\beta) = \alpha\cdot\beta$$

on the unit square. Consequently (compare [9])

$$\lim_{p\to\infty} P\{w_{p-t}(u,v) \leqslant \alpha \,(\mathrm{mod}1)\,,\ w_{p-t-1}(u,v) \leqslant \beta \,(\mathrm{mod}1) \,|\, A, B_p\} = \alpha\beta$$

in the weak sense a.e. in u, v. It is easy to see that if the fractional parts of w_{p-t-1} and w_{p-t} are uniformly distributed on the unit square, then so are the fractional parts of

$$w_{p-t} \quad \text{and} \quad w_{p-t+1} = a_{p-t+1}w_{p-t} + w_{p-t-1}\,.$$

By induction one has therefore

$$\lim_{p\to\infty} P\{w_p(u,v) \leqslant \alpha \,(\mathrm{mod}1)\,,\ w_{p-1}(u,v) \leqslant \beta \,(\mathrm{mod}1) \,|\, A, B_p\} = \alpha\beta$$

in the weak sense a.e. in u, v. This means that for any continuous function $g(\alpha,\beta)$

$$\lim_{p\to\infty} \int_0^1\int_0^1 g(\alpha,\beta)\,dP\{w_p(u,v) \leqslant \alpha \,(\mathrm{mod}1)\,,\ w_{p-1}(u,v) \leqslant \beta \,(\mathrm{mod}1) \,|\, A, B_p\} = \int_0^1\int_0^1 g(\alpha,\beta)\,d\alpha\,d\beta\,.$$

Approximating the characteristic function of $I_1 \times I_2$ by continuous functions, it is now easy to complete the proof.

From now on we take $b-a$ *irrational* and put

$$\theta = b-a\,.$$

Furthermore for any event E we put

$$F_k(I_1, I_2, E) = P\{w_k(\theta,0) \in I_1 \,(\mathrm{mod}1)\,,\ w_{k-1}(\theta,0) \in I_2 \,(\mathrm{mod}1),\, E\}$$

and similarly $F_k(I_1, I_2, E\,|F)$ for the conditional probability, given F.

LEMMA 4. *There exists a finite constant C_{19} such that for any conditions A_k, defined in terms of $a_{k+1}(x), a_{k+2}(x), \ldots$ only and arbitrary intervals I_1, I_2, contained in $[0,1]$*

$$\overline{\lim_{n\to\infty}} \frac{1}{n}\sum_{k=1}^{n} F_k(I_1, I_2, A_k) \leqslant C_{19}|I_1||I_2|\mu(A_0)\,.$$

Proof. Again it suffices to prove (cf. (3.5))

$$\overline{\lim_{n\to\infty}} \frac{1}{n}\sum_{k=1}^{n} F_k(I_1, I_2) \leqslant C_{20}|I_1||I_2|\,.$$

Fix I_1 and I_2 and let

$$M_i = \{m_{i,1} < m_{i,2} < \ldots\}\,, \qquad i = 1, 2$$

be the sequence of all integers for which

$$\text{fractional part of } m\theta \in I_i\,.$$

By the uniform distribution theorem of H. Weyl [9] and the irrationality of θ

$$\lim_{n\to\infty} \frac{1}{n}\{\text{number of } m_{i,j} \leqslant n\} = |I_i|\,, \qquad i = 1, 2\,. \tag{3.15}$$

By (3.2) $w_k(\theta, 0) = q_k(x)\theta$, so that

$$\frac{1}{n}\sum_{k=1}^{n} F_k(I_1, I_2) = \frac{1}{n}\sum_{k=1}^{n} P\{q_k(x) \in M_1,\, q_{k-1}(x) \in M_2\} = \frac{1}{n}\sum_{k=1}^{n}\sum_{m_{i,j}\in M_i} P\{q_k(x) = m_{1,j_1},\, q_{k-1}(x) = m_{2,j_2}\}\,. \tag{3.16}$$

We split the sum in the last member of (3.16) into two parts. The first part over

$$m_{2,j_2} \geqslant e^{2n/\tau}$$

and the second sum over the remaining pairs $m_{i,j}$

$$\sum_{k=1}^{n} \sum_{\substack{m_{i,j}\in M_i \\ m_{2,j_2} \geqslant e^{2n/\tau}}} P\{q_k(x) = m_{1,j_1},\, q_{k-1}(x) = m_{2,j_2}\} \leqslant nP\{q_{n-1}(x) \geqslant e^{2n/\tau}\} \to 0 \qquad (n\to\infty) \tag{3.17}$$

(Lemma (2.1) in [4]). Now for any fixed m_1, m_2, $q_k = m_1$, $q_{k-1} = m_2$ implies

$$\frac{m_1}{m_2} = \frac{q_k(x)}{q_{k-1}(x)} = a_k(x) + [a_{k-1}(x), \ldots, a_1(x)]\,. \tag{3.18}$$

Since m_1/m_2 can at most be expanded in two ways in a finite continued fraction ([3], p. 136, Theorem 162) there are at most two possible choices of k, say k_1 and k_2, and the corresponding $a_1, \ldots, a_k$ in (3.18). But $a_1(x), \ldots, a_k(x)$ determine $p_k(x)$ uniquely, so that there are also at most two choices

for $p_k(x)$, say p_{k_1} and p_{k_2}. Thus $q_k(x) = m_1$, $q_{k-1}(x) = m_2$ is at most possible for $k = k_1$ and

$$\left| x - \frac{p_{k_1}}{m_1} \right| \leqslant \frac{1}{m_1^2}$$

or $k = k_2$ and

$$\left| x - \frac{p_{k_2}}{m_1} \right| \leqslant \frac{1}{m_1^2}.$$

Consequently

$$\sum_{k=1}^{n} P\{q_k(x) = m_{1,j_1},\, q_{k-1}(x) = m_{2,j_2}\} \leqslant \frac{4}{m_{1,j_1}^2} \tag{3.19}$$

whenever $m_{2,j_2} < m_{1,j_1}$. When $m_{2,j_2} \geqslant m_{1,j_1}$ the sum in (3.19) is of course zero. We get now for our second sum

$$\frac{1}{n}\sum_{k=1}^{n} \sum_{\substack{m_{i,j} \in M_i \\ m_{2,j_2} \leqslant e^{2n/\tau}}} P\{q_k(x) = m_{1,j_1},\, q_{k-1}(x) = m_{2,j_2}\} \leqslant \frac{1}{n} \sum_{\substack{m_{2,j_2} \leqslant e^{2n/\tau} \\ m_{i,j} \in M_i}} \sum_{m_{1,j_1} > m_{2,j_2}} \frac{4}{m_{1,j_1}^2}. \tag{3.20}$$

By partial summation one sees from (3.15) for $i = 1$ and sufficiently large m_{2,j_2}

$$\sum_{\substack{m_{1,j_1} > m_{2,j_2} \\ m_{1,j_1} \in M_1}} \frac{4}{m_{1,j_1}^2} \leqslant \frac{8\,|I_1|}{m_{2,j_2}} \tag{3.21}$$

and then from (3.15) for $i = 2$ for large n

$$\sum_{\substack{m_{2,j_2} \leqslant e^{2n/\tau} \\ m_{2,j_2} \in M_2}} \frac{8\,|I_1|}{m_{2,j_2}} \leqslant 16\,|I_1|\,|I_2| \log e^{2n/\tau}. \tag{3.22}$$

The lemma follows from (3.16), (3.17) and (3.20)-(3.22).

LEMMA 5 (4). *Let A_k stand for*

$$a_{k+j}(x) = a_j, \qquad j = 1, \ldots, q.$$

Then for each $r, s \neq 0, 0$

$$\overline{\lim}\, \frac{1}{n} \sum_{k=1}^{n} |\nu_k(r, s;\, \theta, 0\,|\,A_k)|^2 = 0.$$

Proof. Write $B_{k-m}(b_i)$ for

$$a_{k-m+i} = b_i, \qquad i = 1, \ldots, t.$$

(4) Lemma 5 and (3.30) are only used to prove (3.39). It would have been simpler for this to formulate lemma 5 and (3.30) without the conditions A_k resp. $A_k(a_j)$. On the other hand it is interesting to find out whether the limit relation (3.30) holds in general, without the restriction $k \notin K$.

Then for each $t \leqslant m$

$$\begin{aligned}
&\frac{1}{n}\sum_{k=1}^{n} |\nu_k(r, s;\, \theta, 0\,|\,A_k)|^2 \\
&\leqslant \frac{m}{n} + \frac{1}{n} \sum_{k=m+1}^{n} \Big| \sum_{b_1, \ldots, b_t} \int_0^1 \int_0^1 d_{u,v} F_{k-m}([0, u], [0, v], B_{k-m}(b_i)\,|\,A_k) \times \\
&\times E\Big\{\exp\big(2\pi i r w_k(\theta, 0) \\
&\qquad + 2\pi i s w_{k-1}(\theta, 0)\big)\,|\,B_{k-m}(b_i),\, A_k,\, w_{k-m}(x) \equiv u,\, w_{k-m-1}(x) \equiv v\Big\}\Big|^2 \\
&\leqslant \frac{m}{n} + \frac{1}{n} \sum_{k=m+1}^{n} \sum_{b_1, \ldots, b_t} \int_0^1 \int_0^1 d_{u,v} F_{k-m}([0, u], [0, v], B_{k-m}(b_i)\,|\,A_k) \times \\
&\times \Big| E\Big\{\exp\big(2\pi i r w_k(\theta, 0) \\
&\qquad + 2\pi i s w_{k-1}(\theta, 0)\big)\,|\,B_{k-m}(b_i),\, A_k,\, w_{k-m}(x) \equiv u,\, w_{k-m-1} \equiv v\Big\}\Big|^2.
\end{aligned} \tag{3.23}$$

For $w_{k-m} \equiv u$, $w_{k-m-1} \equiv v$ fixed, $w_{k-1}(\theta, 0)$ and $w_k(\theta, 0)$ are determined mod1 by

$$\begin{aligned}
w_{k-m+1}(\theta, 0) &\equiv a_{k-m+1}(x)\, u + v, \\
w_{k-m+2}(\theta, 0) &\equiv a_{k-m+2}(x)\, w_{k-m+1}(\theta, 0) + u, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
w_k(\theta, 0) &\equiv a_k(x)\, w_{k-1}(\theta, 0) + w_{k-2}(\theta, 0),
\end{aligned}$$

that is, $w_{k-1}(\theta, 0)$ and $w_k(\theta, 0)$ are formed in the same way as $w_{m-1}(u, v)$ and $w_m(u, v)$ except for the difference in distribution of $a_{k-m+1}, \ldots, a_k$ and the distribution of $a_1, \ldots, a_m$. However, given $a_{k-m+i}(x) = b_i$, $i = 1, \ldots, t$ the dependence of $a_{k-m+t+1}(x)$, $a_{k-m+t+2}(x)$, ... on $a_1(x), \ldots, a_{k-m}(x)$ is very small as evidenced by (3.6). Using (3.6) we may in fact write

$$\begin{aligned}
&\Big| E\Big\{\exp\big(2\pi i r w_k(\theta, 0) \\
&\qquad + 2\pi i s w_{k-1}(\theta, 0)\big)\,|\,B_{k-m}(b_i),\, A_k,\, w_{k-m}(x) \equiv u,\, w_{k-m-1}(x) \equiv v\Big\} \\
&\qquad - E\Big\{\exp\big(2\pi i r w_m(u, v) + 2\pi i s w_{m-1}(u, v)\big)\,|\,B_0(b_i),\, A_m\Big\}\Big| \leqslant C_{21} \lambda_3^t.
\end{aligned} \tag{3.24}$$

Substituting this in (3.23) one obtains

$$\begin{aligned}
&\frac{1}{n}\sum_{k=1}^{n} |\nu_k(r, s;\, \theta, 0\,|\,A_k)|^2 \\
&\leqslant \frac{m}{n} + 2C_{21}\lambda_3^t + \sum_{b_1, \ldots, b_t} \int_0^1 \int_0^1 d_{u,v} \Big\{\frac{1}{n} \sum_{k=1}^{n-m} F_k([0, u], [0, v], B_k(b_i)\,|\,A_{k+m})\Big\} \times \\
&\qquad \times \big|\nu_m(r, s;\, u, v\,|\,B_0(b_i),\, A_m)\big|^2.
\end{aligned} \tag{3.25}$$

Since for fixed m, $v_m(r, s; u, v \mid B_0(b_i), A_m)$ is a continuous function of u, v and uniformly bounded for all $b_1, \dots, b_t$, one obtains from Lemma 4 and (3.5)

$$(3.26)\quad \overline{\lim} \sum_{b_1,\dots,b_t} \int_0^1\int_0^1 d_{u,v}\Big\{\frac{1}{n}\sum_{k=1}^{n-m} F_k([0,u],[0,v],B_k(b_i)\mid A_{k+m})\Big\}\times \times\big|v_m(r,s;u,v\mid B_0(b_i),A_m)\big|^2 \leqslant C_{19}\sum_{b_1,\dots,b_t}\frac{\mu(B_0(b_i),A_m)}{\mu(A_m)}\int_0^1\int_0^1 du\,dv\,\big|v_m((r,s;u,v\mid B_0(b_i),A_m)\big|^2 .$$

Hence letting $n\to\infty$ in (3.25)

$$(3.27)\quad \overline{\lim}\frac{1}{n}\sum_{k=1}^{n}|v_k(r,s;\theta,0\mid A_k)|^2 \leqslant 2C_{21}\lambda_3^t + C_{19}\sum_{b_1,\dots,b_t}\frac{\mu(B_0(b^i),A_m)}{\mu(A_m)}\int_0^1\int_0^1 du\,dv\,\big|v_m(r,s;u,v\mid B_0(b_i),A_m)\big|^2 .$$

First we choose t large to make $2C_{21}\lambda_3^t$ small. After t is fixed m can be chosen so large that the last term in (3.27) becomes arbitrary small by Lemma 2 and the uniform bound $|v_m|^2\leqslant 1$. This completes the proof.

LEMMA 6. *Let $A_{k,p}(r, a_j)$ stand for*

$$a_{k+j}(x)=a_j,\quad j=1,\dots,q;\qquad a_{k+p+1}(x)=r .$$

For every interval $I\subseteq[0,1]$ there exists a sequence $K=\{k_1<k_2<\dots\}$ of non-negative integers of density zero (i.e. satisfying (2.9)) such that for every fixed q and $a_1,\dots,a_q$

$$\overline{\lim_{p\to\infty}}\ \overline{\lim_{\substack{k\to\infty\\ k\notin K}}}\sum_{b_1,\dots,b_k} P\{a_i(x)=b_i,\ i=1,\dots,k\}\times \times\sup_r\Big|P\{w_{k+p}(\theta,0)\in I\ (\mathrm{mod}\,1)\mid a_i(x)=b_i,\ i=1,\dots,k,\ A_{k,p}(r,a_j)\}-|I|\Big| \leqslant C_{21}\lambda_3^q .$$

Proof. Let us put

$$A_k(a_j)=\{a_{k+j}(x)=a_j,\quad j=1,\dots,q\} .$$

By Lemma 5, for each $r, s\neq 0, 0, a_1,\dots,a_q$

$$\overline{\lim_{n\to\infty}}\frac{1}{n}\sum_{k=1}^{n}|v_k(r,s;\theta,0\mid A_k(a_j)|^2=0 .$$

A familiar argument ([6], pp. 258, 259) then shows that there exists a sequence $K=\{k_i\}$ of density zero such that for all $r, s\neq 0, 0$, and $a_1,\dots,a_q$, $q=1,2,\dots$

$$(3.28)\quad \overline{\lim_{\substack{k\to\infty\\ k\notin K}}} v_k(r,s;\theta,0\mid A_k(a_j))=0 .$$

Of course, also

$$(3.29)\quad v_k(0,0;\theta,0\mid A_k(a_j))\equiv 1 .$$

It was already pointed out in the proof of Lemma 3 (cf. (3.13)) and (3.14) that (3.28) and (3.29) imply

$$(3.30)\quad \lim_{\substack{k\to\infty\\ k\notin K}} F_k([0,u],[0,v]\mid A_k(a_j))=G(u,v)=u\cdot v$$

in the weak sense (4). In other words, the joint distribution of $w_k(\theta,0)$, $w_{k-1}(\theta,0)$ (mod 1) tends to the uniform distribution on the unit square, independent of $a_{k+1}(x), a_{k+2}(x), \dots$ as $k\to\infty$, $k\notin K$. This would practically imply the lemma if r were fixed and no sup over r were taken. In order to show that we have sufficient uniformity in r we write

$$(3.31)\quad \alpha=w_k(\theta,0)\,(\mathrm{mod}1),\quad \beta=w_{k-1}(\theta,0)\,(\mathrm{mod}1) \text{ if } a_i(x)=b_i,\quad i=1,\dots,k .$$

Strictly speaking α and β are functions of $a_1(x),\dots,a_k(x)$. However, just as in (3.24)

$$(3.32)\quad \Big|P\{w_{k+p}(\theta,0)\in I\,(\mathrm{mod}1)\mid A_{k,p}(r,a_j),\ a_i(x)=b_i,\ i=1,\dots,k\} - P\{w_p(\alpha,\beta)\in I\,(\mathrm{mod}1)\mid A_{0,p}(r,a_j)\}\Big|\leqslant C_{21}\lambda_3^q .$$

Let us now estimate

$$\sup_r\Big|P\{w_p(\alpha,\beta)\in I\,(\mathrm{mod}1)\mid A_{0,p}(r,a_j)\}-|I|\Big| .$$

Let $B_p(c_i)$ stand for

$$a_{p-t+i}(x)=c_i,\qquad i=1,\dots,t .$$

Then

$$(3.33)\quad P\{w_p(\alpha,\beta)\in I\mid A_{0,p}(r,a_j)\} = \sum_{c_1,\dots,c_t} P\{w_p(\alpha,\beta)\in I, B_p(c_i)\mid A_{0,p}(r,a_j)\} = \sum_{c_1,\dots,c_t}\frac{P\{w_p(\alpha,\beta)\in I, B_p(c_i), A_0(a_j)\}}{P\{A_0(a_j)\}}\times \times\frac{P\{a_{p+1}(x)=r\mid w_p(\alpha,\beta)\in I, B_p(c_i), A_0(a_j)\}}{P\{a_{p+1}(x)=r\mid A_0(a_j)\}} .$$

Since $B_p(c_i)$ fixes $a_{p-t+1}, \dots, a_p$, one has by (3.6), with k replaced by p, q by t and s by 0,

$$\left|P\{a_{p+1}(x)=r \,|\, w_p(\alpha,\beta)\in I,\, B_p(c_i),\, A_0(a_j)\}-P\{a_{p+1}(x)=r \,|\, B_p(c_i)\}\right| \leqslant C_{15}\lambda_3^t \mu\{a_{p+1}(x)=r\}$$

and by (3.8)

$$\left|P\{a_{p+1}(x)=r \,|\, B_p(c_i)\}-\mu\{a_{p+1}(x)=r \,|\, B_p(c_i)\}\right| \leqslant C_{16}\lambda^t \mu\{a_{p+1}(x)=r\}.$$

Finally by (3.4)

$$\left|P\{a_{p+1}(x)=r \,|\, A_0(a_j)\}-\mu\{a_{p+1}(x)=r\}\right| \leqslant C_{15}\lambda_3^{p-q}\mu\{a_{p+1}(x)=r\}.$$

Thus

$$\text{(3.34)}\qquad \left|\frac{P\{a_{p+1}(x)=r \,|\, w_p(\alpha,\beta)\in I,\, B_p(c_i),\, A_0(a_j)\}}{P\{a_{p+1}(x)=r \,|\, A_0(a_j)\}}-\frac{\mu\{a_{p+1}(x)=r \,|\, B_p(c_i)\}}{\mu\{a_{p+1}(x)=r\}}\right| \leqslant C_{22}(\lambda_3^t+\lambda^t+\lambda_3^{p-q}).$$

Moreover, by (3.5) and (3.34) both ratios in the left hand side of (3.34) are bounded. There exists then for every ε a finite set F of t tuples $c_1, \dots, c_t$ such that uniformly in $\alpha, \beta, a_1, \dots, a_q$ and r

$$\text{(3.35)}\qquad \sum_{c_1,\dots,c_t\notin F} P\{B_p(c_i) \,|\, A_0(a_j)\}\frac{\mu\{a_{p+1}(x)=r \,|\, B_p(c_i)\}}{\mu\{a_{p+1}(x)=r\}} \leqslant \varepsilon$$

as well as

$$\text{(3.36)}\qquad \sum_{c_1,\dots,c_t\notin F} \frac{\mu\{B_p(c_i),\, a_{p+1}(x)=r\}}{\mu\{a_{p+1}(x)=r\}} \leqslant \varepsilon.$$

Combining (3.33)-(3.36)

$$\text{(3.37)}\qquad \sup_r\left|P\{w_p(\alpha,\beta)\in I \,|\, A_{0,p}(r,a_j)-|I|\right| \leqslant \varepsilon + C_{22}(\lambda_3^t+\lambda^t+\lambda_3^{p-q})$$
$$+\sup_r\left|\sum_{c_1,\dots,c_t\in F}\frac{P\{w_p(\alpha,\beta)\in I,\, B_p(c_i) \,|\, A_0(a_j)\}}{\mu\{B_p(c_i)\}}\cdot\frac{\mu\{B_p(c_i),\, a_{p+1}(x)=r\}}{\mu\{a_{p+1}(x)=r\}}-|I|\right|$$
$$\leqslant 2\varepsilon + C_{22}(\lambda_3^t+\lambda^t+\lambda_3^{p-q})$$
$$+\sup_r\sum_{c_1,\dots,c_t\in F}\frac{\mu\{B_p(c_i),\, a_{p+1}(x)=r\}}{\mu\{a_{p+1}(x)=r\}}\left|\frac{P\{w_p(\alpha,\beta)\in I,\, B_p(c_i) \,|\, A_0(a_j)\}}{\mu\{B_p(c_i)\}}-|I|\right|.$$

The lemma will therefore be proved if we show for each $t, c_1, \dots, c_t$

$$\text{(3.38)}\qquad \lim_{p\to\infty}\overline{\lim_{\substack{k\to\infty\\ k\notin K}}}\sum_{b_1,\dots,b_k} P\{a_i(x)=b_i,\ \ ,\ i=1,\dots,k\}\times$$
$$\times\left|\frac{P\{w_p(\alpha,\beta)\in I,\, B_p(c_i) \,|\, A_0(a_j)\}}{\mu\{B_p(c_i)\}}-|I|\right|=0.$$

Let us recall now that α, β are functions of $a_j(x)$, $j=1,\dots,k$ as defined in (3.31), and that (3.30) states

$$\text{(3.39)}\qquad \lim_{\substack{k\to\infty\\ k\notin K}}\sum_{\substack{a\leqslant u\\ \beta\leqslant v}} P\{a_i(x)=b_i,\ i=1,\dots,k\}$$
$$=\lim_{\substack{k\to\infty\\ k\notin K}} P\{w_k(\theta,0)\leqslant u \ (\mathrm{mod}1),\ w_{k-1}(\theta,0)\leqslant v \ (\mathrm{mod}1)\}=u\cdot v$$

in the weak sense. Even though

$$P\{w_p(\alpha,\beta)\in I,\ B_p(c_i) \,|\, A_0(a_j)\}$$

is not strictly a continuous function of α, β it is easy to deduce by a simple approximation argument from (3.39) and (3.12) that

$$\lim_{p\to\infty}\overline{\lim_{\substack{k\to\infty\\ k\notin K}}}\sum_{b_1,\dots,b_k} P\{a_j(x)=b_j,\ j=1,\dots,k\}\times$$
$$\times\left|\frac{P\{w_p(\alpha,\beta)\in I,\, B_p(c_i) \,|\, A_0(a_j)\}}{\mu\{B_p(c_i)\}}-|I|\right|$$
$$=\lim_{p\to\infty}\int_0^1\int_0^1 d\alpha\, d\beta\left|\frac{P\{w_p(\alpha,\beta)\in I,\, B_p(c_i) \,|\, A_0(a_j)\}}{\mu\{B_p(c_i)\}}-|I|\right|=0.$$

Recall that μ is stationary and thus $\mu\{B_p(c_i)\}=\mu\{a_i(x)=c_i,\ i=1,\dots,t\}$. As remarked before this proves the lemma.

Finally we can prove Lemma 1.

Proof of Lemma 1. By (2.7) in [4]

$$P\{a_{k+j}(x)>(\log N)^3 \text{ for some } 1\leqslant j\leqslant p+2 \,|\, a_i(x)=b_i,\ i=1,\dots,k\} \leqslant \frac{2(p+2)}{(\log N)^3},\quad \text{uniformly in } b_1,\dots,b_k,$$

so that we can leave out the condition

$$a_{k+j}(x)\leqslant(\log N)^3,\qquad j=1,\dots,p+2$$

in (2.10). By Lemma 6 there exists for each interval I a sequence K of density zero, such that for all $q, a_1, \dots, a_q$

$$\text{(3.40)}\qquad \overline{\lim_{p\to\infty}}\,\overline{\lim_{\substack{k\to\infty\\ k\notin K}}}\sum P\{a_i(x)=b_i,\ i=1,\dots,k\}\times$$
$$\times\sup_r\left|P\{w_{k+p}(\theta,0)\in I \ (\mathrm{mod}1) \,|\, a_i(x)=b_i,\ i=1,\dots,k,\ A_{k,p}(r,a_j)\}-|I|\right| \leqslant C_{21}\lambda_3^q.$$

Using again the trick of [6], pp. 258, 259 we can choose a sequence $K = \{k_1 < k_2 < \ldots\}$ of non-negative integers of density zero such that (3.40) holds for the countably many intervals $\left[\frac{j}{n}, \frac{j+1}{n}\right)$, $j = 0, 1, \ldots, n-1$; $n = 1, 2, \ldots$ We shall prove that this sequence K satisfies the requirements of the lemma.

Notice that $U(z, q, y) = -U(1-z, q, y)$ so that for any condition A on x

$$(3.41)\quad \int_0^1\int_0^1 dy\,dz\,E\left\{\exp\left(\frac{it}{\log N}(-1)^{k+p+1}a_{k+p+2}(x)\,U\left(z, q_{k+p+1}(x), y\right)\right) \mid A\right\} = \int_0^1\int_0^1 dy\,dz\,E\left\{\cos\left(\frac{t}{\log N}a_{k+p+2}(x)\,U\left(z, q_{k+p+1}(x), y\right)\right) \mid A\right\}.$$

Moreover it is easy to see that the right-hand side of (3.41) depends on q_{k+p+1} only through the value of $q_{k+p+1}(x)\,\theta \pmod 1$. Now

$$(3.42)\quad \int_0^1\int_0^1 dy\,dz E\left\{\cos\left(\frac{t}{\log N}a_{k+p+2}(x)\,U\left(z, q_{k+p+1}(x), y\right)\right)-1 \mid a_i(x) = b_i,\ i = 1, \ldots, k\right\} = \int_0^1\int_0^1 dy\,dz \sum_{a_1,\ldots,a_q}\sum_r P\{a_{k+j}(x) = a_j,\ j = 1, \ldots, q,\ a_{k+p+2}(x) = r \mid a_i(x) = b_i,\ i = 1, \ldots, k\} \times E\left\{\cos\left(\frac{tr}{\log N}U\left(z, q_{k+p+1}(x), y\right)\right)-1 \mid a_i(x) = b_i,\ i = 1, \ldots, k,\ A_{k,p+1}(r, a_j)\right\}.$$

For shortness we shall use the abbreviation

$$B_k(b_i) = \{a_i(x) = b_i,\ i = 1, \ldots, k\}.$$

Let $\varepsilon > 0$ be some small number and choose $n = n(\varepsilon)$ such that

$$(3.43)\quad \sup_{y,z,|u_1-u_2|\leqslant 1/n} |V(z, u_1, y) - V(z, u_2, y)| \leqslant \varepsilon$$

and put

$$I_j = \left[\frac{j}{n}, \frac{j+1}{n}\right).$$

Then

$$(3.44)\quad \int_0^1\int_0^1 dy\,dz E\left\{\cos\left(\frac{tr}{\log N}U\left(z, q_{k+p+1}(x), y\right)\right)-1 \mid B_k(b_i),\ A_{k,p+1}(r, a_j)\right\} = \sum_{j=0}^{n-1} P\{q_{k+p+1}(x)\,\theta \in I_j \pmod 1 \mid B_k(b_i),\ A_{k,p+1}(r, a_j)\} \times n\int_0^1\int_0^1 dy\,dz \int_{I_j} du\left(\cos\left(\frac{tr}{\log N}V(z, u, y)\right)-1\right) + \theta_1 \sup_{y,z,|u_1-u_2|\leqslant 1/n}\left|\cos\frac{tr}{\log N}V(z, u_1, y) - \cos\frac{tr}{\log N}V(z, u_2, y)\right|$$

for some θ_1 with $|\theta_1| \leqslant 1$. Let us first estimate

$$(3.45)\quad \sum_{a_1,\ldots,a_q}\sum_r P\{a_{k+j}(x) = a_j,\ j = 1, \ldots, q,\ a_{k+p+2}(x) = r \mid B_k(b_i)\} \times \sup_{y,z,|u_1-u_2|\leqslant 1/n}\left|\cos\frac{tr}{\log N}V(z, u_1, y) - \cos\frac{tr}{\log N}V(z, u_2, y)\right|.$$

By (3.5)

$$\sum_{a_1,\ldots,a_q} P\{a_{k+j} = a_j,\ j = 1, \ldots, q,\ a_{k+p+2}(x) = r \mid B_k(b_i)\} \leqslant C_{15}\mu\left(a_{k+p+2}(x) = r\right) = \frac{C_{15}}{\log 2}\log\frac{(r+1)^2}{r(r+2)} \leqslant \frac{C_{23}}{r^2}$$

so that (3.45) is at most

$$\sum_{r=1}^{\varepsilon\log N} + \sum_{\varepsilon\log N+1}^{\log N/\varepsilon} + \sum_{\log N/\varepsilon+1}^{\infty} \frac{C_{23}}{r^2}\sup_{y,z,|u_1-u_2|\leqslant 1/n}\left|\cos\frac{tr}{\log N}V(z, u_1, y) - \cos\frac{tr}{\log N}V(z, u_2, y)\right|.$$

In the first sum we use

$$(3.46)\quad \left|\cos\frac{tr}{\log N}V(z, u_1, y) - \cos\frac{tr}{\log N}V(z, u_2, y)\right| \leqslant \frac{t^2r^2}{(\log N)^2}C_{24}$$

which follows from $|\cos x - 1| \leqslant \frac{x^2}{2}$.

In the second sum we use

$$(3.47)\quad \left|\cos\frac{tr}{\log N}V(z, u_1, y) - \cos\frac{tr}{\log N}V(z, u_2, y)\right| \leqslant \left|\frac{tr}{\log N}\right||V(z, u_1, y) - V(z, u_2, y)| \leqslant \frac{\varepsilon|t|r}{\log N} \quad \text{if} \quad |u_1-u_2| \leqslant \frac{1}{n}$$

which follows from $\left|\frac{d}{dx}\cos x\right| \leqslant 1$ and (3.43). In the last sum we use

$$(3.48)\quad \left|\cos\frac{tr}{\log N}V(z, u_1, y) - \cos\frac{tr}{\log N}V(z, u_2, y)\right| \leqslant 2.$$

Using (3.46)-(3.48), the left hand side of (3.45) is seen to be bounded by

$$(3.49)\quad \sum_{r=1}^{\infty}\frac{C_{23}}{r^2}\sup_{y,z,|u_1-u_2|\leqslant 1/n}\left|\cos\frac{tr}{\log N}V(z, u_1, y) - \cos\frac{tr}{\log N}V(z, u_2, y)\right| \leqslant C_{25}\frac{\varepsilon\log\frac{1}{\varepsilon}}{\log N}.$$

C_{25} depends on t. Later on, we shall also need the estimate

$$(3.50)\qquad \sum_{r=1}^{\infty} \mu(a_1(x)=r) \sup_{y,z,u}\left|\cos\left(\frac{tr}{\log N}V(z,u,y)\right)-1\right| \leqslant \frac{C_{26}}{\log N}\,.$$

Again C_{26} depends on t. For the proof split the sum into two parts: $1 \leqslant r \leqslant \log N$ and $\log N < r$. Use (3.46) and (3.48) respectively to estimate the sums. We now return to (3.44)

$$(3.51)\qquad \Big|\sum_{j=0}^{n-1} P\{q_{k+p+1}(x)\,\theta \in I_j \,(\mathrm{mod}1)\,|\,B_k(b_i)\,,\ A_{k,p+1}(r,a_j)\} \times$$

$$\times n\int_{I_j} du\left(\cos\left(\frac{tr}{\log N}V(z,u,y)\right)-1\right)-\int_0^1 du\left(\cos\left(\frac{tr}{\log N}V(z,u,y)\right)-1\right)\Big|$$

$$\leqslant C_{27}\sup_{y,z,u}\left|\cos\left(\frac{tr}{\log N}V(z,u,y)\right)-1\right| \times$$

$$\times \sup_r \sum_{j=0}^{n-1}\left| P\{q_{k+p+1}(x)\,\theta \in I_j \,(\mathrm{mod}1)\,|\,B_k(b_i)\,,\ A_{k,p+1}(r,a_j)\}-\frac{1}{n}\right|.$$

Using (3.5), the stationarity of μ and (3.50) we obtain for the error term

$$(3.52)\qquad \sum_{a_1,\dots,a_q}\sum_r P\{a_{k+j}=a_j\,,\ j=1,\dots,q\,,\ a_{k+p+2}=r\,|\,B_k(b_i)\} \times$$

$$\times \sup_{y,z,u}\left|\cos\left(\frac{tr}{\log N}V(z,u,y)\right)-1\right| \times$$

$$\times \sup_r \sum_{j=0}^{n-1}\left| P\{q_{k+p+1}(x)\,\theta \in I_j \,(\mathrm{mod}1)\,|\,B_k(b_i)\,,\ A_{k,p+1}(r,a_j)\}-\frac{1}{n}\right|$$

$$\leqslant \frac{C_{28}}{\log N}\sum_{a_1,\dots,a_q}\mu(a_j(x)=a_j\,,\ j=1,\dots,q)\times$$

$$\times \sup_r \sum_{j=0}^{n-1}\left| P\{q_{k+p+1}(x)\,\theta \in I_j \,(\mathrm{mod}1)\,|\,B_k(b_i)\,,\ A_{k,p+1}(r,a_j)\}-\frac{1}{n}\right|.$$

Since

$$(3.53)\qquad \sup_r \sum_{j=0}^{n-1}\left| P\{q_{k+p+1}(x)\,\theta \in I_j \,(\mathrm{mod}1)\,|\,B_k(b_i)\,,\ A_{k,p+1}(r,a_j)\}-\frac{1}{n}\right| \leqslant 2\,,$$

uniformly in b_j, a_j, and

$$\sum_{a_1,\dots,a_q}\mu(a_j(x)=a_j\,,\ j=1,\dots,q)=1\,,$$

one obtains from (3.40) and (3.2)

$$(3.54)\qquad \varlimsup_{p\to\infty}\varlimsup_{\substack{k\to\infty\\ k\notin K}}\sum_{b_1,\dots,b_k} P\{B_k(b_i)\}\sum_{a_1,\dots,a_q}\mu(a_j(x)=a_j\,,\ j=1,\dots,q)\times$$

$$\times \sup_r \sum_{j=0}^{n-1}\left| P\{q_{k+p+1}(x)\,\theta \in I_j \,(\mathrm{mod}1)\,|\,B_k(b_i)\,,\ A_{k,p+1}(r,a_j)\}-\frac{1}{n}\right| \leqslant nC_{21}\lambda_3^q\,.$$

Combining (3.41), (3.42), (3.44), (3.45), (3.49), (3.51), (3.54) gives

$$(3.55)\qquad \varlimsup_{p\to\infty}\varlimsup_{\substack{k\to\infty\\ k\notin K\\ N\to\infty}}\sum_{b_1,\dots,b_k} P\{B_k(b_i)\}\cdot\log N\times$$

$$\times\left|E\exp\left\{\frac{it}{\log N}(-1)^{k+p+1}a_{k+p+2}(x)\,U(z,q_{k+p+1}(x),y)\,|\,B_k(b_i)\right\}-\left(1-\frac{C|t|}{\log N}\right)\right|$$

$$\leqslant C_{25}\,\varepsilon\log\frac{1}{\varepsilon}+C_{27}C_{28}C_{21}n\lambda_3^q+\varlimsup_{p\to\infty}\varlimsup_{\substack{k\to\infty\\ k\notin K\\ N\to\infty}}\sum_{b_1,\dots,b_k} P\{B_k(b_i)\}\log N\times$$

$$\times\Big|\sum_r P\{a_{k+p+2}(x)=r\,|\,B_k(b_i)\}\int_0^1\int_0^1\int_0^1 dz\,du\,dy\left(\cos\left(\frac{tr}{\log N}V(z,u,y)\right)-1\right)$$

$$+\frac{C|t|}{\log N}\Big|\,.$$

Finally, by (3.4) and (3.50)

(3.56)

$$\sum_r P\{a_{k+p+2}(x)=r\,|\,B_k(b_i)\}\log N\int_0^1\int_0^1\int_0^1 dz\,du\,dy\left(\cos\left(\frac{tr}{\log N}V(z,u,y)\right)-1\right)$$

$$=\log N\sum_r(\mu a_1(x)=r)\int_0^1\int_0^1\int_0^1 dz\,du\,dy\left(\cos\left(\frac{tr}{\log N}V(z,u,y)\right)-1\right)+\vartheta_2 C_{29}\lambda_3^p$$

for some $|\vartheta_2|\leqslant 1$. From pp. 455, 456 in [4] we see easily that

(3.57)

$$\lim_{N\to\infty}\log N\sum_r\mu(a_1(x)=r)\int_0^1\int_0^1\int_0^1 dz\,du\,dy\left(\cos\left(\frac{tr}{\log N}V(z,u,y)\right)-1\right)=-C|t|$$

with C given by (2.11). Lemma 1 follows then from (3.55)-(3.57) since $\varepsilon\log\frac{1}{\varepsilon}+n(\varepsilon)\lambda_3^q$ can be made arbitrarily small by first choosing ε small and then q large. This completes the proof of Lemma 1 and the theorem.

4. The function ϱ of $(b-a)$. We shall give here an expression for ϱ which involves only some definite integrals and, for $(b-a)$ rational, also a number theoretical function. For this purpose we define the following function, for integers u, v, $0 \leqslant u \leqslant v-1$:

$$p(u;v) = \frac{\text{number of } s,\ 0 \leqslant s \leqslant v-1 \text{ with } (s,u,v)=1}{\text{number of } s,t,\ 0 \leqslant s,t \leqslant v-1 \text{ with } (s,t,v)=1}. \tag{4.1}$$

Here (s,t,v) is the greatest common divisor of s, t and u. We then have the following

PROPOSITION 2. *For* $b-a = \frac{w}{v}$, w, v *integer*, $(w,v)=1$, ϱ *in* (1.2) *is given by*

$$\frac{\pi}{6}\left\{\sum_{u=0}^{v-1} p(u;v) \int_0^1\int_0^1 dz\,dy \left| V\left(z, \frac{uw}{v}, y\right)\right|\right\}^{-1}. \tag{4.2}$$

If $b-a$ *is irrational then* ϱ *in* (1.2) *is given by*

$$\frac{\pi}{6}\left\{\int_0^1\int_0^1\int_0^1 dz\,du\,dy\,|V(z,u,y)|\right\}^{-1}. \tag{4.3}$$

Remark. The integral

$$\int_0^1\int_0^1 dy\,dz\,|V(z,u,y)|$$

can be evaluated by means of the relation

$$\sum_{k=1}^{\infty} \frac{\cos 2\pi kx}{2\pi^2 k^2} = \frac{\{x\}(\{x\}-1)}{2} + \frac{1}{12}$$

where $\{x\}$ is the fractional part of x.

Proof. It was proved in [7], p. 320, formula (42) that τ^{-1} in (2.10) of [4] is given by

$$\tau^{-1} = \frac{\pi^2}{12\log 2}.$$

The value (4.3) of ϱ for irrational $(b-a)$ was proved in proposition 1 while in [4] (cf. Lemma 2.6 and formula (3.33)) it was proved that for $b-a = \frac{w}{v}$, $(w,v)=1$

$$\varrho = \tau^{-1}\left\{\sum_{u=0}^{v-1} \frac{\pi p(u)}{2\log 2} \int_0^1\int_0^1 dz\,dy \left| V\left(z, \frac{uw}{v}, y\right)\right|\right\}^{-1}$$

with $p(u)$ defined as follows: We say that (s,t), $0 \leqslant s,t \leqslant v-1$ is a possible pair if there exist a ξ and k with

$$q_{k-1}(\xi) \equiv s \pmod v$$
$$q_k(\xi) \equiv t \pmod v.$$

Then

$$p(u) = \frac{\text{number of possible pairs } (s,u),\ 0 \leqslant s \leqslant v-1}{\text{total number of possible pairs}}.$$

It remains to show therefore that $p(u) = p(u;v)$ or the following: [5]

$$(s,t) \text{ is a possible pair if and only if } (s,t,v)=1. \tag{4.4}$$

Since ([3], p. 131, Theorem 150)

$$p_k(\xi)q_{k-1}(\xi) - p_{k-1}(\xi)q_k(\xi) = (-1)^{k-1}$$

one must have $\left(q_{k-1}(\xi), q_k(\xi)\right) = 1$ and hence $(s,t,v)=1$ if (s,t) is a possible pair. To show the converse, let $(s,t,v)=1$ and $(s,t)=r$. Let

$$s' = \prod_{\substack{p|s \\ p\nmid r}} p$$

be the product of all primes dividing s but not r and put $t' = t+s'v$. We claim $(s,t')=1$. Firstly no factor of s' can divide t (or equivalently t') for then it divides r. Secondly any prime factor of s which is not a factor of s' must be a factor of r and hence divides t but not $s'v$ since $(s,t,v)=1$. Hence no factor of s divides t' and $(s,t')=1$. Moreover $s<t'$ and by Euclid's algorithm ([3], p. 136) there exists a k and integers $t_1, \ldots, t_{k-2}$, $a_2, \ldots, a_k$ such that

$$\begin{aligned} t' &= a_k s + t_{k-2} \\ s &= a_{k-1}t_{k-2} + t_{k-3} \\ t_{k-2} &= a_{k-2}t_{k-3} + t_{k-4} \\ &\vdots \\ t_2 &= a_2 t_1 + 1. \end{aligned} \tag{4.5}$$

Taking $\xi = [t_1, a_2, \ldots, a_k]$ we have $q_0(\xi)=1$, $q_1(\xi)=t_1$, $q_2(\xi) = a_2t_1+1 = t_2$, $\ldots$, $q_{k-1}(\xi) = s$, $q_k(\xi) = t' \equiv t \pmod v$. This shows (4.4) and completes the proof.

Errata to paper [4]

p. 447, line 6, add "for $n \geqslant 2$" after In particular.
p. 448, lines 5 and 8 from bottom change $2t^2\lambda_4$ into $2t^2\lambda_4^2$ in the exponent.

[5] Recently, P. Szüsz (Acta Arithmetica 7 (1962), pp. 149-160) gave another proof of lemma 2.5 in [4], even sharpening the estimate of the error. At the same time the proof of Szüsz implies $p(u) = p(u;v)$.

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p. 452; the first two lines of Lemma 2.4 should read: "If $a_{m+1}(x), \ldots, a_{m+p}(x)$ are fixed (mod v), then one can find for each i ($1 \leqslant i \leqslant k$) exactly one $j(i) \leqslant k$ such that $x(m+p) \epsilon S'_i$ is equivalent".

p. 452, line 6 from bottom should start with "$\geqslant \varepsilon h P\{\ldots$".

p. 453, line 9 should have at the end "(take $a_1 = 1$ and $r = 1$)".

p. 454, line 4 from bottom should read

$$\Delta_{m,p} \leqslant \lambda_5 \Delta_{m+M+n,p-M-n} + C_5 \lambda_5^p .$$

p. 460, lines 2,3 from bottom should read "(Σ'' is over k from 1 to $a_{m+1}(x)$ but includes only those k with $kq_m \leqslant N$)".

p. 464, formula (3.19) should be: $\dfrac{q_s(x)}{q_n(x)} \dfrac{1}{(q_n(x)+q_{n-1}(x))}$.

p. 464, line 2 from bottom $a_1(x), \ldots, a_{m+1}(x)$ should be $a_1(x), \ldots, a_{m+2}(x)$.

p. 466, line 3 $A(\eta)\exp(-(m-p)\eta)$ should be $A(\eta)\exp(-(m-p)B(\eta))$.

p. 469, line 7 $C_8 \dfrac{t}{\log N}$ should be $-C_8 \dfrac{|t|}{\log N}$.

p. 469, formula (3.31). The sum over m should run from $m = 2p$ to $m = (\tau-\varepsilon)\log N$. This requires corresponding changes on p. 470.

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Reçu par la Rédaction le 6. 2. 1962

Binomial coefficients in an algebraic number field *

by

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1. Let $K = R(\Theta)$ denote an algebraic number field of degree n over the rationals. Let $\mathfrak{p}$ be a prime ideal of K and let p be the rational prime divisible by $\mathfrak{p}$. Let $K_\mathfrak{p}$ denote the set of numbers of K that are integral (mod$\mathfrak{p}$). Put

$$\binom{\alpha}{m} = \frac{\alpha(\alpha-1)\ldots(\alpha-m+1)}{m!} .$$

We shall prove the following result.

Theorem 1. *The binomial coefficients* $\binom{\alpha}{m}$ *are integral* (mod$\mathfrak{p}$) *for all* $\alpha \epsilon K_\mathfrak{p}$ *and all* $m \geqslant 1$ *if and only if* $\mathfrak{p}$ *is a prime ideal of the first degree and moreover* p *does not divide the discriminant of* K.

Proof. To prove the necessity of the stated conditions suppose first that $\mathfrak{p}$ is of degree $f > 1$. Then the residue class ring $K_\mathfrak{p}/\mathfrak{p}$ is a finite field of order p^f. Since $f > 1$ there exists a number $\alpha \epsilon K_\mathfrak{p}$ such that

$$\alpha \not\equiv r \ (\mathrm{mod}\mathfrak{p}) \qquad (r = 0, 1, \ldots, p-1) .$$

Therefore the binomial coefficient $\binom{\alpha}{p}$ is not integral (mod$\mathfrak{p}$).

Next let $\mathfrak{p}$ be of the first degree but let p divide the discriminant of K. Then by Dedekind's theorem on discriminantal divisors, $\mathfrak{p}^2 | p$. Also there exists an integer a of K such that ([3], p. 97, Theorem 74]

$$(a, p) = \mathfrak{p} . \tag{1.1}$$

Since $\mathfrak{p}$ is of the first degree, the numbers

$$a, a-1, \ldots, a-p+1$$

constitute a complete residue system (mod$\mathfrak{p}$). Clearly only the first of these numbers is divisible by $\mathfrak{p}$. Therefore by (1.1) the product

$$a(a-1)\ldots(a-p+1)$$

* Supported in part by National Science Foundation grant G 16485.