ACTA ARITHMETICA
VII (1962)

Contributions to the theory of the distribution of prime numbers in arithmetical progressions II

by

S. KNAPOWSKI (Poznań)

1. The subject of this paper will be some further questions concerning the distribution of prime powers in two different progressions mod k. I shall keep throughout the notation of the previous paper:

$$\psi(x,k,l) = \sum_{\substack{p^m \equiv l \pmod k \\ p^m \leqslant x}} \log p = \sum_{\substack{n \equiv l \pmod k \\ n \leqslant x}} \Lambda(n), \tag{1.1}$$

$$\Pi(x,k,l) = \sum_{\substack{p^m \equiv l \pmod k \\ p^m \leqslant x}} \frac{1}{m}, \tag{1.2}$$

$$\pi(x) = \sum_{\substack{p \equiv l \pmod k \\ p \leqslant x}} 1, \tag{1.3}$$

p—primes, p^m—prime powers, $c_1, c_2, \ldots$ positive numerical constants.

Similarly to [1] the following conjecture will be of importance in the sequel:

(1.4) *In the rectangle* $0 < \sigma < 1$, $|t| \leqslant \max(c_1, k^7)$, $s = \sigma + it$, *L-functions* mod k *may vanish only at points of the line* $\sigma = \frac{1}{2}$.

On this conjecture I shall prove the following

THEOREM 1. *Let* $k \geqslant 3$, $0 < l_1, l_2 < k$, $l_1 \neq l_2$, $(l_1, k) = (l_2, k) = 1$. *Suppose* (1.4) *to be satisfied. Then*

$$\int_X^T \frac{|\psi(x,k,l_1) - \psi(x,k,l_2)|}{x} dx > T^{1/2} \exp\left(-2\frac{\log T}{\log\log T}\right) \tag{1.5}$$

and

$$\int_X^T \frac{|\Pi(x,k,l_1) - \Pi(x,k,l_2)|}{x} dx > T^{1/2} \exp\left(-2\frac{\log T}{\log\log T}\right) \tag{1.6}$$

with

$$X = T\exp\left(-(\log T)^{3/4}\right)$$

for

(1.7) $$T \geqslant \max\left(c_2, \exp(k^{40})\right)$$

where c_2 is a calculable numerical constant.

Further, similarly to [1], we have a result holding without any conjectures.

THEOREM 2. *Let* $k \geqslant 3$, $0 < l_1$, $l_2 < k$, $l_1 \neq l_2$, $(l_1, k) = (l_2, k) = 1$. *Then*

(1.8) $$\int_X^T \frac{|\psi(x,k,l_1) - \psi(x,k,l_2)|}{x}\,dx > T^{1/4}$$

and

(1.9) $$\int_X^T \frac{|\Pi(x,k,l_1) - \Pi(x,k,l_2)|}{x}\,dx > T^{1/4}$$

with

$$X = T\exp\left(-(\log T)^{0.9}\right)$$

for

(1.10) $$T \geqslant \max\left(c_3, \exp(k^{30L_0})\right)$$

where L_0 is Linnik's constant [1] *and c_3 is numerically calculable.*

Theorems 1 and 2 refer to the distribution of prime powers in arithmetical progressions. Results concerning the distribution of primes $\equiv l_1 \pmod k$ compared to those $\equiv l_2 \pmod k$, i.e. concerning the behaviour of the function

(1.11) $$\int_X^T \frac{|\pi(x,k,l_1) - \pi(x,k,l_2)|}{x}\,dx,$$

as $T \to \infty$, will be presented in the third paper of this series. However, in one case we can state immediately a result for the function (1.11), deriving it directly from Theorem 1. This is

THEOREM 3. *Let* $k \geqslant 3$, $0 < l_1$, $l_2 < k$, $l_1 \neq l_2$, $(l_1, k) = (l_2, k) = 1$. *Let the congruences*

(1.12) $$\begin{cases} x^2 \equiv l_1 \pmod k, \\ x^2 \equiv l_2 \pmod k \end{cases}$$

[1] By *Linnik's constant* I understand, as in [1], such a number that to arbitrary given integers l, k, $0 < l < k$, there exists a prime number $P \equiv l \pmod k$ with $k < P \leqslant k^{L_0}$.

have no solutions and suppose (1.4) *to be satisfied. Then*

(1.13) $$\int_X^T \frac{|\pi(x,k,l_1) - \pi(x,k,l_2)|}{x}\,dx > T^{1/2}\exp\left(-3\frac{\log T}{\log\log T}\right)$$

with

$$X = T\exp\left(-(\log T)^{3/4}\right)$$

for

(1.14) $$T \geqslant \max\left(c_4, \exp(k^{40})\right).$$

In fact, by our assumption concerning the congruences (1.12), we have

$$\Pi(x,k,l_1) = \pi(x,k,l_1) + O(x^{1/3})$$

and

$$\Pi(x,k,l_2) = \pi(x,k,l_2) + O(x^{1/3})$$

which together with (1.6) yield (1.13).

2. Proofs of Theorems 1 and 2 will rest on the following lemmas.

LEMMA 1. *Let m be a non-negative number and $z_1, z_2, \ldots, z_N$ complex numbers such that*

$$1 = |z_1| \geqslant |z_2| \geqslant \ldots \geqslant |z_h| \geqslant \ldots \geqslant |z_N|, \qquad |z_h| > 2\frac{N}{m+N}.$$

Then there exists an integer ν with $m \leqslant \nu \leqslant m+N$ such that

(2.1) $$\frac{|b_1 z_1^\nu + b_2 z_2^\nu + \ldots + b_N z_N^\nu|}{\left(\frac{1}{2}|z_h|\right)^\nu} \geqslant \min_{h \leqslant j < h_1} |b_1 + b_2 + \ldots + b_j| \left(\frac{1}{24e}\cdot\frac{N}{2N+m}\right)^N,$$

where $h_1 \leqslant N$ is any integer for which $|z_{h_1}| < |z_h| - N/(m+N)$. In that case when there do not exist numbers h_1 satisfying the latter inequality, we put at the right-hand side of (2.1) $\min\limits_{h \leqslant j \leqslant N} |b_1 + b_2 + \ldots + b_j|$ *instead.*

This lemma—which is a modification of Turán's second main theorem—has been proved in [1].

LEMMA 2. *Let* $k \geqslant 3$, $0 < l_1$, $l_2 < k$, $l_1 \neq l_2$, $(l_1, k) = (l_2, k) = 1$. *Suppose* (1.4) *to be satisfied. Then there exists a number D,* $\frac{1}{2}\max(c_5, k^3) \leqslant D \leqslant \max(c_5, k^3)$ *such that*

(2.2) $$\left|\frac{1}{\varphi(k)}\sum_{(\chi)}\left(\bar\chi(l_1) - \bar\chi(l_2)\right)\sum_{\varrho(\chi)} D^\varrho \left(\frac{e^{\psi\varrho} - e^{-\psi\varrho}}{2\psi\varrho}\right)^2\right| \geqslant c_6 D\log D$$

where $\psi = 1/3D$, χ runs through all characters mod k *and $\varrho = \varrho(\chi)$ through the zeros of $L(s,\chi)$ lying in the strip $0 < \sigma < 1$.*

LEMMA 3. *Let* $k \geqslant 3$, $0 < l_1, l_2 < k$, $l_1 \neq l_2$, $(l_1, k) = (l_2, k) = 1$. *Let* L_0 *be the constant of Linnik. There exists a number* D_1, $\max(c_7, k^7) \leqslant D_1 \leqslant \max(c_8, k^{L_0})$ *such that*

$$\left|\frac{1}{\varphi(k)}\sum_{(\chi)}\left(\bar{\chi}(l_1)-\bar{\chi}(l_2)\right)\sum_{\varrho(\chi)} D_1^{\varrho}\left(\frac{e^{\psi_1\varrho}-e^{-\psi_1\varrho}}{2\psi_1\varrho}\right)^2\right| \geqslant c_9 D_1 \log D_1 \tag{2.3}$$

with $\psi_1 = 1/3\,D_1$ *and* χ, $\varrho(\chi)$ *running as in lemma* 2.

Proof of lemma 2. As in [1], proof of lemma 2, we shall confine ourselves to $k \geqslant c_{10}$ and, correspondingly, suppose $\prod_{(\chi)} L(s,\chi) \neq 0$ in $\sigma > \frac{1}{2}$, $|t| \leqslant k^7$. Let D be a prime or a prime square with $\frac{1}{2}k^3 \leqslant D \leqslant k^3$, $D \equiv l_1 (\operatorname{mod} k)$ (the existence of which has been proved in [1], p. 423). Let, further, χ_0 be the principal character and χ_1 an arbitrary non-principal character mod$k$. We have then (see [1], formulae (3.3), (3.4))

$$-\bar{\chi}_1(l_1)\sum_{\varrho(\chi_1)} D^{\varrho}\left(\frac{e^{\psi\varrho}-e^{-\psi\varrho}}{2\psi\varrho}\right)^2 = \frac{A(D)}{2\psi}+O(D^{1/2}\log D) \tag{2.4}$$

and

$$-\bar{\chi}_0(l_1)\sum_{\varrho(\chi_0)} D^{\varrho}\left(\frac{e^{\psi\varrho}-e^{-\psi\varrho}}{2\psi\varrho}\right)^2 = \frac{A(D)}{2\psi}-D+O(D^{1/2}\log D)\,. \tag{2.5}$$

Multiplying (2.4) and (2.5) by $\frac{1}{\varphi(k)}\chi_1(l_1)\bar{\chi}_1(l_2)$, $\frac{1}{\varphi(k)}\chi_0(l_1)\bar{\chi}_0(l_2)$, respectively and summing up we obtain

$$\frac{1}{\varphi(k)}\sum_{(\chi)}\bar{\chi}(l_2)\sum_{\varrho(\chi)} D^{\varrho}\left(\frac{e^{\psi\varrho}-e^{-\psi\varrho}}{2\psi\varrho}\right)^2 = -\frac{A(D)}{2\psi}\cdot\frac{1}{\varphi(k)}\sum_{(\chi)}\chi(l_1)\bar{\chi}(l_2)+O(D)\,.$$

Hence

$$\frac{1}{\varphi(k)}\sum_{(\chi)}\bar{\chi}(l_2)\sum_{\varrho(\chi)} D^{\varrho}\left(\frac{e^{\psi\varrho}-e^{-\psi\varrho}}{2\psi\varrho}\right)^2 = O(D) \tag{2.6}$$

since

$$\frac{1}{\varphi(k)}\sum_{(\chi)}\chi(l_1)\bar{\chi}(l_2) = 0\,,$$

this being equivalent to $l_1 \not\equiv l_2 (\operatorname{mod} k)$.

Similarly, multiplying (2.4) and (2.5) by $1/\varphi(k)$ and summing up, we have

$$\frac{1}{\varphi(k)}\sum_{(\chi)}\bar{\chi}(l_1)\sum_{\varrho(\chi)} D^{\varrho}\left(\frac{e^{\psi\varrho}-e^{-\psi\varrho}}{2\psi\varrho}\right)^2 = -\frac{A(D)}{2\psi}+O(D)\,. \tag{2.7}$$

Subtracting (2.6) and (2.7) we come to (2.2).

Proof of lemma 3 does not essentially differ from that of lemma 2 (compare [1], p. 425) and can be dropped.

3. Proof of Theorem 1. As in [1] it will be enough to consider only the case of sufficiently large k. Consequently the rectangle in (1.4) can be taken as being $0 < \sigma < 1$, $|t| \leqslant k^7$. Further, one can content oneself only with a proof of e.g. (1.6); proof of (1.5), in fact, would run completely parallel.

Let us write

$$T_1 = \frac{T}{D}e^{-2\psi} \quad (D,\psi \text{ as in lemma } 2)\,,$$

$$A = 0.6\log\log T_1\,,\quad B = (\log T_1)^{-0.25}\,,\quad m = \frac{\log T_1}{A+B}-\log^{0.6}T_1\,(\log\log T_1)^2\,,$$

r an integer, to be fixed later, satisfying

$$m \leqslant r \leqslant \frac{\log T_1}{A+B}\quad\left(<\frac{5}{3}\cdot\frac{\log T_1}{\log\log T_1}\right). \tag{3.1}$$

Putting further

$$F_{l_1 l_2}(s) = \frac{1}{\varphi(k)}\sum_{(\chi)}\left(\bar{\chi}(l_2)-\bar{\chi}(l_1)\right)\frac{L'}{L}(s,\chi)$$

we consider the integral

$$\begin{aligned} J_{l_1 l_2} &= \frac{1}{2\pi i}\int_{(2)} D^s\left(\frac{e^{\psi s}-e^{-\psi s}}{2\psi s}\right)^2\left(e^{As}\frac{e^{Bs}-e^{-Bs}}{2Bs}\right)^r F_{l_1 l_2}(s)\,ds \\ &= \sum_{n\equiv l_1 (\operatorname{mod} k)}\frac{A(n)}{2\pi i}\int_{(2)} D^s\left(\frac{e^{\psi s}-e^{-\psi s}}{2\psi s}\right)^2\left(e^{As}\frac{e^{Bs}-e^{-Bs}}{2Bs}\right)^r\frac{ds}{n^s} - \\ &\quad - \sum_{n\equiv l_2 (\operatorname{mod} k)}\frac{A(n)}{2\pi i}\int_{(2)} D^s\left(\frac{e^{\psi s}-e^{-\psi s}}{2\psi s}\right)^2\left(e^{As}\frac{e^{Bs}-e^{-Bs}}{2Bs}\right)^r\frac{ds}{n^s}\,. \end{aligned} \tag{3.2}$$

Noting that terms of the series in (3.2) are $\neq 0$ only in

$$(X_1 \stackrel{\text{def}}{=})\quad De^{-2\psi}e^{(A-B)r} \leqslant n \leqslant De^{2\psi}e^{(A+B)r}\quad(\stackrel{\text{def}}{=} X_2)$$

we obtain

$$\begin{aligned} J_{l_1 l_2} &= \sum_{\substack{X_1\leqslant n\leqslant X_2\\ n\equiv l_1 (\operatorname{mod} k)}}\frac{A(n)}{2\pi i}\int_{(0)} D^s\left(\frac{e^{\psi s}-e^{-\psi s}}{2\psi s}\right)^2\left(e^{As}\frac{e^{Bs}-e^{-Bs}}{2Bs}\right)^r\frac{ds}{n^s} - \\ &\quad - \sum_{\substack{X_1\leqslant n\leqslant X_2\\ n\equiv l_2 (\operatorname{mod} k)}}\frac{A(n)}{2\pi i}\int_{(0)} D^s\left(\frac{e^{\psi s}-e^{-\psi s}}{2\psi s}\right)^2\left(e^{As}\frac{e^{Bs}-e^{-Bs}}{2Bs}\right)^r\frac{ds}{n^s}\,. \end{aligned}$$

Using Stieltjes integral we have further

$$J_{l_1l_2} =$$

$$= \int_{X_1}^{X_2} \left\{\frac{\log x}{2\pi i} \int_{(0)} D^s \left(\frac{e^{\psi s}-e^{-\psi s}}{2\psi s}\right)^2 \left(e^{As}\frac{e^{Bs}-e^{-Bs}}{2Bs}\right)^r \frac{ds}{x^s}\right\} d\big(\Pi(x,k,l_1)-\Pi(x,k,l_2)\big)$$

$$= \left\{\big(\Pi(x,k,l_1)-\Pi(x,k,l_2)\big)\frac{\log x}{2\pi i}\int_{(0)} D^s\left(\frac{e^{\psi s}-e^{-\psi s}}{2\psi s}\right)^2\left(e^{As}\frac{e^{Bs}-e^{-Bs}}{2Bs}\right)^r\frac{ds}{x^s}\right\}_{X_1}^{X_2} -$$

$$-\int_{X_1}^{X_2}\big(\Pi(x,k,l_1)-\Pi(x,k,l_2)\big)\, d\left\{\frac{\log x}{2\pi i}\int_{(0)} D^s\left(\frac{e^{\psi s}-e^{-\psi s}}{2\psi s}\right)^2\left(e^{As}\frac{e^{Bs}-e^{-Bs}}{2Bs}\right)^r\frac{ds}{x^s}\right\}$$

$$= -\int_{X_1}^{X_2}\big(\Pi(x,k,l_1)-\Pi(x,k,l_2)\big)\, d\left\{\frac{\log x}{\pi}\int_0^\infty \cos\big(t(\log D+Ar-\log x)\big)\times\right.$$

$$\left.\times\left(\frac{\sin\psi t}{\psi t}\right)^2\left(\frac{\sin Bt}{Bt}\right)^r dt\right\} = \int_{X_1}^{X_2}\big(\Pi(x,k,l_1)-\Pi(x,k,l_2)\big)\times$$

$$\times\left\{\frac{-1}{\pi x}\int_0^\infty \cos\big(t(\log D+Ar-\log x)\big)\left(\frac{\sin\psi t}{\psi t}\right)^2\left(\frac{\sin Bt}{Bt}\right)^r dt+\right.$$

$$\left.+\frac{\log x}{\pi}\int_0^\infty \sin\big(t(\log D+Ar-\log x)\big)\left(-\frac{t}{x}\right)\left(\frac{\sin\psi t}{\psi t}\right)^2\left(\frac{\sin Bt}{Bt}\right)^r dt\right\} dx\,.$$

Hence

$$|J_{l_1l_2}| \leqslant \int_{X_1}^{X_2} \frac{|\Pi(x,k,l_1)-\Pi(x,k,l_2)|}{x}\log x\, dx \int_0^\infty \frac{t+1}{\pi}\left(\frac{\sin\psi t}{\psi t}\right)^2\left|\frac{\sin Bt}{Bt}\right|^r dt\,.$$

But

$$\int_0^\infty t\left(\frac{\sin\psi t}{\psi t}\right)^2\left|\frac{\sin Bt}{Bt}\right|^r dt \leqslant \int_0^\infty t\left|\frac{\sin Bt}{Bt}\right|^r dt = \frac{1}{B^2}\int_0^\infty\left|\frac{\sin u}{u}\right|^r u\,du$$

$$\leqslant \frac{1}{B^2}\left(1+\int_1^\infty\frac{du}{u^{r-1}}\right) < \frac{2}{B^2} < 2\sqrt{\log T}$$

and similarly

$$\int_0^\infty\left(\frac{\sin\psi t}{\psi t}\right)^2\left|\frac{\sin Bt}{Bt}\right|^r dt < 2\sqrt[4]{\log T}\,.$$

Further by (3.1)

$$X_2 = De^{(A+B)r+2\psi} \leqslant De^{2\psi}T_1 = T$$

and

$$X_1 = De^{(A-B)r-2\psi}$$

$$\geqslant D\exp\big(-2\psi-2Br+\log T_1-(A+B)\log^{0.6}T_1(\log\log T_1)^2\big)$$

$$> T\exp\left(-4\psi-\frac{10}{3}\cdot\frac{\log T_1}{\log\log T_1}(\log T_1)^{-1/4}-(\log T_1)^{0.6}(\log\log T_1)^3\right)$$

$$> Te^{-(\log T)^{3/4}},$$

whence

$$(3.3)\qquad |J_{l_1l_2}| \leqslant (\log T)^{1.5}\int_X^T \frac{|\Pi(x,k,l_1)-\Pi(x,k,l_2)|}{x}\,dx$$

with

$$X = T\exp\big(-(\log T)^{3/4}\big)\,.$$

Similarly to [1] there exists an infinite connected broken line U consisting of segments alternately parallel to the axes, all lying in

$$\tfrac{1}{30} \leqslant \sigma \leqslant \tfrac{1}{20}\,,$$

and such that

$$\left|\frac{L'}{L}(s,\chi)\right| \leqslant c_{11}k\log^2\big(k(|t|+1)\big)\,,\qquad \chi\ (\mathrm{mod}\ k)\,,$$

over U.

Applying Cauchy's theorem of residues to the integral (3.2) we obtain

$$J_{l_1l_2} = \frac{1}{\varphi(k)}\sum_{(\chi)}\big(\overline{\chi}(l_2)-\overline{\chi}(l_1)\big)\sum_{\varrho=\varrho(\chi)>U} D^\varrho\left(\frac{e^{\psi\varrho}-e^{-\psi\varrho}}{2\psi\varrho}\right)^2\left(e^{A\varrho}\frac{e^{B\varrho}-e^{-B\varrho}}{2B\varrho}\right)^r +$$

$$+\frac{1}{2\pi i}\int_{(U)} D^s\left(\frac{e^{\psi s}-e^{-\psi s}}{2\psi s}\right)^2\left(e^{As}\frac{e^{Bs}-e^{-Bs}}{2Bs}\right)^r F_{l_1l_2}(s)\,ds$$

(here, as in [1], $\varrho > U$ means that the ϱ's are to be taken to the right of U).

The above formula can be converted, as in [1], to

(3.4)

$$J_{l_1l_2} = \frac{1}{\varphi(k)}\sum_{(\chi)}\big(\overline{\chi}(l_2)-\overline{\chi}(l_1)\big)\sum_{\substack{\varrho>U\\ |\Im\varrho|\leqslant Y}} D^\varrho\left(\frac{e^{\psi\varrho}-e^{-\psi\varrho}}{2\psi\varrho}\right)^2\left(e^{A\varrho}\frac{e^{B\varrho}-e^{-B\varrho}}{2B\varrho}\right)^r + O(T^{0.48})\,,$$

where

$$Y \overset{\mathrm{def}}{=} (\log T_1)^{0.6}\,.$$

Let $\varrho_1 = \frac{1}{2}+i\gamma_1$ be that zero from $0<\sigma<1$, $|t|\leqslant k^{6.5}$ which has the greatest imaginary part. We have (see [1], (4.8))

$$(3.5)\qquad \left|\frac{e^{B\varrho}-e^{-B\varrho}}{2B\varrho}\right| \geqslant \left|\frac{e^{B\varrho_1}-e^{-B\varrho_1}}{2B\varrho_1}\right|$$

for all zeros $\varrho=\frac{1}{2}+i\gamma$ with $|\gamma|\leqslant|\gamma_1|-1$. Denoting, further, that zero from $0<\sigma<1$, $|t|\leqslant Y$ at which $\left|e^{A\varrho}\frac{e^{B\varrho}-e^{-B\varrho}}{2B\varrho}\right|$ is maximal by $\varrho_2=\beta_2+i\gamma_2$ and putting

$$b_j=\frac{1}{\varphi(k)}\big(\overline{\chi}(l_2)-\overline{\chi}(l_1)\big)D^{\varrho}\left(\frac{e^{\psi\varrho}-e^{-\psi\varrho}}{2\psi\varrho}\right)^2,$$

$$z_j=e^{A(\varrho-\varrho_2)}\frac{e^{B\varrho}-e^{-B\varrho}}{e^{B\varrho_2}-e^{-B\varrho_2}}\cdot\frac{\varrho_2}{\varrho},$$

$$z_h=e^{A(\varrho_1-\varrho_2)}\frac{e^{B\varrho_1}-e^{-B\varrho_1}}{e^{B\varrho_2}-e^{-B\varrho_2}}\cdot\frac{\varrho_2}{\varrho_1},$$

we make the double sum in (3.4) equal to

$$\left(\frac{e^{A\varrho_2}(e^{B\varrho_2}-e^{-B\varrho_2})}{2B\varrho_2}\right)^r\sum_{j=1}^{N}b_jz_j^r$$

with

$$N=[k\log^{0.6}T_1(\log\log T_1)^2]$$

(if $N>\sum_{(\chi)}\sum_{\substack{\varrho>U\\|\Im\varrho|\leqslant Y}}1$, we put $z_j=b_j=0$ for the remaining j's). We check as in [1]

$$|z_h|>\frac{2N}{N+m},$$

so that by lemma 1, (2.1), we have with an appropriate r

(3.6)

$$|J_{l_1l_2}|+c_{12}T^{0.48}\geqslant\left|\frac{z_h}{2}\right|^r\left|e^{A\varrho_2}\frac{e^{B\varrho_2}-e^{-B\varrho_2}}{2B\varrho_2}\right|^r\left(\frac{1}{24e}\cdot\frac{N}{2N+m}\right)^N\min_{h\leqslant j\leqslant N}|b_1+b_2+\ldots+b_j|.$$

Owing to (3.5)

$$b_1+b_2+\ldots+b_j$$

$$=\frac{1}{\varphi(k)}\sum_{(\chi)}\big(\overline{\chi}(l_2)-\overline{\chi}(l_1)\big)\sum_{|\Im\varrho|\leqslant|\gamma_1|-1}D^{\varrho}\left(\frac{e^{\psi\varrho}-e^{-\psi\varrho}}{2\psi\varrho}\right)^2+O\Big(\sum_{n\geqslant|\gamma_1|-2}\frac{D}{\psi^2}\cdot\frac{\log kn}{n^2}\Big)$$

$$=\frac{1}{\varphi(k)}\sum_{(\chi)}\big(\overline{\chi}(l_2)-\overline{\chi}(l_1)\big)\sum_{\varrho(\chi)}D^{\varrho}\left(\frac{e^{\psi\varrho}-e^{-\psi\varrho}}{2\psi\varrho}\right)^2+O(k^{2.5}\log^2k),$$

so that by lemma 2, (2.2),

(3.7)
$$\min_{h\leqslant j\leqslant N}|b_1+b_2+\ldots+b_j|>c_{13}k^3\log k.$$

This and (3.6) give, similarly to [1],

$$|J_{l_1l_2}|>T^{1/2}\exp\left(-\frac{23}{12}\cdot\frac{\log T}{\log\log T}\right),$$

which combined with (3.3) yields (1.6).

4. Proof of Theorem 2. As in the preceding section we shall limit ourselves to a proof of (1.9). Again there is no loss of generality in supposing k to be sufficiently large. We write

$$T_2=\frac{T}{D_1}e^{-2\psi_1}\quad(D_1,\ \psi_1\text{ as in lemma 3}),$$

$$A_1=0.4\log\log T_2,\qquad B_1=(\log T_2)^{-0.1},$$

$$m_1=\frac{\log T_2}{A_1+B_1}-\log^{0.4}T_2(\log\log T_2)^2,$$

integer r_1 with

$$m_1\leqslant r_1\leqslant\frac{\log T_2}{A_1+B_1}\quad\left(<\frac{5}{2}\cdot\frac{\log T_2}{\log\log T_2}\right)$$

and consider the integral

$$\tilde{J}_{l_1l_2}=\frac{1}{2\pi i}\int_{(2)}D_1^s\left(\frac{e^{\psi_1s}-e^{-\psi_1s}}{2\psi_1s}\right)^2\left(e^{A_1s}\frac{e^{B_1s}-e^{-B_1s}}{2B_1s}\right)^{r_1}F_{l_1l_2}(s)\,ds.$$

Similarly to the preceding section we obtain

(4.1)
$$|\tilde{J}_{l_1l_2}|<(\log T)^{1.5}\int_X^T\frac{|\Pi(x,k,l_1)-\Pi(x,k,l_2)|}{x}dx$$

with

$$X=T\exp\big(-(\log T)^{0.9}\big).$$

On the other hand we get as before (cf. (3.4))

(4.2)
$$\tilde{J}_{l_1l_2}=\frac{1}{\varphi(k)}\sum_{(\chi)}\big(\overline{\chi}(l_2)-\overline{\chi}(l_1)\big)\sum_{\substack{\varrho>U_1\\|\Im\varrho|\leqslant Y_1}}D_1^{\varrho}\left(\frac{e^{\psi_1\varrho}-e^{-\psi_1\varrho}}{2\psi_1\varrho}\right)^2\left(e^{A_1\varrho}\frac{e^{B_1\varrho}-e^{-B_1\varrho}}{2B_1\varrho}\right)^r+$$
$$+O(T^{1/4+1/150}),$$

where U_1 is a certain broken line lying in

$$\tfrac{1}{300}\leqslant\sigma\leqslant\tfrac{1}{200}$$

and $Y_1=(\log T_2)^{0.4}$.

Let $\varrho_3=\beta_3+i\gamma_3$ be that zero from $0<\sigma<1$, $|t|\leqslant Y_1$ at which $\left|e^{A_1\varrho}\frac{e^{B_1\varrho}-e^{-B_1\varrho}}{2B_1\varrho}\right|$ is maximal. Denoting, further, that zero from the rectangle

$$\tfrac{1}{27}\leqslant\sigma<1,\qquad|t|\leqslant D_1^{2.5},$$

at which $\left|e^{A_1\varrho}\frac{e^{B_1\varrho}-e^{-B_1\varrho}}{2B_1\varrho}\right|$ is minimal by $\varrho_4=\beta_4+i\gamma_4$ and putting

$$b_j=\frac{1}{\varphi(k)}\big(\overline{\chi}(l_2)-\overline{\chi}(l_1)\big)D_1^{\varrho}\left(\frac{e^{\psi_1\varrho}-e^{-\psi_1\varrho}}{2\psi_1\varrho}\right)^2,$$

$$z_j = e^{A_1(\varrho-\varrho_3)}\frac{e^{B_1\varrho}-e^{-B_1\varrho}}{e^{B_1\varrho_3}-e^{-B_1\varrho_3}}\cdot\frac{\varrho_3}{\varrho},$$

$$z_h = e^{A_1(\varrho_4-\varrho_3)}\frac{e^{B_1\varrho_4}-e^{-B_1\varrho_4}}{e^{B_1\varrho_3}-e^{-B_1\varrho_3}}\cdot\frac{\varrho_3}{\varrho_4},$$

we make the double sum in (4.2) equal to

$$\left(\frac{e^{A_1\varrho_3}(e^{B_1\varrho_3}-e^{-B_1\varrho_3})}{2B_1\varrho_3}\right)^{r_1}\sum_{j=1}^{N_1} b_j z_j^{r_1},$$

with

$$N_1 = [k\log^{0.4}T_2\,(\log\log T_2)^2]$$

(again, if N_1 is greater than the actual number of zeros in the considered domain, we can put $z_j = b_j = 0$ for the remaining j's).

The inequality

$$|z_h| > \frac{2N_1}{N_1+m_1}$$

can be easily verified, whence by lemma 1, (2.1), we have with an appropriate r_1

$$|\tilde{J}_{l_1 l_2}| + c_{14}T^{1/4+1/150}$$
$$\geqslant \left|\frac{z_h}{2}\right|^{r_1}\left|\frac{e^{A_1\varrho_3}(e^{B_1\varrho_3}-e^{-B_1\varrho_3})}{2B_1\varrho_3}\right|^{r_1}\left(\frac{1}{24e}\cdot\frac{N_1}{2N_1+m_1}\right)^{N_1}\min_{h\leqslant j\leqslant N_1}|b_1+b_2+\ldots+b_j|$$

and further

$$(4.3)\qquad |\tilde{J}_{l_1 l_2}| + c_{14}T^{1/4+1/150} \geqslant T^{7/27}\exp\left(-3\frac{\log T}{\log\log T}\right)\min_{h\leqslant j\leqslant N_1}|b_1+b_2+\ldots+b_j|.$$

In order to estimate $\min\limits_{h\leqslant j\leqslant N_1}|b_1+b_2+\ldots+b_j|$ we use the following density theorem (see [2], Satz 1.1, p. 299 and p. 323).

Let $0<\alpha<1$ and $N(\alpha, T) = N(\alpha, T, k)$ stand for the number of zeros of all L-functions mod k *in the rectangle*

$$\alpha<\sigma<1, \qquad |t|\leqslant T.$$

Then, if $T\geqslant k$,

$$(4.4)\qquad N(\alpha, T) < c_{15}(k^4T^{8/3})^{1-\alpha}\log^8 T.$$

We have (with $h\leqslant j\leqslant N_1$)

$$b_1+b_2+\ldots+b_j = \frac{1}{\varphi(k)}\sum_{(\chi)}\big(\overline{\chi}(l_2)-\overline{\chi}(l_1)\big)\sum_{\varrho(\chi)}D_1^{\varrho}\left(\frac{e^{\psi_1\varrho}-e^{-\psi_1\varrho}}{2\psi_1\varrho}\right)^2+$$
$$+O\Bigg(\sum_{n\geqslant D_1^{2.5}-1}\frac{\log kn}{n^2}D_1^3\Bigg)+O\Bigg(\frac{1}{\varphi(k)}\sum_{(\chi)}\sum_{\substack{\varrho(\chi)\\ \Re\varrho<7/27}}|D_1^{\varrho}|\left|\frac{e^{\psi_1\varrho}-e^{-\psi_1\varrho}}{2\psi_1\varrho}\right|^2\Bigg).$$

The first error term is (cf. [1], p. 433) $O(D_1^{1/2}\log^2 D_1)$, the second one can be estimated using (4.4). In fact it is (cf. [1], p. 434)

$$O\left(\frac{D_1^{7/27}}{\varphi(k)}N\left(\frac{20}{27}, D_1\right)+\frac{D_1^{7/27}}{\varphi(k)}\int_{D_1}^{\infty}\frac{1}{\psi_1^2}\cdot\frac{dN(20/27, x)}{x^2}\right) = O(D_1).$$

This and lemma 3, (2.3), give

$$\min_{h\leqslant j\leqslant N_1}|b_1+b_2+\ldots+b_j| > c_{16}D_1\log D_1,$$

so that by (4.1) and (4.3) we obtain the desired (1.9).

References

[1] S. Knapowski, *Contributions to the theory of the distribution of prime numbers in arithmetical progressions I*, Acta Arith. 6 (1961), pp. 415-434.

[2] K. Prachar, *Primzahlverteilung*, Berlin 1957.

Reçu par la Rédaction le 14. 12. 1961