ACTA ARITHMETICA
VII (1962)

Mean-value estimations for the Möbius function II

by

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1. In this paper I shall prove the following (announced in [1])

Theorem. *Suppose that all the zeros of $\zeta(s)$ in the rectangle*

$$0<\sigma<1, \qquad |t|\leqslant\omega$$

are simple and have $\sigma=\frac{1}{2}$. In that case we have (¹)

$$\int_X^T \frac{|M(x)|}{x}\,dx > T^{1/2}\exp\left(-12\frac{\log T}{\log\log T}\log\log\log T\right) \tag{1.1}$$

with

$$X = T\exp\left(-100\frac{\log T}{\log\log T}\log\log\log T\right)$$

for (²)

$$c_1\leqslant T\leqslant \exp(\omega^{10}). \tag{1.2}$$

Remark. It may be noted that the assumptions of this theorem have been checked [3] at least up to $\omega=10^4$.

The following lemma (see [2], p. 419), being a slightly modified version of a theorem of P. Turán (see [5], Satz X, p. 52), will be essential in a proof of (1.1).

Lemma 1. *Let m be a non-negative number and $z_1, z_2, \ldots, z_N$ complex numbers such that*

$$1=|z_1|\geqslant|z_2|\geqslant\ldots\geqslant|z_h|\geqslant\ldots\geqslant|z_N|, \qquad |z_h|>2\frac{N}{N+m}.$$

Then there exists an integer ν with $m\leqslant\nu\leqslant m+N$ such that

$$\frac{|b_1z_1^\nu+b_2z_2^\nu+\ldots+b_Nz_N^\nu|}{(\frac{1}{2}|z_h|)^\nu}\geqslant\min_{h\leqslant j<h_1}|b_1+b_2+\ldots+b_j|\left(\frac{1}{24e}\cdot\frac{N}{2N+m}\right)^N, \tag{1.3}$$

(¹) As in [1] $M(x)\overset{\text{def}}{=}\sum_{n\leqslant x}\mu(n)$, $\mu(n)$ the Möbius function.

(²) Here and in what follows c_ν denote positive numerical constants.

where $h_1 \leqslant N$ *is any integer for which* $|z_{h_1}| < |z_h| - \frac{N}{(m+N)}$. *In that case when there do not exist numbers* h_1 *satisfying the latter inequality, we put at the right-hand side of* (1.3) $\min\limits_{h\leqslant j\leqslant N} |b_1+b_2+\ldots+b_j|$ *instead.*

2. Let integer, numerical $r \geqslant 2$ be large enough to satisfy the following statement (see [4], p. 185, Theorem 9.7):

for every $T \geqslant 2$, *there exists some* $t = t(T)$, $T \leqslant t \leqslant T+1$ *such that*

$$|\zeta(\sigma+it)| > t^{-r+1} \qquad (-1 \leqslant \sigma \leqslant 2)\,. \tag{2.1}$$

Having fixed r we prove the following

LEMMA 2. *There exist positive numerical* D, ψ *such that*

$$\Big|\sum_{|\Im\varrho|<t(Y)} D^{\varrho}\Big(\frac{e^{\psi\varrho}-e^{-\psi\varrho}}{2\psi\varrho}\Big)^r \frac{1}{\zeta'(\varrho)}\Big| > \frac{r}{4^r(r-1)!}\,, \tag{2.2}$$

where ϱ *runs through non-trivial zeros of* $\zeta(s)$, *provided that* $Y > c_2$ *and that all the* ϱ*'s with* $|\Im\varrho| \leqslant Y+1$ *are simple* ζ*-zeros.*

Proof. Let N_1 be a positive integer to be fixed later. We write

$$\psi = \frac{1}{2N_1 r}\,, \qquad D = N_1 e^{-(r-1/2)\psi}$$

and consider the integral

$$I_0 = \frac{1}{2\pi i}\int_{2-it(Y)}^{2+it(Y)} D^s\Big(\frac{e^{\psi s}-e^{-\psi s}}{2\psi s}\Big)^r \frac{ds}{\zeta(s)}\,. \tag{2.3}$$

We have obviously

$$I_0 = \frac{1}{2\pi i}\int_{(2)} D^s\Big(\frac{e^{\psi s}-e^{-\psi s}}{2\psi s}\Big)^r \frac{ds}{\zeta(s)} + O\Big(\frac{N_1^{r+2}}{Y^{r-1}}\Big)$$

$$= \sum_{n=1}^{\infty}\frac{\mu(n)}{2\pi i}\int_{(2)} D^s\Big(\frac{e^{\psi s}-e^{-\psi s}}{2\psi s}\Big)^r \frac{ds}{n^s} + O\Big(\frac{N_1^{r+2}}{Y^{r-1}}\Big)\,.$$

Let us note that

$$\int_{(2)} D^s\Big(\frac{e^{\psi s}-e^{-\psi s}}{2\psi s}\Big)^r \frac{ds}{n^s} = 0 \quad \text{for} \quad n \geqslant De^{r\psi}\,,$$

and also, moving the line of integration e.g. to $\sigma = -1$,

$$\int_{(2)} D^s\Big(\frac{e^{\psi s}-e^{-\psi s}}{2\psi s}\Big)^r \frac{ds}{n^s} = 0 \quad \text{for} \quad n \leqslant De^{-r\psi}\,.$$

We have then

$$N_1 - 1 = N_1(1-1/N_1) < N_1 e^{\psi/2}\cdot e^{-1/N_1} = De^{-r\psi}$$

and

$$De^{r\psi} = N_1 e^{\psi/2} < N_1(1+1/N_1 r) < N_1+1\,,$$

so that

$$I_0 = \frac{\mu(N_1)}{2\pi i}\int_{(2)}\Big(\frac{e^{\psi s}-e^{-\psi s}}{2\psi s}\Big)^r\Big(\frac{D}{N_1}\Big)^r ds + O\Big(\frac{N_1^{r+2}}{Y^{r-1}}\Big)\,.$$

Now

$$\frac{De^{r\psi}}{N_1} < 1 \quad \text{for} \quad \nu = 1, 2, \ldots, r-1\,,$$

whence

$$I_0 = \frac{\mu(N_1)}{2\pi i}\int_{(2)}\Big(\frac{De^{r\psi}}{N_1}\Big)^s \frac{ds}{(2\psi)^r s^r} + O\Big(\frac{N_1^{r+2}}{Y^{r-1}}\Big) = \frac{2\mu(N_1)}{4^r\psi(r-1)!} + O\Big(\frac{N_1^{r+2}}{Y^{r-1}}\Big)\,. \tag{2.4}$$

On the other hand by Cauchy's theorem of residues

$$I_0 = \sum_{|\Im\varrho|<t(Y)} D^{\varrho}\Big(\frac{e^{\psi\varrho}-e^{-\psi\varrho}}{2\psi\varrho}\Big)^r\frac{1}{\zeta'(\varrho)} + O\Big(\frac{N_1^{r+2}}{Y}\Big) + \frac{1}{2\pi i}\int_{-1-it(Y)}^{-1+it(Y)} D^s\Big(\frac{e^{\psi s}-e^{-\psi s}}{2\psi s}\Big)^r\frac{ds}{\zeta(s)}$$

$$= \sum_{|\Im\varrho|<t(Y)} D^{\varrho}\Big(\frac{e^{\psi\varrho}-e^{-\psi\varrho}}{2\psi\varrho}\Big)^r\frac{1}{\zeta'(\varrho)} + O\Big(\frac{N_1^{r+2}}{Y}\Big) + O\Big(\frac{1}{N_1}\int_{-\infty}^{+\infty}\frac{dt}{|\zeta(-1+it)|}\Big)$$

$$= \sum_{|\Im\varrho|<t(Y)} D^{\varrho}\Big(\frac{e^{\psi\varrho}-e^{-\psi\varrho}}{2\psi\varrho}\Big)^r\frac{1}{\zeta'(\varrho)} + O\Big(\frac{N_1^{r+2}}{Y}\Big) + O(1/N_1)\,.$$

This combined with (2.4) gives

$$\sum_{|\Im\varrho|<t(Y)} D^{\varrho}\Big(\frac{e^{\psi\varrho}-e^{-\psi\varrho}}{2\psi\varrho}\Big)^r\frac{1}{\zeta'(\varrho)} = \mu(N_1)\frac{N_1 r}{4^{r-1}(r-1)!} + O\Big(\frac{N_1^{r+2}}{Y}\Big) + O(1/N_1)\,. \tag{2.5}$$

We choose N_1 with $\mu(N_1) \neq 0$ (e.g. N_1 a prime number) to be so large as to make the error term $O(1/N_1)$ in (2.5) less than $\frac{1}{2}\cdot\frac{N_1 r}{4^{r-1}(r-1)!}$. Having done that we take Y to be large enough to make $O(N_1^{r+2}/Y)$ less than $\frac{1}{4}\cdot\frac{N_1 r}{4^{r-1}(r-1)!}$. (2.2) then easily follows.

3. We introduce the following notation:

$$T_1 = \frac{T}{D}e^{-r\psi} \qquad (D, r, \psi \text{ as in (2.1) and in lemma 2})\,,$$

$$A = 0.1\log\log T_1\,, \qquad B = 3r\log\log\log T_1\,, \qquad Z = t\big((\log T_1)^{0.1}-1\big)\,,$$

$$m = \frac{\log T_1}{A+B} - \log^{0.1} T_1(\log\log T_1)^2\,.$$

Let, further, k be an integer varying in the interval

$$m \leqslant k \leqslant \frac{\log T_1}{A+B} \quad \left(< 10\frac{\log T_1}{\log\log T_1} \right). \tag{3.1}$$

We consider the integral

$$I = \frac{1}{2\pi i} \int_{4/3-iZ}^{4/3+iZ} D^s \left(\frac{e^{\psi s} - e^{-\psi s}}{2\psi s} \right)^r \left(e^{As} \frac{e^{Bs} - e^{-Bs}}{2Bs} \right)^k \frac{ds}{\zeta(s)} \tag{3.2}$$

and evaluate it as follows:

$$I = \frac{1}{2\pi i} \int_{(4/3)} D^s \left(\frac{e^{\psi s} - e^{-\psi s}}{2\psi s} \right)^r \left(e^{As} \frac{e^{Bs} - e^{-Bs}}{2Bs} \right)^k \frac{ds}{\zeta(s)} + O(T^{4/3}/Z^k)$$

$$= \sum_{n=1}^{\infty} \frac{\mu(n)}{2\pi i} \int_{(4/3)} D^s \left(\frac{e^{\psi s} - e^{-\psi s}}{2\psi s} \right)^r \left(e^{As} \frac{e^{Bs} - e^{-Bs}}{2Bs} \right)^k \frac{ds}{n^s} + O(T^{0.4}).$$

Similarly as in Section 2 we have

$$I = \sum_{X_1 \leqslant n \leqslant X_2} \frac{\mu(n)}{2\pi i} \int_{(4/3)} D^s \left(\frac{e^{\psi s} - e^{-\psi s}}{2\psi s} \right)^r \left(e^{As} \frac{e^{Bs} - e^{-Bs}}{2Bs} \right)^k \frac{ds}{n^s} + O(T^{0.4}),$$

where

$$X_1 = De^{-r\psi + (A-B)k}, \quad X_2 = De^{r\psi + (A+B)k}.$$

Further

$$I = \int_{X_1}^{X_2} \left\{ \frac{1}{2\pi i} \int_{(0)} D^s \left(\frac{e^{\psi s} - e^{-\psi s}}{2\psi s} \right)^r \left(e^{As} \frac{e^{Bs} - e^{-Bs}}{2Bs} \right)^k \frac{ds}{x^s} \right\} dM(x) + O(T^{0.4})$$

$$= -\int_{X_1}^{X_2} M(x)\, d\left\{ \frac{1}{2\pi i} \int_{(0)} D^s \left(\frac{e^{\psi s} - e^{-\psi s}}{2\psi s} \right)^r \left(e^{As} \frac{e^{Bs} - e^{-Bs}}{2Bs} \right)^k \frac{ds}{x^s} \right\} + O(T^{0.4})$$

$$= -\int_{X_1}^{X_2} M(x)\, d\left\{ \frac{1}{\pi} \int_0^{\infty} \cos\big(t(\log D + Ak - \log x)\big) \left(\frac{\sin \psi t}{\psi t} \right)^r \left(\frac{\sin Bt}{Bt} \right)^k dt \right\} + O(T^{0.4})$$

$$= \int_{X_1}^{X_2} M(x) \left\{ \frac{1}{\pi} \int_0^{\infty} \sin\big(t(\log D + Ak - \log x)\big) \left(-\frac{t}{x} \right) \left(\frac{\sin \psi t}{\psi t} \right)^r \left(\frac{\sin Bt}{Bt} \right)^k dt \right\} dx + O(T^{0.4}).$$

Hence we obtain

$$|I| \leqslant \int_{X_1}^{X_2} \frac{|M(x)|}{x} dx \, \frac{1}{\pi} \int_0^{\infty} t \left| \frac{\sin \psi t}{\psi t} \right|^r \left| \frac{\sin Bt}{Bt} \right|^k dt + O(T^{0.4})$$

$$< \int_{X_1}^{X_2} \frac{|M(x)|}{x} dx + O(T^{0.4}).$$

We have also

$$X_2 = De^{r\psi + (A+B)k} \leqslant De^{r\psi} \cdot T_1 = T$$

and

$$X_1 = De^{-r\psi + (A-B)k} = De^{(A+B)k} e^{r\psi} \cdot e^{-2kB - 2\psi r}$$

$$> T \exp\left(-c_3 \frac{\log T}{\log\log T} \log\log\log T \right),$$

so that

$$|I| \leqslant \int_X^T \frac{|M(x)|}{x} dx + O(T^{0.4}). \tag{3.3}$$

We apply Cauchy's theorem to the rectangle with vertices $\frac{4}{3} \pm iZ$, $-1 \pm iZ$. Our assumption (1.2) ensures that all the ϱ's inside this rectangle are simple and have $\sigma = \frac{1}{2}$. Hence

$$I = \sum_{|\Im\varrho| < Z} D^\varrho \left(\frac{e^{\psi\varrho} - e^{-\psi\varrho}}{2\psi\varrho} \right)^r \left(e^{A\varrho} \frac{e^{B\varrho} - e^{-B\varrho}}{2B\varrho} \right)^k \frac{1}{\zeta'(\varrho)} + O\left(\frac{Z^{r-1}}{Z^r} \cdot \frac{T^{4/3}}{Z^k} \right) +$$

$$+ \frac{1}{2\pi i} \int_{-1-iZ}^{-1+iZ} D^s \left(\frac{e^{\psi s} - e^{-\psi s}}{2\psi s} \right)^r \left(e^{As} \frac{e^{Bs} - e^{-Bs}}{2Bs} \right)^k \frac{ds}{\zeta(s)}.$$

The O-term is obviously $O(T^{0.4})$ and the latter integral is

$$O\left(e^{-(A-B)k} \int_{-\infty}^{+\infty} \frac{dt}{|\zeta(-1+it)|} \right) = O(1).$$

Hence and by (3.3) we get

$$\int_X^T \frac{|M(x)|}{x} dx \geqslant \left| \sum_{|\Im\varrho| < Z} D^\varrho \left(\frac{e^{\psi\varrho} - e^{-\psi\varrho}}{2\psi\varrho} \right)^r \left(e^{A\varrho} \frac{e^{B\varrho} - e^{-B\varrho}}{2B\varrho} \right)^k \frac{1}{\zeta'(\varrho)} \right| + O(T^{0.4}). \tag{3.4}$$

4. In this section we shall estimate the absolute value of the sum

$$S \stackrel{\text{def}}{=} \sum_{|\Im\varrho| < Z} D^\varrho \left(\frac{e^{\psi\varrho} - e^{-\psi\varrho}}{2\psi\varrho} \right)^r \left(e^{A\varrho} \frac{e^{B\varrho} - e^{-B\varrho}}{2B\varrho} \right)^k \frac{1}{\zeta'(\varrho)}$$

from below. We put it in the form

$$S = \left(e^{A\varrho_0} \frac{e^{B\varrho_0} - e^{-B\varrho_0}}{2B\varrho_0} \right)^k \left\{ \sum_{|\Im\varrho| < Z} D^\varrho \left(\frac{e^{\psi\varrho} - e^{-\psi\varrho}}{2\psi\varrho} \right)^r \frac{1}{\zeta'(\varrho)} \left(e^{A(\varrho - \varrho_0)} \frac{e^{B\varrho} - e^{-B\varrho}}{e^{B\varrho_0} - e^{-B\varrho_0}} \cdot \frac{\varrho_0}{\varrho} \right)^k \right\},$$

where ϱ_0 is that zero with $0 < \Im\varrho_0 < Z$ at which $\left| \frac{e^{B\varrho_0} - e^{-B\varrho_0}}{\varrho_0} \right|$ attains maximum, and define

$$b_j = D^\varrho \left(\frac{e^{\psi\varrho} - e^{-\psi\varrho}}{2\psi\varrho} \right)^r \frac{1}{\zeta'(\varrho)}, \quad z_j = e^{A(\varrho - \varrho_0)} \frac{e^{B\varrho} - e^{-B\varrho}}{e^{B\varrho_0} - e^{-B\varrho_0}} \cdot \frac{\varrho_0}{\varrho}, \tag{4.1}$$

arranging them so as required in lemma 1. The number of terms in S is, as easy to see, $\leqslant N = [\log^{0.1} T_1 (\log\log T_1)^2]$, so that putting $b_j = z_j = 0$ for those $j \leqslant N$ which do not occur in (4.1) (if such j's exist) we have defined b_j, z_j (with $1 = |z_1| \geqslant |z_2| \geqslant \ldots \geqslant |z_N|$) for all $1 \leqslant j \leqslant N$.

Put $Y_1 = \log\log T_1$. Let

$$z_h = e^{A(\varrho_h - \varrho_0)} \frac{e^{B\varrho_h} - e^{-B\varrho_h}}{e^{B\varrho_0} - e^{-B\varrho_0}} \cdot \frac{\varrho_0}{\varrho_h}$$

denote this of our z_j's corresponding to ϱ's with $|\Im\varrho| < t(Y_1)$, which has the maximal index $j = h$. Let, further,

$$z_{h_1} = e^{A(\varrho_{h_1} - \varrho_0)} \frac{e^{B\varrho_{h_1}} - e^{-B\varrho_{h_1}}}{e^{B\varrho_0} - e^{-B\varrho_0}} \cdot \frac{\varrho_0}{\varrho_{h_1}}$$

be any of z_j's with $|\Im\varrho_{h_1}| > t(Y_1)$. Writing $\varrho_h = \frac{1}{2} + i\gamma_h$, $\varrho_{h_1} = \frac{1}{2} + i\gamma_{h_1}$, we assert

$$|\gamma_{h_1}| - |\gamma_h| > \frac{1}{Y_1^{2r}}. \tag{4.2}$$

In fact, we can obviously suppose $|\gamma_{h_1}| < 2Y_1$ and have then by (2.1)

$$\frac{1}{Y_1^r} < \left|\zeta\left(\tfrac{1}{2} + it(Y_1)\right)\right| = \left|\int\limits_{1/2+it(Y_1)}^{1/2+i|\gamma_{h_1}|} \zeta'(s)\,ds\right| \leqslant \left(|\gamma_{h_1}| - t(Y_1)\right) Y_1^r,$$

whence (4.2).

We shall show now that the condition of lemma 1 is satisfied for every h_1 with $|\Im\varrho_{h_1}| > t(Y_1)$. Writing

$$|z_h| - |z_{h_1}| = \frac{|\varrho_0|}{|e^{B\varrho_0} - e^{-B\varrho_0}|}\left(\frac{|e^{B\varrho_h} - e^{-B\varrho_h}|}{|\varrho_h|} - \frac{|e^{B\varrho_{h_1}} - e^{-B\varrho_{h_1}}|}{|\varrho_{h_1}|}\right)$$
$$> e^{-B/2}\left\{\left(\frac{e^B + e^{-B} - 2}{\frac{1}{4} + \gamma_h^2}\right)^{1/2} - \left(\frac{e^B + e^{-B} + 2}{\frac{1}{4} + \gamma_{h_1}^2}\right)^{1/2}\right\},$$

we shall distinguish the following two cases. If $|\gamma_{h_1}| > 2Y_1$, we obtain

$$|z_h| - |z_{h_1}| > 1/3\, Y_1.$$

In the case of $|\gamma_{h_1}| \leqslant 2Y_1$ we get

$$|z_h| - |z_{h_1}| > e^{-B/2} \frac{\dfrac{e^B + e^{-B} - 2}{\frac{1}{4} + \gamma_h^2} - \dfrac{e^B + e^{-B} + 2}{\frac{1}{4} + \gamma_{h_1}^2}}{\left(\dfrac{e^B + e^{-B} - 2}{\frac{1}{4} + \gamma_h^2}\right)^{1/2} + \left(\dfrac{e^B + e^{-B} + 2}{\frac{1}{4} + \gamma_{h_1}^2}\right)^{1/2}}$$

$$= \frac{-1 - 2(\gamma_h^2 + \gamma_{h_1}^2) + (e^B + e^{-B})(\gamma_{h_1}^2 - \gamma_h^2)}{e^{B/2}\left\{\left(\dfrac{e^B + e^{-B} - 2}{\frac{1}{4} + \gamma_h^2}\right)^{1/2} + \left(\dfrac{e^B + e^{-B} + 2}{\frac{1}{4} + \gamma_{h_1}^2}\right)^{1/2}\right\}(\frac{1}{4} + \gamma_h^2)(\frac{1}{4} + \gamma_{h_1}^2)}$$

$$> \frac{e^B(\gamma_{h_1}^2 - \gamma_h^2)}{c_3 e^B Y_1^3} > c_4 \frac{(|\gamma_{h_1}| - |\gamma_h|)}{Y_1^2} > \frac{c_4}{Y_1^{2r+2}}.$$

Hence in either case

$$|z_h| - |z_{h_1}| > \frac{c_5}{Y_1^{2r+2}} > \frac{N}{m+N} \tag{4.3}$$

and also

$$|z_h| > \frac{2N}{m+N}, \tag{4.4}$$

so that we are entitled to apply lemma 1 with $h_1 = h+1$. We get then with a suitable k

$$|S| \geqslant \left|e^{A\varrho_0} \frac{e^{B\varrho_0} - e^{-B\varrho_0}}{2B\varrho_0}\right|^k \left|\sum_{|\Im\varrho| < t(Y_1)} D^\varrho \left(\frac{e^{\psi\varrho} - e^{-\psi\varrho}}{2\psi\varrho}\right)^r \frac{1}{\zeta'(\varrho)}\right| \times$$
$$\times \left|\frac{e^{A(\varrho_h - \varrho_0)}}{2} \cdot \frac{e^{B\varrho_h} - e^{-B\varrho_h}}{e^{B\varrho_0} - e^{-B\varrho_0}} \cdot \frac{\varrho_0}{\varrho_h}\right|^k \left(\frac{1}{24e} \cdot \frac{N}{2N+m}\right)^N$$
$$> c_6 e^{Ak/2} \left|\frac{e^{B\varrho_h} - e^{-B\varrho_h}}{4B\varrho_h}\right|^k \exp\left(-\log^{0.1} T (\log\log T)^3\right)$$
$$= c_6 e^{Ak/2} \frac{|e^B + e^{-B} - 2\cos(2B\gamma_h)|^{k/2}}{(4B|\varrho_h|)^k} \exp\left(-\log^{0.1} T (\log\log T)^3\right)$$
$$> c_7 e^{(A+B)k/2} \exp\left(-11 \frac{\log T}{\log\log T} \log\log\log T\right)$$
$$> 2T^{1/2} \exp\left(-12 \frac{\log T}{\log\log T} \log\log\log T\right).$$

This and (3.4) give the result.

References

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Reçu par la Rédaction le 14. 12. 1961