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Almost even functions of finite abelian groups

by

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*To Professor H. S. Vandiver
on his eigtieth birthday*

1. Introduction. In his monograph [8] of 1943, Wintner developed a Fourier theory of almost periodic arithmetical functions; a little later (1945) the same subject was treated undependently by Delsarte [4]. Although the investigations of both authors were brief, and on the surface quite simple, they were in fact based upon the Besicovitch theory of almost periodic functions. In view of their essential dependence upon real function theory, both may be described as non-arithmetical in nature.

In the Wintner and Delsarte theories, a central part was played by the trigonometric sum of Ramanujan. More generally, the additivity of the integers appeared prominetly, in spite of the fact that the functions treated were all defined on the multiplicative semigroup J^* of the positive integers. It seems natural to ask if a purely arithmetical theory might be developed, which at the same time makes no appeal to additive characters or periodicity concepts. In attempting to generalize his theory to algebraic number fields, Delsarte surmised the need of some such development, asking "Quelle est la propriété remplaçant la presque-périodicité ... quand l'idéal A n'est pas principal?"

It is the aim of this paper to indicate an answer to Delsarte's question by initiating the rudiments of a Fourier theory of arithmetical functions which is of a transparently arithmetical character and which makes no use of additivity properties or periodicity notions. In place of the positive integers, the domain of the investigation will be the multiplicative semigroup X of the finite abelian groups. This is to stress the multiplicative nature of the theory; in X additive characters are not available for an investigation of the Wintner-Delsarte type. Moreover, the arguments required for a Fourier theory in X are only relatively more complicated than those needed in the subsemigroup J of the completely reducible groups of X, equivalently ([2], § 1) the semigroup J^*. A corresponding discussion for J^* is therefore omitted.

With respect to Delsarte's question, the notion of almost periodicity is replaced by that of "almost parity". The precise definition of an *almost even function* $f(G)$, $G \in X$, is given in § 4. This class of functions extends the class of even functions $(\bmod H)$, $H \in X$, studied in [3] and characterized by the property, $f(G, H) = f((G, H), H)$, where (G, H) denotes the greatest common direct factor of G and H in X. The concept of even function $(\bmod H)$ helps to fill a vacuum created by the absence of a periodicity property in X.

The rôle played by the Ramanujan sum $c(n, r)$ in the theory of Wintner and Delsarte is assumed by a multiplicatively defined analogue of $c(n, r)$ in X. This function is the uniquely defined function $c(G, H) = c_\varrho(G, H)$, satisfying the relation ([3], § 3),

$$(1.1)\qquad \sum_{D|H} c(G, H) = \eta(G, H) \equiv \begin{cases} \varrho(H) & \text{if} \quad H|G, \\ 0 & \text{if} \quad H \nmid G, \end{cases}$$

where $\varrho(H)$ is the order of H and the summation is over all direct factors D in X of H. In [3], § 4, it was proved that a function $f(G, H)$ is even $(\bmod H)$ *if and only if it possesses an expansion of the form,*

$$(1.2)\qquad f(G, H) = \sum_{D|H} a(D, H) c(G, D).$$

This result suggests a means for extending the concept of even function $(\bmod H)$ to that of almost even function. The representation (1.2) is, in fact, the Fourier series of $f(G, H)$ in the sense of § 4 (see Theorem 4.1); this is a consequence of the orthogonality property of $c(G, H)$ proved in Theorem 3.1.

In § 5 we determine Fourier expansions for a wide class of almost even functions. The convergence of some of these expansions is investigated in § 6. Application is made to generalized divisor functions; for example, it is proved ($s = 1$ in (6.4)) that

$$(1.3)\qquad \frac{\sigma(G)}{\varrho(G)} = Z(2) \sum_{H \in X} \frac{c(G, H)}{\varrho^2(H)},$$

where $\sigma(G)$ denotes the sum of the orders of the direct factors of G in X, and $Z(s)$ is the zeta-function of X ([2], § 2). This result is the abelian analogue of a classical result of Ramanujan ([6], Theorem 293) on the sum $\sigma(n)$ of the divisors of n.

In § 7 we introduce an important subclass of the almost even functions which itself contains as a subclass the primitive functions $(\bmod H)$ of [3]. A convergence criterion is proved in Theorem 7.3.

Finally, we mention that Theorems 5.1 and 6.1 (b) are analogues, respectively of the two principal results in Wintner's theory ([8], Theorems XVI and XVIII). The convergence criteria contained in Theorem 6.1 (a) and Theorem 7.3 seem to be of a new type.

2. Preliminaries. We recall some known results. The function defined by (1.1) has the evaluation ([3], (3.2))

$$(2.1)\qquad c(G, H) = \sum_{\substack{DE=H \\ D|G}} \varrho(D)\mu(E),$$

where $\mu(H)$ is the Möbius function of X ([2], § 2). We note that $c(I, H) = \mu(H)$, where I is the identity of X. The function $\varphi(H) = c(H, H)$ is the "order totient" of X ([3], § 3). The functions $\mu(H)$, $\varphi(H)$, and $c(G, H)$ are multiplicative functions of H; by (1.1)

$$(2.2)\qquad \sum_{D|H} \mu(D) = e(H) \equiv \begin{cases} 1 & (H = I), \\ 0 & (H \neq I), \end{cases}$$

while by virtue of (2.1)

$$(2.3)\qquad \varphi(H) = \sum_{DE=H} \varrho(D)\mu(E).$$

It can be easily verified that $\varphi(H) > 0$ for all $H \in X$. We note that $\mu(H)$ has the value 0 for inseparable groups H of X and has the value $(-1)^k$ if H is separable with k (distinct) indecomposable factors.

Let $\zeta(s)$ denote the Riemann ζ-function; the function $Z(s)$ has the expansion, $Z(s) = \prod_{i=1}^{\infty} \zeta(is)$, $s > 1$. With $a = Z(2)/\zeta(2)$, we recall the following estimate of Erdös and Szekeres [5] for the number $A(x)$ of groups in X of order $\leqslant x$:

$$(2.4)\qquad A(x) = ax + O(\sqrt{x}).$$

It is assumed here, and throughout the paper, that $x \geqslant 1$.

3. Orthogonality. For a complex-valued function $f(G)$, we define the average or mean value $M(f) = M(f(G))$ of $f(G)$, by

$$M(f) = \lim_{x \to \infty} \frac{1}{A(x)} \sum_{\varrho(G) \leqslant x} f(G),$$

provided this limit exists.

We consider now the average behavior of $c(G, H)$. We shall need the following simple extension of (2.2), whose proof is the same as the corresponding result in J^* [1, Lemma 4].

LEMMA 3.1.

$$(3.1)\qquad \sum_{\substack{DE=H \\ (E,G)=I}} \mu(D) = \begin{cases} \mu(H) & \textit{if } H|G, \\ 0 & \textit{otherwise}. \end{cases}$$

Remark 3.1. In the following, vacuous sums will be assumed to have the value 0.

LEMMA. 3.2. *Let D and H denote groups of X, H fixed; then*

$$\sum_{\substack{\varrho(G)\leqslant x\\ D|G}} c(G,H)=\begin{cases}\dfrac{a\varphi(H)x}{\varrho(D)}+O\left(\sqrt{\dfrac{x}{\varrho(D)}}\right) & \text{if } H|D,\\ O\left(\sqrt{\dfrac{x}{\varrho(D)}}\right) & \text{if } H\nmid D.\end{cases}\tag{3.2}$$

Proof. Denoting the left member of (3.2) by Σ_D, it follows from (2.1) that

$$\Sigma_D=\sum_{\varrho(DE)\leqslant x} c(DE,H)=\sum_{\varrho(D)\varrho(E)\leqslant x}\ \sum_{\substack{D_1E_1=H\\ D_1|DE}} \varrho(D_1)\mu(E_1)$$

$$=\sum_{\varrho(E)\leqslant \frac{x}{\varrho(D)}}\ \sum_{\substack{D_1E_1=H\\ D=\Delta D',D_1=\Delta D_1'\\ (D',D_1')=I,D_1'|E}} \varrho(D_1)\mu(E_1)=\sum_{\substack{\varrho(E)\leqslant \frac{x}{\varrho(D)}\\ \Delta D'=D}}\ \sum_{\substack{\Delta D_1'E_1=H\\ (D',D_1')=I\\ E=D_1'E'}} \varrho(\Delta D_1')\mu(E_1),$$

so that from (2.4) and the hypothesis on H

$$\Sigma_D=\sum_{\substack{\Delta D'=D\\ \Delta|H}}\varrho(\Delta)\sum_{\substack{D_1'E_1=H/\Delta\\ (D_1',D')=I}}\varrho(D_1')\mu(E_1)A\left(\frac{x}{\varrho(DD_1')}\right)=\Sigma_D'+O\left(\sqrt{\frac{x}{\varrho(D)}}\right),\tag{3.3}$$

where

$$\Sigma_D'=\frac{ax}{\varrho(D)}\sum_{\substack{\Delta D'=D\\ \Delta\Delta'=H}}\varrho(\Delta)\sum_{\substack{D_1'E_1=\Delta'\\ (D_1',D')=I}}\mu(E_1).$$

Appealing to Lemma 3.1, one obtains

$$\Sigma_D'=\frac{ax}{\varrho(D)}\sum_{\substack{\Delta D'=D\\ \Delta\Delta'=H,\Delta'|D'}}\varrho(\Delta)\mu(\Delta')=\frac{ax}{\varrho(D)}\sum_{\substack{\Delta\Delta'=H\\ H|D}}\varrho(\Delta)\mu(\Delta');$$

hence by (2.3)

$$\Sigma_D'=\begin{cases}\dfrac{a\varphi(H)x}{\varrho(D)} & \text{if } H|D,\\ 0 & \text{if } H\nmid D.\end{cases}\tag{3.4}$$

The lemma results from (3.3) and (3.4).

Remark 3.2. In applying Lemma 3.2 to prove our next result, the quantity D will assume values in a finite set so that the O-terms of (3.2) can be replaced by $O(\sqrt{x})$. The complete form of the lemma will, however, be required for a later application (Theorem 5.1).

THEOREM 3.1. *For fixed groups H_1,H_2 of X,*

$$\sum_{\varrho(G)\leqslant x} c(G,H_1)c(G,H_2)=\begin{cases}a\varphi(H_1)x+O(\sqrt{x}) & \text{if } H_1=H_2,\\ O(\sqrt{x}) & \text{if } H_1\neq H_2.\end{cases}\tag{3.5}$$

Proof. Denoting the left side of (3.5) by S, it follows from (2.1) that

$$S=\sum_{\varrho(G)\leqslant x}c(G,H_1)\sum_{\substack{DE=H_2\\ D|G}}\varrho(D)\mu(E)=\sum_{DE=H_2}\varrho(D)\mu(E)\sum_{\substack{\varrho(G)\leqslant x\\ D|G}}c(G,H_1);$$

hence by Lemma 3.2 and Remark 3.2, $S=S'+O(\sqrt{x})$, where

$$S'=a\varphi(H_1)x\sum_{\substack{DE=H_2\\ H_1|D\\ (D=H_1D')}}\mu(E)=\begin{cases}a\varphi(H_1)x\sum\limits_{D'E=H_2/H_1}\mu(E) & \text{if } H_1|H_2,\\ 0 & \text{if } H_1\nmid H_2,\end{cases}$$

and the theorem results on application of (2.2).

The theorem yields the following result on the mean value of $c(G,H_1)c(G,H_2)$.

COROLLARY 3.1.1.

$$\lim_{x\to\infty}\frac{1}{A(x)}\sum_{\varrho(G)\leqslant x}c(G,H_1)c(G,H_2)=\begin{cases}\varphi(H_1) & \text{if } H_1=H_2,\\ 0 & \text{if } H_1\neq H_2.\end{cases}\tag{3.6}$$

Proof. Apply (2.4) in connection with (3.5).

The weaker form (3.6) of the orthogonality property contained in (3.5) is sufficient for the application considered in the next section.

4. Basic concepts. First we introduce the convention that a series $\sum a(G)$ summed over all $G\in X$ denotes the series,

$$\sum_{n=1}^{\infty}b(n),\qquad b(n)=\sum_{\varrho(G)=n}a(G),$$

and that its k-th partial sum is $\sum_{n\leqslant k}b(n)$. We shall also use the notation,

$$\overline{M}(f)=\limsup_{x\to\infty}\frac{1}{A(x)}\sum_{\varrho(G)\leqslant x}f(G).$$

DEFINITION 4.1. A function $f_k(G)$ will be called an *even function of order not exceeding* k if $f_k(G)$ is representable as a sum of even functions $(\mathrm{mod}\,H)$, $\varrho(H)\leqslant k$.

DEFINITION 4.2. A function $f(G)$ will be said to be *almost even* (B) if there exists a sequence of even functions $f_k(G)$ of order $\leqslant k$ $(k=1,2,\ldots)$ such that

$$\lim_{k\to\infty}\overline{M}\big(|f(G)-f_k(G)|\big)=0.\tag{4.1}$$

DEFINITION 4.3. Let $f(G)$ be an arbitrary function; if $M_H(f)\equiv M\big(f(G)c(G,H)\big)$ exists for each $H\in X$, then

$$a_H=\frac{M_H(f)}{\varphi(H)}\tag{4.2}$$

is defined to be the *Fourier coefficient* of $f(G)$ corresponding to H, and the expansion

$$f(G) \sim \sum_{H \in X} a_h c(G, H) \tag{4.3}$$

the *Fourier series* of $f(G)$.

These definitions are justified by the following theorem.

THEOREM 4.1. *Let H denote an arbitrary fixed group of X and let $f(G, H)$ be an even function* $(\bmod H)$. *Then $f(G, H)$ is almost even* (B) *and the expansion occuring in its representation* (1.2) *is its Fourier series.*

Proof. Since $c(G, H)$ is even $(\bmod H)$, the k-th partial sum $f_k(G, H)$ of the finite series in (1.2) is an even function of order $\leqslant k$. By [3, Theorem 4.2] one is assured that $f(G, H)$ has a representation (1.2). Moreover, $f(G, H) - f_k(G, H) = 0$ for $k \geqslant \varrho(H)$, and therefore $f(G, H)$ is almost even (B), by (4.1).

For a fixed E in X, multiply (1.2) by $c(G, H)$ and sum over all G of order $\leqslant x$, to obtain

$$\sum_{\varrho(G) \leqslant x} f(G, H) c(G, E) = \sum_{D|H} a(D, H) \sum_{\varrho(G) \leqslant x} c(G, D) c(G, E) .$$

By Corollary 3.1.1, it follows on dividing by $A(x)$ and letting $x \to \infty$, that

$$M_E\big(f(G, H)\big) = \begin{cases} \varphi(H) a(E, H) & \text{if} \quad E | H , \\ 0 & \text{if} \quad E \nmid H . \end{cases}$$

This completes the proof.

5. Fourier expansions. We prove first

LEMMA 5.1. *Let $g(n)$ denote a function of the integral variable n, and suppose that $\sum\limits_1^\infty g(n)/n$ converges. Then*

$$T(x) \equiv \sum_{n \leqslant x} \frac{g(n)}{\sqrt{n}} = o(\sqrt{x}) .$$

Proof. Let β denote the sum of the series, so that $S(x) \equiv \sum\limits_{n \leqslant x} g(n)/n = \beta + o(1)$. By partial summation,

$$T(x) = \sum_{n \leqslant x} S(n)(\sqrt{n} - \sqrt{n+1}) + S(x)\sqrt{[x]+1} \equiv \Sigma_1 + \Sigma_2 , \tag{5.1}$$

let us say. Moreover, (cf. [7], p. 77),

$$\Sigma_1 = \sum_{n \leqslant x} S(n)\sqrt{n}\left(-\frac{1}{2n} + O\left(\frac{1}{n^2}\right)\right) = -\frac{1}{2}\sum_{n \leqslant x} \frac{S(n)}{\sqrt{n}} + O(1)$$
$$= -\frac{\beta}{2}\sum_{n \leqslant x} \frac{1}{\sqrt{n}} + o\left(\sum_{n \leqslant x} \frac{1}{\sqrt{n}}\right) + O(1);$$

hence by a well-known elementary estimate,

$$\Sigma_1 = -\beta\sqrt{x} + o(\sqrt{x}) . \tag{5.2}$$

Furthermore,

$$\Sigma_2 = S(x)\sqrt{x}\big(1 + o(1)\big) = \beta\sqrt{x} + o(\sqrt{x}) . \tag{5.3}$$

The lemma is a consequence of (5.1), (5.2), and (5.3).

We obtain now the Fourier expansions for a broad class of almost even functions (B).

THEOREM 5.1. *Let $f'(G)$ denote the Möbius image of the function $f(G)$,*

$$f(G) = \sum_{D|G} f'(D) , \tag{5.4}$$

and suppose that

$$\sum_{G \in X} \frac{|f'(G)|}{\varrho(G)} = \sum_{n=1}^{\infty} \frac{g(n)}{n} \qquad \Big(g(n) = \sum_{\varrho(G)=n} |f'(G)|\Big) \tag{5.5}$$

converges. Then $f(G)$ is almost even (B) with Fourier expansion,

$$f(G) \sim \sum_{H \in X} a_H c(G, H) , \qquad a_H = \sum_{\substack{G \in X \\ H|G}} \frac{f'(G)}{\varrho(G)} . \tag{5.6}$$

Remark 5.1. That $f(G)$ is representable in the form (5.4) is a consequence of ([2], Theorem 2.6).

Proof. The function $\eta(G, H)$ defined by (1.1) is clearly an even function of $G \,(\bmod H)$. Moreover, for fixed $H \in X$

$$M\big(\eta(G, H)\big) = \lim_{x \to \infty} \frac{1}{A(x)} \sum_{\varrho(L) \leqslant x} \eta(L, H) = \lim_{x \to \infty} \frac{1}{A(x)} \sum_{\substack{\varrho(L) \leqslant x \\ H|L}} \varrho(H)$$

so that by (2.4)

$$\overline{M}\big(\eta(G, H)\big) = M\big(\eta(G, H)\big) \equiv \lim_{x \to \infty} \frac{\varrho(H)}{A(x)} A\left(\frac{x}{\varrho(H)}\right) = 1 . \tag{5.7}$$

We may write, on the basis of (5.4),

$$f(G) = \sum_{D \in X} \frac{f'(D)\eta(G, D)}{\varrho(D)} , \qquad f_k(G) = \sum_{\varrho(D) \leqslant k} \frac{f'(D)\eta(G, D)}{\varrho(D)} ; \tag{5.8}$$

evidently, $f_k(G)$ is even of order $\leqslant k$. It follows that

$$\overline{M}_k \equiv \overline{M}\big(|f(G) - f_k(G)|\big) = \overline{M}\left(\left|\sum_{\varrho(D) > k} \frac{f'(D)\eta(G, D)}{\varrho(D)}\right|\right) .$$

Moreover, by (5.7) and the convergence of (5.5),

$$\bar{M}_k \leqslant \bar{M}\Big(\sum_{\varrho(D)>k} \frac{|f'(D)|\,\eta(G,D)}{\varrho(D)}\Big) \leqslant \sum_{\varrho(D)>k} \frac{|f'(D)|\,\bar{M}(\eta(G,D))}{\varrho(D)}$$

$$= \sum_{\varrho(D)>k} \frac{|f'(D)|}{\varrho(D)} \to 0 \quad \text{as} \quad k\to\infty .$$

This proves that $f(G)$ is almost even (B).

We next consider $M_H(f)$ for a fixed H in X. By (5.4)

$$M(x) \equiv \sum_{\varrho(D)\leqslant x} f(G)c(G,H) = \sum_{\varrho(G)\leqslant x} \sum_{D|H} f'(D)c(G,H) ,$$

so that

$$M(x) = \sum_{\varrho(D)\leqslant x} f'(D) \sum_{\substack{\varrho(G)\leqslant x\\ D|G}} c(G,H) .$$

Application of Lemma 3.2 yields

$$M(x) = a\varphi(H)x \sum_{\substack{\varrho(D)\leqslant x\\ H|D}} \frac{f'(D)}{\varrho(D)} + O\Big(\sqrt{x} \sum_{\varrho(D)\leqslant x} \frac{|f'(D)|}{\varrho^{1/2}(D)}\Big) .$$

By Lemma 5.1, the O-term becomes, using the notation of (5.5)

$$O\Big(\sqrt{x}\sum_{n\leqslant x} \frac{g(n)}{\sqrt{n}}\Big) = o(x) ,$$

and therefore, by the convergence of the series in (5.5),

$$M(x) = a\varphi(H)x \sum_{\substack{D\in X\\ H|G}} \frac{f'(D)}{\varrho(D)} + o(x) .$$

Finally, using (2.4),

$$M_H(f) = \lim_{x\to\infty} \frac{M(x)}{A(x)} = \varphi(H) \sum_{\substack{G\in X\\ H|G}} \frac{f'(G)}{\varrho(G)} . \tag{5.9}$$

This proves the theorem.

6. Convergence criteria. In this section we determine certain sufficient conditions for the convergence of the Fourier series in (5.6). Let $\tau(G)$ denote the number of direct factors of G in X.

THEOREM 6.1. *Let $f(G)$ be defined by* (5.4). *In case either* (a) *the series in* (5.5) *converges and $f'(G)$ is completely multiplicative, or in case* (b) *the series*

$$\sum_{G\in X} \frac{\tau(G)\,|f'(G)|}{\varrho(G)} \tag{6.1}$$

converges, then the Fourier series (5.6) *of $f(G)$ converges absolutely and represents $f(G)$ for every $G\in X$,*

$$f(G) = \sum_{G\in X} a_H c(G,H) , \qquad a_H = \sum_{\substack{G\in X\\ H|G}} \frac{f'(G)}{\varrho(G)} . \tag{6.2}$$

Proof. (a) Disregarding convergence questions, in this case one obtains from (5.6) and (2.1),

$$\Sigma \equiv \sum_{H\in X} a_H c(G,H) = \sum_{H\in X}\Big(\sum_{\substack{H|L\\ L=HB}} \frac{f'(L)}{\varrho(L)}\Big) \sum_{\substack{DE=H\\ D|G}} \varrho(D)\mu(E)$$

$$= \sum_{D|G} \varrho(D) \sum_{\substack{H\in X\\ DE=H}} \mu(E) \sum_{B\in X} \frac{f'(HB)}{\varrho(HB)} = \sum_{D|G} \varrho(D) \sum_{E\in X} \mu(E) \sum_{B\in X} \frac{f'(DEB)}{\varrho(DEB)} ,$$

so that

$$\Sigma = \sum_{D|G} f'(D) \sum_{E\in X} \frac{\mu(E)f'(E)}{\varrho(E)} \sum_{B\in X} \frac{f'(B)}{\varrho(B)} . \tag{6.3}$$

Applying Dirichlet multiplication in connection with (2.2), the hypothesis of complete multiplicativity yields

$$\sum_{G\in X} a_H c(G,H) = \sum_{D|G} f'(D) \sum_{\substack{H\in X\\ BE=H}} \frac{\mu(E)f'(BE)}{\varrho(BE)}$$

$$= \sum_{D|G} f'(D) \sum_{H\in X} \frac{f'(H)}{\varrho(H)} \sum_{BE=H} \mu(E) = \sum_{D|G} f'(D) = f(G) .$$

The formal steps of the proof are justified by the convergence of (5.5), and hence (a) is proved.

(b) In this case we again proceed formally. Appealing to (1.1) one obtains by (5.6),

$$\sum_{H\in X} a_H c(G,H) = \sum_{H\in X} c(G,H) \sum_{\substack{L\in X\\ H|L}} \frac{f'(L)}{\varrho(L)}$$

$$= \sum_{L\in X} \frac{f'(L)}{\varrho(L)} \sum_{H|L} c(G,H) = \sum_{L|G} f'(L) = f(G) .$$

To justify the steps, we prove the double series absolutely convergent. By (2.1),

$$|c(G,H)| \leqslant \sum_{D|G} \varrho(D) = \sigma(G) .$$

Therefore, for fixed G in X,

$$\left|\sum_{H\in X} c(G,H)\sum_{H|L}\frac{f'(L)}{\varrho(L)}\right| \leqslant \sigma(G)\sum_{H\in X}\left|\sum_{H|L}\frac{f'(L)}{\varrho(L)}\right|$$
$$\leqslant \sigma(G)\sum_{H\in X}\sum_{H|L}\frac{|f'(L)|}{\varrho(L)} = \sigma(G)\sum_{L\in X}\frac{\tau(L)\,|f'(L)|}{\varrho(L)}.$$

By the hypothesis of case (6), the double series is therefore absolutely convergent and the proof is complete.

Defining for real s,

$$\frac{\sigma_s(G)}{\varrho^s(G)} = \sum_{D|G}\frac{1}{\varrho^s(D)},$$

we have the following corollary of Theorem 5.1 (a).

COROLLARY 5.1.1. *If $s>0$, then $\sigma_s(G)/\varrho^s(G)$ is almost even (B), and is represented for all $G\in X$ by its (absolutely convergent) Fourier expansion,*

$$\frac{\sigma_s(G)}{\varrho^s(G)} = Z(s+1)\sum_{H\in X}\frac{c(G,H)}{\varrho^{s+1}(H)}. \tag{6.4}$$

7. Almost primitive functions over X. Let $\gamma(G)$ denote the separable (square-free) factor of G in X of maximal order. As in [3], a function $f(G)$ is called *separable* if $f(G)=f(\gamma(G))$ for all G in X. We shall say that $f(G)$ is *simple* if $f(G)=0$ for all inseparable $G\in X$. We recall the following result.

LEMMA 7.1. ([3], Lemma 3.3). *The function $f(G)$ is separable if and only if its Möbius image is simple.*

We also recall [3] that a function $f(G,H)$ is defined to be primitive $(\operatorname{mod} H)$ if $f(G,H)=f(\gamma(G,H),H)$ for all $G\in X$. This class of functions was characterized as follows.

Remark 7.1. (cf. [3], Theorem 5.2). *A function $f(G,H)$ is primitive $(\operatorname{mod} H)$ if and only if it is even $(\operatorname{mod} H)$ with a representation* (1.2) *such that $a(D,H)=0$ provided D is inseparable.*

We can extend the class of primitive functions $(\operatorname{mod} H)$ just as the even functions $(\operatorname{mod} H)$ were extended in § 4. Let us call a function $f_k(G)$ *primitive of order* $\leqslant k$ if $f_k(G)$ is representable as a sum of primitive functions $(\operatorname{mod} H)$, $\varrho(H)\leqslant k$. A function $f(G)$ will be termed *almost primitive* (B) if there exists a sequence $\{f_k(G)\}$ of primitive functions of order $\leqslant k$ such that (4.1) holds. Evidently, the class of almost primitive functions (B) is a subset of the class of almost even functions (B).

We have the following analogue of Theorem 4.1.

THEOREM 7.1. *If $f(G,H)$ is primitive $(\operatorname{mod} H)$ then $f(G,H)$ is almost primitive (B) with Fourier series* (1.2).

This is a consequence of Remark 7.1. Analogous to Theorem 5.1, one obtains

THEOREM 7.2. *If $f(G)$ is separable with Möbius image $f'(G)$ and if* (5.5) *converges, then $f(G)$ is almost primitive (B) with Fourier expansion determined by* (5.6).

The first part of the theorem is proved just like the corresponding part of Theorem 5.1, if it is observed by Remark 7.1 that $f'(G)$ is simple and hence that the functions $f_k(G)$ in (5.8) are primitive of order $\leqslant k$.

We now prove a convergence criterion which complements the one proved in Theorem 6.1 (a). First we prove a preliminary lemma. Let $\sum'$ indicate summations restricted to separable groups, and define

$$k(G) = {\sum_{D|G}}'\frac{f'(D)}{\varrho(D)},\qquad C_1 = {\sum_{G\in X}}'\frac{f'(G)}{\varrho(G)}, \tag{7.1}$$

if the series converges.

LEMMA 7.2. *Let $f(G)$ be a separable function with Möbius image $f'(G)$, such that $f'(G)$ is multiplicative, $f'(P)\neq-\varrho(P)$ for all indecomposable $P\in X$, and such that* (5.5) *converges. If $k(G)$ and C_1 are defined by* (7.1), *then $C_1\neq 0$ and $k(G)\neq 0$ for all separable G in X; moreover, $k(G)$ is multiplicative, and if H is a separable group of X,*

$$\sum_{D|H}\frac{\mu(D)f'(D)}{\varrho(D)k(D)} = \frac{1}{k(H)},\qquad {\sum_{G\in X}}'\frac{\mu(G)f'(G)}{\varrho(G)k(G)} = \frac{1}{C_1}, \tag{7.2}$$

and

$$\sum_{\substack{G\in X\\(G,H)=I}}\frac{f'(G)}{\varrho(G)} = \frac{C_1}{k(h)},\qquad {\sum_{\substack{G\in X\\(G,H)=I}}}'\frac{\mu(G)f'(G)}{\varrho(G)k(G)} = \frac{k(H)}{C_1}, \tag{7.3}$$

the series in (7.2) *being absolutely convergent.*

Proof. The multiplicativity of $k(G)$ is a consequence of the multiplicativity of $f'(G)$, (cf. [3], Lemma 2.1). By Lemma 7.1, $f'(G)$ is simple, and hence

$$k(G) = \prod_{P|G}\left(1+\frac{f'(P)}{\varrho(P)}\right),\qquad C_1 = \prod_{P}\left(1+\frac{f'(P)}{\varrho(P)}\right), \tag{7.4}$$

the first product being extended over the indecomposable factors of G and the second over all indecomposable groups $P\in X$ (cf. [6], § 17.4). Since $f'(P)\neq-\varrho(P)$ for all P, $k(G)\neq 0$ for all separable G in X. Also, since $\sum f'(P)/\varrho(P)$ converges absolutely, it must follow that $C_1\neq 0$.

The absolute convergence of the product on the right of (7.4) justifies the following steps:

$$(7.5)\qquad \sum_{G\in X}{}' \frac{\mu(G)f'(G)}{\varrho(G)k(G)} = \prod_P \left(1-\frac{f'(P)}{\varrho(P)k(P)}\right) = \prod_P \left(1-\frac{f'(P)}{\varrho(P)+f'(P)}\right)$$
$$= \prod_P \left(\frac{\varrho(P)}{\varrho(P)+f'(P)}\right) = \frac{1}{\prod_P \left(1+\frac{f'(P)}{\varrho(P)}\right)} = \frac{1}{C_1};$$

thus the series in (7.2) converges absolutely to $1/C_1$. The proof of the first formula in (7.2) is similar except that no convergence questions arise.

We also have by (7.4),

$$\sum_{\substack{G\in X\\(G,H)=1}} \frac{f'(G)}{\varrho(G)} = \prod_P \left(1+\frac{f'(P)}{\varrho(P)}\right) \Big/ \prod_{P|H} \left(1+\frac{f'(P)}{\varrho(P)}\right) = \frac{C_1}{k(H)}.$$

The proof of the second formula in (7.3) proceeds similarly, by virtue of (7.5) and (7.2).

THEOREM 7.3. *Let $f(G)$ and $f'(G)$ be defined as in Lemma 7.2. Then $f(G)$ is almost primitive (B) and is represented for all $G\in X$ by its (absolutely convergent) Fourier series* (6.2).

Proof. The primitivity of $f(G)$ is a consequence of Theorem 7.2. Just as in the proof of Theorem 6.1 (a), we obtain (in place of (6.3)), now using Theorem 7.2 in connection with the multiplicativity and simplicity of $f'(G)$,

$$(7.6)\qquad \sum_{H\in X} a_H c(G,H) = \sum_{D|G}{}' f'(D) \sum_{\substack{E\in X\\(E,D)=I}}{}' \frac{\mu(E)f'(E)}{\varrho(E)} \sum_{\substack{B\in X\\(B,DE)=I}} \frac{f'(B)}{\varrho(B)},$$

the omitted steps being justified by the convergence of (5.5). By Lemma 7.2, it now results from (7.6) that

$$\sum_{H\in X} a_H c(G,H) = C_1 \sum_{D|G}{}' \frac{f'(D)}{k(D)} \sum_{\substack{E\in X\\(E,D)=I}} \frac{\mu(E)f'(E)}{\varrho(E)k(E)} = \sum_{D|G}{}' f'(D),$$

which is $f(G)$. The theorem is proved.

We apply the above results to the generalized totient,

$$\frac{\varphi_s(G)}{\varrho_s(G)} = \sum_{D|G} \frac{\mu(D)}{\varrho^s(D)}.$$

In particular, it is easily verified on the basis of Theorem 7.3 that

COROLLARY 7.3.1. *If $s>0$, then $\varphi_s(G)/\varrho_s(G)$ is almost primitive (B) and is represented by the absolutely convergent Fourier expansion,*

$$(7.7)\qquad \frac{\varphi_\varrho(G)}{\varrho^s(G)} = \frac{1}{Z(s+1)} \sum_{H\in X} \left(\frac{\mu(H)}{\varphi_{s+1}(H)}\right) c(G,H).$$

Finally, we note that the expansions (6.4) and (7.7) can also be deduced from Theorem 6.1 (b), if one appeals to the order property, $\tau(G) = O\left(\varrho^\varepsilon(G)\right)$ for all $\varepsilon>0$.

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