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On the zeros of Hecke's L-functions III

by

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Introduction

1. In the two previous papers (see [1], [2]) some results were proved about the location of the zeros near the line $\sigma = 1$ of the Hecke-Landau function $\zeta(s, \chi)$ (corresponding to a field K of degree $n \geqslant 1$) with characters χ modulo $\mathfrak{f}$ for $D = |\Delta| \cdot N\mathfrak{f} \to \infty$ (where Δ denotes the discriminant of the field) and $n \ll 1$. In the present paper we will prove the following

Theorem. *Let $N(\alpha, T)$ denote the number of the zeros of the function $Z(s) = \prod\limits_{\chi} \zeta(s, \chi)$ in the rectangle $(1-\alpha \leqslant \sigma \leqslant 1,\ |t| \leqslant T)$. Then there is a constant $C > 0$ (which depends only on n) such that for any $\lambda \in [0, 2\log D]$, $D \geqslant D_0 > 1$*

$$N(\lambda/\log D,\ e^{\lambda}/\log D) < \exp(C\lambda).$$

The method used in this paper is in the outline the same as that employed by Rodosskiĭ for Dirichlet's L-functions (see [9], pp. 332-341 or [8] X § 2). There is however an essential digression from his final arguments as will be shown at the end of the paper.

The notation used here is the same as in [1], [2].

We shall need the known estimate $h \ll D$ (see [4] § 3 and [5] p. 66) for the number of classes $\mathfrak{H}$, the proof of which must be postponed to the next paper of this series, where it fits in more conveniently.

Estimates for the number of ideals with the norm $\leqslant x$

2. By the fundamental properties of group characters (see, for example, [12] § 13)

$$\sum_{\chi} \chi(\mathfrak{H}) = \begin{cases} h & \text{if } \mathfrak{H} \text{ is the principal class,} \\ 0 & \text{otherwise,} \end{cases}$$

whence

$$(1)\qquad \sum_{\chi} \overline{\chi}(\mathfrak{a})\chi(\mathfrak{b}) = \begin{cases} h & \text{if } \mathfrak{a} \sim \mathfrak{b} \text{ (i.e. } \mathfrak{a} \text{ and } \mathfrak{b} \text{ are in the same class } \mathfrak{H}), \\ 0 & \text{otherwise.} \end{cases}$$

Let us introduce the function

$$(2)\qquad \zeta(s, \mathfrak{H}) = \sum_{\mathfrak{a}\in\mathfrak{H}} N\mathfrak{a}^{-s} (\sigma > 1).$$

For a fixed $\mathfrak{c}\in\mathfrak{H}$ we have, by (1), (2), [1] (8)

$$(3)\qquad \zeta(s, \mathfrak{H}) = h^{-1}\sum_{\chi}\overline{\chi}(\mathfrak{c})\zeta(s, \chi).$$

Hence $\zeta(s, \mathfrak{H})$ is regular in the whole plane, except for a simple pole at $s=1$ with the residue

$$\operatorname*{Res}_{s=1}\zeta(s, \mathfrak{H}) = h^{-1}\operatorname*{Res}_{s=1}\zeta(s, \chi_0)$$

which does not depend on $\mathfrak{H}$.

(3) and [1] (32) implies that for $|s-1| > \frac{1}{8}$ we have uniformly in σ

$$(4)\qquad \zeta(s, \mathfrak{H}) \ll \delta^{-n}D^{(1-\sigma)/2}(1+|t|)^{(1+\delta-\sigma)n/2}$$
$$(-\delta\leqslant\sigma\leqslant 1+\delta,\ 0<\delta\leqslant 1/\log D).$$

The results of this paragraph are based on the following lemma.

Let $\alpha(x)$ for $x > x_0$ be a positive non-decreasing function of x such that $\alpha(2x)\ll\alpha(x)$ and for all $m > x_0$

$$a_m \ll \alpha(m).$$

Further suppose the series

$$(5)\qquad f(s) = \sum_{m=1}^{\infty} a_m m^{-s}$$

is convergent for $\sigma > 1$, and for some fixed l

$$(6)\qquad (\eta-1)^l\sum_m |a_m| m^{-\eta} \ll 1 \quad \text{if} \quad 1<\eta<2.$$

Then we have uniformly in $1<\eta<2$, $T>0$, $x>2$

$$(7)\qquad \sum_{m\leqslant x} a_m - \frac{1}{2\pi i}\int_{\eta-iT}^{\eta+iT}\frac{x^s}{s}f(s)\,ds \ll \frac{1}{T}\{x^{\eta}(\eta-1)^{-l} + x\alpha(x)\log x\} + \alpha(x).$$

For $l=1$ and $\alpha(x) = \log x$ this lemma is proved by Landau ([3], Hilfssatz 3) and the proof still holds under the generalised conditions.

Lemma 1. *Let $\nu(x, \mathfrak{H})$ denote the number of ideals of the class $\mathfrak{H}$ having the norm $\leqslant x$ $(x\geqslant 1)$, and let $\nu(x) = \sum_{\mathfrak{H}}\nu(x, \mathfrak{H})$. Then*

$$(8)\qquad \nu(x, \mathfrak{H}) = \mu x + O(D^{2/3}x^{1-1/k})$$

and

$$(9)\qquad \nu(x) = h\mu x + O(D^{2/3}x^{1-1/k}),$$

where $\mu = h^{-1}\operatorname*{Res}_{s=1}\zeta(s, \chi_0)$, $k = \frac{1}{2}(n+3)$.

Proof. Since there are only $\ll 1$ ideals of the field having the norm $\leqslant 3$, for $x\leqslant 3$ the lemma is evidently true. In what follows we suppose $x>3$.

For any of the functions $f(s) = \zeta(s, \chi_0), \zeta(s, \mathfrak{H})$ the coefficients a_m in (5) are $\leqslant d_n(m)$ (in the notation of [2] § 13). By [1] (3) and induction (with respect to n) $d_n(m)\leqslant c(\varepsilon)m^{\varepsilon}$ for any $\varepsilon>0$. By (2), [1] (8), [1] (14), we have, for $\sigma>1$, $\zeta(\sigma, \mathfrak{H})\leqslant\zeta(\sigma, \chi_0)\ll(\sigma-1)^{-n}$. Hence (6) holds for $l=n$.

Taking $\eta = 1+1/\log x$, $T = x^{2/(n+2)}$ we have, by (7),

$$(10)\qquad \nu(x, \mathfrak{H}) = \frac{1}{2\pi i}\int_{\eta-iT}^{\eta+iT}\frac{x^s}{s}\zeta(s, \mathfrak{H})\,ds + O(x^{1-2/(n+2)+\varepsilon}\log x)$$

(the constant in O depending on ε). For $\varepsilon = 1/(n+2)(n+3)$ the remaining term in (10) is $\ll x^{1-2/(n+3)} = x^{1-1/k}$.

Passing over a pole of the integrand at $s=1$ with the residue μx we replace the path of integration in (10) by $L_1+L_2+L_3$, where L_1, L_2, L_3 denote straight lines joining the points $(\eta-iT,\ 1/\log Dx - iT)$, $(1/\log Dx - iT,\ 1/\log Dx + iT)$, $(1/\log Dx+iT,\ \eta+iT)$, respectively. By (4) (with $\delta = 1/\log Dx$)

$$\int_{L_2}\frac{x^s}{s}\zeta(s, \mathfrak{H})\,ds \ll D^{1/2}T^{n/2}\log^n Dx \ll D^{2/3}x^{1-1/k},$$

$$\int_{L_1, L_3} \ll \max(D^{1/2}T^{n/2}\log^n Dx,\ xT^{-1}\log^n Dx) + D^{1/2}x^{1-1/k} \ll D^{2/3}x^{1-1/k}.$$

From this and (10) we deduce (8). Using [1] (32) instead of (4) by the same argument we get (9).

Lemma 2. *If $\nu(x; \mathfrak{H}, \mathfrak{b})$ denotes the number of such ideals $\mathfrak{a}$ of the class $\mathfrak{H}$ which are divisible by $\mathfrak{b}$ and have the norm $\leqslant x$ $(x\geqslant 1$, $(\mathfrak{b}, \mathfrak{f}) = \mathfrak{o}$, $N\mathfrak{b}\leqslant x)$, then*

$$(11)\qquad \nu(x; \mathfrak{H}, \mathfrak{b}) = \mu x/N\mathfrak{b} + O\big(D^{2/3}(x/N\mathfrak{b})^{1-1/k}\big),$$

where μ, k are defined by (9).

Proof. Write $\mathfrak{a} = \mathfrak{b}\mathfrak{c}$. If $\mathfrak{a}$ runs through the ideals of the lemma and $\mathfrak{b}\in\mathfrak{H}'$ (say), then $\mathfrak{c}$ runs through the ideals of the class $\mathfrak{H}/\mathfrak{H}'$ having the norms $\leqslant x/N\mathfrak{b}$, and conversely. Hence (11) follows from (8).

An application of Selberg's sieve method

3. Lemma 3. *Let*

$$(12)\qquad \mathfrak{a}_m \quad (m = 1, \ldots, X)$$

be a set of ideals in K such that for any $\mathfrak{b}$

$$(13)\qquad \sum_{\substack{\mathfrak{a}_m\\ \mathfrak{b}|\mathfrak{a}_m}} 1 = X/f(\mathfrak{b}) + R_{\mathfrak{b}},$$

where $f(\mathfrak{b})$ *is a multiplicative function* $\neq 0$, *i.e.*

$$f(\mathfrak{a}\mathfrak{b}) = f(\mathfrak{a})f(\mathfrak{b}) \quad \text{if} \quad (\mathfrak{a}, \mathfrak{b}) = \mathfrak{o}. \tag{14}$$

Further let Q *be a fixed set (empty or not) of prime ideals, generally denoted by* $\mathfrak{q}$, *and let for* $z > 1$ N_z *denote the number of ideals* (12) *not divisible by any prime ideal* $\mathfrak{p} \neq \mathfrak{q}$ *having the norm* $N\mathfrak{p} \leqslant z$. *Writing*

$$F(\mathfrak{b}) = \sum_{\mathfrak{d}|\mathfrak{b}} \mu(\mathfrak{d}) f\left(\frac{\mathfrak{b}}{\mathfrak{d}}\right), \tag{15}$$

$$S_z = \sum_{\substack{\mathfrak{d} \\ N\mathfrak{d} \leqslant z \\ \mathfrak{q} \nmid \mathfrak{d}}} \frac{\mu^2(\mathfrak{d})}{F(\mathfrak{d})}, \qquad S_z(\mathfrak{c}) = \sum_{\substack{\mathfrak{d} \\ (\mathfrak{d}, \mathfrak{c}) = \mathfrak{o} \\ N\mathfrak{d} \leqslant z/N\mathfrak{c} \\ \mathfrak{q} \nmid \mathfrak{d}\mathfrak{c}}} \frac{\mu^2(\mathfrak{d})}{F(\mathfrak{d})}, \tag{16}$$

$$\lambda_{\mathfrak{c}} = \begin{cases} \mu(\mathfrak{c}) \prod_{\mathfrak{p}|\mathfrak{c}} \left(1 - 1/f(\mathfrak{p})\right)^{-1} S_z(\mathfrak{c})/S_z & \text{if} \quad \mathfrak{q} \nmid \mathfrak{c},\ N\mathfrak{c} \leqslant z, \\ 0 & \text{otherwise}, \end{cases} \tag{17}$$

we have

$$N_z \leqslant X/S_z + \sum_{\substack{\mathfrak{c}_1, \mathfrak{c}_2 \\ N\mathfrak{c}_1 \leqslant z, N\mathfrak{c}_2 \leqslant z \\ \mathfrak{q} \nmid \mathfrak{c}_1\mathfrak{c}_2}} |\lambda_{\mathfrak{c}_1} \lambda_{\mathfrak{c}_2} R_{\mathfrak{c}_1\mathfrak{c}_2/(\mathfrak{c}_1, \mathfrak{c}_2)}|. \tag{18}$$

It is understood that all the ideals used in this lemma and further on are prime to $\mathfrak{f}$.

Proof. (The method used in this proof is borrowed from Selberg's paper [10] where an analogous result is obtained for a set of natural numbers. But there are no exceptional primes q in his paper corresponding to the ideals $\mathfrak{q}$ of the present lemma.)

We begin the proof by taking $\lambda_{\mathfrak{o}} = 1$ and $\lambda_{\mathfrak{c}} = 0$ whenever $\mathfrak{q}|\mathfrak{c}$ or $N\mathfrak{c} > z$ (or both). For all other $\mathfrak{c}$ let for the present $\lambda_{\mathfrak{c}}$ be any real numbers. Then

$$N_z \leqslant \sum_{\mathfrak{a}_m} \Big(\sum_{\mathfrak{c}|\mathfrak{a}_m} \lambda_{\mathfrak{c}} \Big)^2, \tag{19}$$

since the inner sum is 1 if $\mathfrak{a}_m$ has no other prime divisors than $\mathfrak{q}$ or $\mathfrak{p} \neq \mathfrak{q}$ with $N\mathfrak{p} > z$ (or both), and $\geqslant 0$ otherwise. By $\mathfrak{d}_1, \ldots, \mathfrak{d}_v$ denoting all the divisors of $\mathfrak{a}_m$ we have

$$\Big(\sum_{\substack{\mathfrak{c} \\ \mathfrak{c}|\mathfrak{a}_m}} \lambda_{\mathfrak{c}} \Big)^2 = (\lambda_{\mathfrak{d}_1} + \ldots + \lambda_{\mathfrak{d}_v})^2 = \lambda_{\mathfrak{d}_i} \lambda_{\mathfrak{d}_j} \sum_{\frac{\mathfrak{d}_i\mathfrak{d}_j}{(\mathfrak{d}_i, \mathfrak{d}_j)} | \mathfrak{a}_m} 1,$$

since '$\mathfrak{d}_i|\mathfrak{a}_m$ and $\mathfrak{d}_j|\mathfrak{a}_m$' is equivalent to

$$\frac{\mathfrak{d}_i\mathfrak{d}_j}{(\mathfrak{d}_i, \mathfrak{d}_j)} \Big| \mathfrak{a}_m.$$

Hence, by (19)

$$N_z \leqslant \sum_{\substack{\mathfrak{c}_1, \mathfrak{c}_2 \\ N\mathfrak{c}_1 \leqslant z, N\mathfrak{c}_2 \leqslant z \\ \mathfrak{q} \nmid \mathfrak{c}_1\mathfrak{c}_2}} \lambda_{\mathfrak{c}_1} \lambda_{\mathfrak{c}_2} \sum_{\frac{\mathfrak{c}_1\mathfrak{c}_2}{(\mathfrak{c}_1, \mathfrak{c}_2)} | \mathfrak{a}_m} 1. \tag{20}$$

By (14) $f(\mathfrak{o}) = 1$. It may be proved by induction with respect to the number of different prime factors of $\mathfrak{d} = (\mathfrak{a}, \mathfrak{b})$ that

$$f(\mathfrak{a}\mathfrak{b}/\mathfrak{d}) = f(\mathfrak{a})f(\mathfrak{b})/f(\mathfrak{d}). \tag{21}$$

(This is evident for $\mathfrak{d} = \mathfrak{o}$. Let $\mathfrak{a} = \mathfrak{a}'\mathfrak{p}^m$, $\mathfrak{b} = \mathfrak{b}'\mathfrak{p}^m$ and let $\mathfrak{a}'$ (for example) be prime to $\mathfrak{p}$; then $(\mathfrak{a}, \mathfrak{b}) = (\mathfrak{a}'\mathfrak{p}^m, \mathfrak{b}'\mathfrak{p}^m) = \mathfrak{p}^m(\mathfrak{a}', \mathfrak{b}') = \mathfrak{p}^m(\mathfrak{a}', \mathfrak{b})$. Supposing (21) is true for $\mathfrak{a}', \mathfrak{b}$ we have, by (14),

$$f\left(\frac{\mathfrak{a}\mathfrak{b}}{(\mathfrak{a}, \mathfrak{b})}\right) = f\left(\frac{\mathfrak{p}^{2m}\mathfrak{a}'\mathfrak{b}'}{\mathfrak{p}^m(\mathfrak{a}', \mathfrak{b}')}\right) = f\left(\frac{\mathfrak{a}'(\mathfrak{p}^m\mathfrak{b}')}{(\mathfrak{a}', \mathfrak{b}')}\right) = f\left(\frac{\mathfrak{a}'\mathfrak{b}}{(\mathfrak{a}', \mathfrak{b})}\right) = \frac{f(\mathfrak{a}')f(\mathfrak{b})}{f\left((\mathfrak{a}', \mathfrak{b})\right)}$$
$$= \frac{f(\mathfrak{p}^m)f(\mathfrak{a}')f(\mathfrak{b})}{f(\mathfrak{p}^m)f\left((\mathfrak{a}', \mathfrak{b})\right)} = \frac{f(\mathfrak{a})f(\mathfrak{b})}{f\left(\mathfrak{p}^m(\mathfrak{a}', \mathfrak{b})\right)} = \frac{f(\mathfrak{a})f(\mathfrak{b})}{f\left((\mathfrak{a}, \mathfrak{b})\right)},$$

whence (21).)

By (21), (13),

$$\sum_{\substack{\mathfrak{a}_m \\ \frac{\mathfrak{c}_1\mathfrak{c}_2}{(\mathfrak{c}_1, \mathfrak{c}_2)} | \mathfrak{a}_m}} 1 = X/f\left(\frac{\mathfrak{c}_1\mathfrak{c}_2}{(\mathfrak{c}_1, \mathfrak{c}_2)}\right) + R_{\mathfrak{c}_1\mathfrak{c}_2/(\mathfrak{c}_1, \mathfrak{c}_2)} = \frac{Xf\left((\mathfrak{c}_1, \mathfrak{c}_2)\right)}{f(\mathfrak{c}_1) \cdot f(\mathfrak{c}_2)} + R_{\mathfrak{c}_1\mathfrak{c}_2/(\mathfrak{c}_1, \mathfrak{c}_2)}.$$

Hence, by (20),

$$N_z \leqslant XU + \sum_{\substack{\mathfrak{c}_1, \mathfrak{c}_2 \\ N\mathfrak{c}_1, N\mathfrak{c}_2 \leqslant z \\ \mathfrak{q} \nmid \mathfrak{c}_1\mathfrak{c}_2}} |\lambda_{\mathfrak{c}_1} \lambda_{\mathfrak{c}_2} R_{\mathfrak{c}_1\mathfrak{c}_2/(\mathfrak{c}_1, \mathfrak{c}_2)}|, \tag{22}$$

if we write

$$U = \sum_{\substack{\mathfrak{c}_1, \mathfrak{c}_2 \\ N\mathfrak{c}_1, N\mathfrak{c}_2 \leqslant z \\ \mathfrak{q} \nmid \mathfrak{c}_1\mathfrak{c}_2}} \frac{\lambda_{\mathfrak{c}_1}}{f(\mathfrak{c}_1)} \frac{\lambda_{\mathfrak{c}_2}}{f(\mathfrak{c}_2)} f\left((\mathfrak{c}_1, \mathfrak{c}_2)\right). \tag{23}$$

From (15) by inversion (cf. [6] Satz 38),

$$f(\mathfrak{b}) = \sum_{\mathfrak{d}|\mathfrak{b}} F(\mathfrak{d})$$

whence

$$f\left((\mathfrak{c}_1, \mathfrak{c}_2)\right) = \sum_{\substack{\mathfrak{d}|\mathfrak{c}_1 \\ \mathfrak{d}|\mathfrak{c}_2}} F(\mathfrak{d})$$

and, by (23),

$$U = \sum_{\substack{\mathfrak{d} \\ N\mathfrak{d} \leqslant z \\ \mathfrak{q} \nmid \mathfrak{d}}} F(\mathfrak{d}) \Big(\sum_{\substack{\mathfrak{c} \\ \mathfrak{d}|\mathfrak{c} \\ N\mathfrak{c} \leqslant z \\ \mathfrak{q} \nmid \mathfrak{c}}} \frac{\lambda_{\mathfrak{c}}}{f(\mathfrak{c})} \Big)^2 = \sum_{\substack{\mathfrak{d} \\ N\mathfrak{d} \leqslant z \\ \mathfrak{q} \nmid \mathfrak{d}}} F(\mathfrak{d}) y_{\mathfrak{d}}^2, \tag{24}$$

if we write

$$y_{\mathfrak{d}} = \sum_{\substack{\mathfrak{c}\\ \mathfrak{d}|\mathfrak{c}\\ N\mathfrak{c}\leqslant z\\ \mathfrak{q}\nmid\mathfrak{c}}} \frac{\lambda_{\mathfrak{c}}}{f(\mathfrak{c})}.$$

Hence (since $\sum_{\mathfrak{d}|\mathfrak{a}} \mu(\mathfrak{d}) = 1$ if $\mathfrak{a} = \mathfrak{o}$, and $=0$ otherwise)

$$\sum_{\substack{\mathfrak{d}\\ N\mathfrak{d}\leqslant z/N\mathfrak{c}\\ \mathfrak{q}\nmid\mathfrak{c}\mathfrak{d}}} \mu(\mathfrak{d})\, y_{\mathfrak{c}\mathfrak{d}} = \sum_{\substack{\mathfrak{d}\\ N\mathfrak{d}\leqslant z/N\mathfrak{c}\\ \mathfrak{q}\nmid\mathfrak{d}}} \mu(\mathfrak{d}) \sum_{\substack{\mathfrak{g}\\ \mathfrak{c}\mathfrak{d}|\mathfrak{g}\\ N\mathfrak{g}\leqslant z,\, \mathfrak{q}\nmid\mathfrak{g}}} \frac{\lambda_{\mathfrak{g}}}{f(\mathfrak{g})}$$

$$= \sum_{\substack{\mathfrak{d}\\ N\mathfrak{d}\leqslant z/N\mathfrak{c}\\ \mathfrak{q}\nmid\mathfrak{d}}} \mu(\mathfrak{d}) \sum_{\substack{\mathfrak{b}\\ \mathfrak{d}|\mathfrak{b}\\ N\mathfrak{b}\leqslant z/N\mathfrak{c}\\ \mathfrak{q}\nmid\mathfrak{b}}} \frac{\lambda_{\mathfrak{b}\mathfrak{c}}}{f(\mathfrak{b}\mathfrak{c})} = \sum_{\substack{\mathfrak{b}\\ N\mathfrak{b}\leqslant z/N\mathfrak{c}\\ \mathfrak{q}\nmid\mathfrak{b}}} \frac{\lambda_{\mathfrak{b}\mathfrak{c}}}{f(\mathfrak{b}\mathfrak{c})} \sum_{\mathfrak{d}|\mathfrak{b}} \mu(\mathfrak{d}) = \frac{\lambda_{\mathfrak{c}}}{f(\mathfrak{c})}$$

or

(25) $$\frac{\lambda_{\mathfrak{c}}}{f(\mathfrak{c})} = \sum_{\substack{\mathfrak{d}\\ N\mathfrak{d}\leqslant z/N\mathfrak{c}\\ \mathfrak{q}\nmid\mathfrak{c}\mathfrak{d}}} \mu(\mathfrak{d})\, y_{\mathfrak{c}\mathfrak{d}},$$

whence, putting $\mathfrak{c} = \mathfrak{o}$,

(26) $$1 = \sum_{\substack{\mathfrak{d}\\ N\mathfrak{d}\leqslant z\\ \mathfrak{q}\nmid\mathfrak{d}}} \mu(\mathfrak{d})\, y_{\mathfrak{d}}.$$

Now we choose the numbers $\lambda_{\mathfrak{c}}$ $(1 < N\mathfrak{c} \leqslant z,\ \mathfrak{q}\nmid\mathfrak{c})$ satisfying

(27) $$\frac{\mu(\mathfrak{d})}{F(\mathfrak{d})} S_z^{-1} = y_{\mathfrak{d}},$$

i.e.

$$\sum_{\substack{\mathfrak{c}\\ \mathfrak{d}|\mathfrak{c}\\ N\mathfrak{c}\leqslant z,\, \mathfrak{q}\nmid\mathfrak{c}}} \frac{\lambda_{\mathfrak{c}}}{f(\mathfrak{c})} = \frac{\mu(\mathfrak{d})}{F(\mathfrak{d})} S_z^{-1},$$

whence, by the argument used in proving (25)

(28) $$\lambda_{\mathfrak{c}} = f(\mathfrak{c}) \sum_{\substack{\mathfrak{d}\\ N\mathfrak{d}\leqslant z/N\mathfrak{c}\\ \mathfrak{q}\nmid\mathfrak{c}\mathfrak{d}}} \mu(\mathfrak{d}) \frac{\mu(\mathfrak{c}\mathfrak{d})}{F(\mathfrak{c}\mathfrak{d})} S_z^{-1}.$$

If $\mathfrak{c}$ is a quadratfrei ideal (i.e. not divisible by a square $\neq \mathfrak{o}$), then by (15), (14),

(29) $$F(\mathfrak{c}) = f(\mathfrak{c}) \prod_{\mathfrak{p}|\mathfrak{c}} \left(1 - \frac{1}{f(\mathfrak{p})}\right).$$

Owing to the factors $\mu(\mathfrak{d})$, $\mu(\mathfrak{c}\mathfrak{d})$ we may suppose that the ideals $\mathfrak{c}$, $\mathfrak{d}$ in (28) are quadratfrei and prime to each other. Then, by (29), $F(\mathfrak{c}\mathfrak{d}) = F(\mathfrak{c})F(\mathfrak{d})$ and we have

$$\lambda_{\mathfrak{c}} = \mu(\mathfrak{c}) \prod_{\substack{\mathfrak{p}|\mathfrak{c}\\ \mathfrak{q}\nmid\mathfrak{c}}} \left(1 - \frac{1}{f(\mathfrak{p})}\right)^{-1} \sum_{\substack{\mathfrak{d}\\ (\mathfrak{d},\mathfrak{c})=\mathfrak{o}\\ N\mathfrak{d}\leqslant z/N\mathfrak{c}\\ \mathfrak{q}\nmid\mathfrak{d}\mathfrak{c}}} \frac{\mu^2(\mathfrak{d})}{F(\mathfrak{d})} S_z^{-1} = \mu(\mathfrak{c}) \prod_{\mathfrak{p}|\mathfrak{c}} \left(1 - \frac{1}{f(\mathfrak{p})}\right)^{-1} S_z(\mathfrak{c})/S_z$$

(cf. (16)), which is (17). For $\lambda_{\mathfrak{c}}$ satisfying this equation we have, by (24), (27), (26)

$$U = \sum_{\substack{\mathfrak{d}\\ N\mathfrak{d}\leqslant z\\ \mathfrak{q}\nmid\mathfrak{d}}} F(\mathfrak{d})\, y_{\mathfrak{d}} \frac{\mu(\mathfrak{d})}{F(\mathfrak{d})} S_z^{-1} = S_z^{-1} \sum_{\substack{\mathfrak{d}\\ N\mathfrak{d}\leqslant z\\ \mathfrak{q}\nmid\mathfrak{d}}} \mu(\mathfrak{d})\, y_{\mathfrak{d}} = S_z^{-1}.$$

From this and (22) we obtain (18).

4. LEMMA 4. *Let (12) be the set of ideals $\mathfrak{a}$ of a class $\mathfrak{H}$ having the norms $N\mathfrak{a} \leqslant x$, $x > D^{3k}$, $k = \frac{1}{2}(n+3)$, and let*

(30) $$z \geqslant \max(x^{1/4}, D^{3k}),$$

(31) $$\sum_{\mathfrak{q}} 1/N\mathfrak{q} \ll 1$$

for $\mathfrak{q}$ running through all the prime ideals $\mathfrak{q} \in Q$ of Lemma 3. Then the main term in (18) satisfies

$$X/S_z \ll x/h \log x.$$

Proof. Since $h \ll D$, we have, by (8), [1] (13),

(32) $$\nu(y, \mathfrak{H}) = \mu y + O(D^{2/3} y^{1-1/k}), \qquad \mu = h^{-1} \operatorname*{Res}_{s=1} \zeta(s, \chi_0) > D^{-9/8}$$

if $D > D_0$. Hence for

$$y \geqslant y_0 = D^{2k}$$

the order of magnitude of the principal term in (32) is larger than that of the remaining term as $D \to \infty$. By (32)

(33) $$X = \mu x + O(D^{2/3} x^{1-1/k}).$$

Hence, by (11),

$$\nu(x; \mathfrak{H}, \mathfrak{b}) = \mu x/N\mathfrak{b} + O\left(D^{2/3}\left(\frac{x}{N\mathfrak{b}}\right)^{1-1/k}\right) = \frac{X + O(D^{2/3}x^{1-1/k})}{N\mathfrak{b}} + O\left(D^{2/3}\left(\frac{x}{N\mathfrak{b}}\right)^{1-1/k}\right)$$

$$= X/N\mathfrak{b} + O\left(D^{2/3}\left(\frac{x}{N\mathfrak{b}}\right)^{1-1/k}\right).$$

Comparing with (13) we deduce

(34) $$f(\mathfrak{b}) = N\mathfrak{b}, \qquad R_{\mathfrak{b}} \ll D^{2/3}(x/N\mathfrak{b})^{1-1/k}.$$

Let $\nu(y)$ denote (as in § 2) the number of all the ideals of the field K prime to $\mathfrak{f}$ and having the norm $\leqslant y$. We have, by (32),

(35) $$\nu(y) > \tfrac{1}{2} h\mu y \quad \text{for} \quad y \geqslant y_0 .$$

By (16), (29), (34),

(36) $$S_z = \sum_{\substack{\mathfrak{b}\\ N\mathfrak{b}\leqslant z\\ \mathfrak{q}\nmid\mathfrak{b}}} \frac{\mu^2(\mathfrak{b})}{N\mathfrak{b}\prod_{\mathfrak{p}|\mathfrak{b}}(1-1/N\mathfrak{p})} = \sum_{\substack{\mathfrak{b}\\ N\mathfrak{b}\leqslant z\\ \mathfrak{q}\nmid\mathfrak{b}}} \mu^2(\mathfrak{b})\prod_{\mathfrak{p}|\mathfrak{b}}\left(\frac{1}{N\mathfrak{p}}+\frac{1}{N\mathfrak{p}^2}+\dots\right) = \sum_{\substack{\mathfrak{b}\in(z)\\ \mathfrak{q}\nmid\mathfrak{b}}} 1/N\mathfrak{b},$$

where (z) denotes the set of ideals $\mathfrak{b}$ (prime to $\mathfrak{f}$) in K such that the product of all different prime factors of any $\mathfrak{b}$ is in norm $\leqslant z$. Let $\mathfrak{g}$ denote a typical ideal in K which is divisible by no other prime ideals than the $\mathfrak{q}\in Q$; then

$$1 \leqslant \prod_{\mathfrak{q}}(1+1/N\mathfrak{q}+1/N\mathfrak{q}^2+\dots) = \sum_{\mathfrak{g}} 1/N\mathfrak{g} = V,$$

say. Hence, by (31), $V \ll 1$. (If Q is an empty set, then, by definition, $V=1$).

By (35), (36), (30), [1] (6),

$$S_z\cdot V \geqslant \sum_{\mathfrak{b}\in(z)} 1/N\mathfrak{b} \geqslant \sum_{\substack{\mathfrak{b}\\ N\mathfrak{b}\leqslant z\\ (\mathfrak{b},\mathfrak{f})=\mathfrak{o}}} 1/N\mathfrak{b} > \int_{y_0}^{z}\frac{\nu(y)\,dy}{y^2} > c_1 h\mu\log z$$

(since $z/y_0 \geqslant z^{1/3}$). Considering that $\log z \geqslant \frac{1}{4}\log x$ we deduce

$$S_z\cdot V > c_2 h\mu\log x,$$

whence

$$S_z > c_3 h\mu\log x,$$

(since $V\ll 1$). Combining this with (33) (where the remaining term may be dropped, since $x>y_0$) we obtain

$$X/S_z \ll \mu x/h\mu\log x = x/h\log x,$$

the desired result.

5. LEMMA 5. *Let W denote the remaining term in (18) and let in Lemma 4*

(37) $$z^{2/k} = \frac{x^{1/k}}{D^{4/3}(h\mu)^2 h\log x} \geqslant D^6;$$

then

$$W \ll x/h\log x .$$

Proof. If $\mathfrak{b}_1$ denotes the product of all the different prime divisors of $\mathfrak{b}$, then, by (16), (34), (29) and the argument used in proving (36)

$$\prod_{\substack{\mathfrak{p}|\mathfrak{c}\\ \mathfrak{q}\nmid\mathfrak{c}}}\left(1-\frac{1}{N\mathfrak{p}}\right)^{-1} S_z(\mathfrak{c}) = \prod_{\substack{\mathfrak{p}|\mathfrak{c}\\ \mathfrak{q}\nmid\mathfrak{c}}}\left(1+\frac{1}{N\mathfrak{p}}+\frac{1}{N\mathfrak{p}^2}+\dots\right)\sum_{\substack{\mathfrak{b}\in(z/N\mathfrak{c})\\ (\mathfrak{b},\mathfrak{c})=\mathfrak{o}\\ \mathfrak{q}\nmid\mathfrak{b}}}\frac{1}{N\mathfrak{b}}$$

$$= \sum_{\mathfrak{b}_1|\mathfrak{c}}\frac{1}{N\mathfrak{b}}\sum_{\substack{\mathfrak{b}\in(z/N\mathfrak{c})\\ (\mathfrak{b},\mathfrak{c})=\mathfrak{o}\\ \mathfrak{q}\nmid\mathfrak{b}}}\frac{1}{N\mathfrak{b}} \leqslant \sum_{\substack{N\mathfrak{b}_1\leqslant z\\ \mathfrak{q}\nmid\mathfrak{b}}}\frac{1}{N\mathfrak{b}} = S_z .$$

Hence, by (17), for all $\mathfrak{c}$

(38) $$|\lambda_{\mathfrak{c}}| \leqslant 1 .$$

By (38), (18) and (34)

(39) $$W \ll D^{2/3}x^{1-1/k}\sum_{\substack{\mathfrak{c}_1,\mathfrak{c}_2\\ N\mathfrak{c}_1,N\mathfrak{c}_2\leqslant z\\ \mathfrak{q}\nmid\mathfrak{c}_1\mathfrak{c}_2}}\left(\frac{N(\mathfrak{c}_1,\mathfrak{c}_2)}{N\mathfrak{c}_1\cdot N\mathfrak{c}_2}\right)^{1-1/k}.$$

Using (9), (37), [1] (6), [1] (13) we obtain

(40) $$\sum_{\substack{\mathfrak{c}\\ N\mathfrak{c}\leqslant z}} N\mathfrak{c}^{-(1-1/k)} = (1-1/k)\int_1^z\frac{\nu(x)\,dx}{x^{2-1/k}}+\frac{\nu(z)}{z^{1-1/k}}$$
$$\ll h\mu\int_1^z\frac{dx}{x^{1-1/k}}+D^{2/3}\int_1^z\frac{dx}{x}+\frac{h\mu z+D^{2/3}z^{1-1/k}}{z^{1-1/k}}$$
$$\ll h\mu z^{1/k}+D^{2/3}\log z \ll h\mu z^{1/k},$$

(41) $$\sum_{\substack{\mathfrak{c}\\ N\mathfrak{c}\leqslant z}} N\mathfrak{c}^{-1-1/k} \ll \int_1^z\frac{\nu(x)\,dx}{x^{2+1/k}}+\frac{\nu(z)}{z^{1+1/k}}$$
$$\ll h\mu\int_1^z\frac{dx}{x^{1+1/k}}+D^{2/3}\int_1^z\frac{dx}{x^{1+1/k}}+\frac{h\mu z+D^{2/3}z^{1-1/k}}{z^{1+1/k}}$$
$$\ll h\mu+D^{2/3} \ll D^{2/3}.$$

By (40)

(42) $$\sum_{\substack{\mathfrak{c}_1,\mathfrak{c}_2\\ N\mathfrak{c}_1,N\mathfrak{c}_2\leqslant z\\ (\mathfrak{c}_1,\mathfrak{c}_2)=\mathfrak{o}}}\left(\frac{N(\mathfrak{c}_1,\mathfrak{c}_2)}{N\mathfrak{c}_1\cdot N\mathfrak{c}_2}\right)^{1-1/k} \leqslant \left(\sum_{\substack{\mathfrak{c}\\ N\mathfrak{c}\leqslant z}} N\mathfrak{c}^{-1+1/k}\right)^2 \ll (h\mu)^2 z^{2/k}.$$

Denote the left-hand side in (42) by $S_0(z)$. Writing $(\mathfrak{c}_1,\mathfrak{c}_2)=\mathfrak{c}$, $\mathfrak{c}_1=\mathfrak{c}\mathfrak{a}$, $\mathfrak{c}_2=\mathfrak{c}\mathfrak{b}$ we have $(\mathfrak{a},\mathfrak{b})=\mathfrak{o}$ and, by (42),

$$\sum_{\substack{\mathfrak{c}_1,\mathfrak{c}_2\\ N\mathfrak{c}_1,N\mathfrak{c}_2\leqslant z\\ (\mathfrak{c}_1,\mathfrak{c}_2)=\mathfrak{c}}}\left(\frac{N(\mathfrak{c}_1,\mathfrak{c}_2)}{N\mathfrak{c}_1\cdot N\mathfrak{c}_2}\right)^{1-1/k} = N\mathfrak{c}^{-1+1/k}\sum_{\substack{\mathfrak{a},\mathfrak{b}\\ N\mathfrak{a},N\mathfrak{b}\leqslant z/N\mathfrak{c}\\ (\mathfrak{a},\mathfrak{b})=\mathfrak{o}}}\left(\frac{1}{N\mathfrak{a}\cdot N\mathfrak{b}}\right)^{1-1/k} = N\mathfrak{c}^{-1+1/k}S_0\left(\frac{z}{N\mathfrak{c}}\right)$$
$$\ll (h\mu)^2 z^{2/k}N\mathfrak{c}^{-1+1/k-2/k} = (h\mu)^2 z^{2/k}N\mathfrak{c}^{-1-1/k}.$$

Denoting the first term in this sequence of equations by $S_c(z)$, we have

$$\sum_{\substack{\mathfrak{c}_1,\mathfrak{c}_2\\ N\mathfrak{c}_1,N\mathfrak{c}_2\leqslant z}} \left(\frac{N(\mathfrak{c}_1,\mathfrak{c}_2)}{N\mathfrak{c}_1\cdot N\mathfrak{c}_2}\right)^{1-1/k} \leqslant \sum_{\substack{\mathfrak{c}\\ N\mathfrak{c}\leqslant z}} S_{\mathfrak{c}}(z) \ll (h\mu)^2 z^{2/k} \sum_{\substack{\mathfrak{c}\\ N\mathfrak{c}\leqslant z}} N\mathfrak{c}^{-1-1/k} \ll D^{2/3}(h\mu)^2 z^{2/k},$$

by (41). Hence, by (39), (37),

$$W \ll D^{4/3}(h\mu)^2 x^{1-1/k} z^{2/k} = x/h\log x .$$

This is the desired result.

Taking $x \geqslant D^{5k}$ we have (since $h \ll D$, $h\mu \ll D^{1/16}$)

$$D^{4/3}(h\mu)^2 h\log x < x^{1/2k} \qquad (D > D_0)$$

whence, by (37),

$$z \geqslant x^{1/4}, \qquad z \geqslant D^{3k}$$

which is (30).

6. Lemma 6. *If $\pi(x,\mathfrak{H})$ denotes the number of prime ideals of the class $\mathfrak{H}$ having the norms $\leqslant x$, then*

(43) $$\pi(x,\mathfrak{H}) \ll x/h\log x \quad \textit{for} \quad x \geqslant D^{5k},\ k = \tfrac{1}{2}(n+3).$$

Proof. Let in Lemma 3 (12) be the set of ideals $\mathfrak{a}$ of the class $\mathfrak{H}$ with $N\mathfrak{a} \leqslant x$ and let the set Q be empty. Since $z < \sqrt{x}$, by (37), among the N_z ideals (18) there are all the prime ideals of $\mathfrak{H}$ with the norm $> z$ and $\leqslant x$. Hence, by (18), Lemmas 4, 5 and (32) (where the remaining term may be dropped, for $y = \sqrt{x} > y_0$)

$$\pi(x,\mathfrak{H}) \leqslant N_z + \pi(z,\mathfrak{H}) \ll x/h\log x + \pi(\sqrt{x},\mathfrak{H}) \ll x/h\log x ,$$

the desired result.

Corollary. *We have*

(44) $$\sum_{\substack{\mathfrak{p}^m\in\mathfrak{H}\\ N\mathfrak{p}^m\leqslant x,\, m\geqslant 1}} \log N\mathfrak{p} \ll x/h \quad \textit{for} \quad x \geqslant D^{5k+1},\ k = \tfrac{1}{2}(n+3),\ D > D_0 .$$

Proof. Writing $x_0 = D^{5k}$ we have $x \geqslant x_0 D$. By (43), [2] (2), [1] (6)

(45) $$\sum_{\substack{N\mathfrak{p}\leqslant x\\ \mathfrak{p}\in\mathfrak{H}}} \log N\mathfrak{p} \ll \sum_{p\leqslant x_0} \log p + \sum_{\substack{x_0<N\mathfrak{p}\leqslant x\\ \mathfrak{p}\in\mathfrak{H}}} \log N\mathfrak{p} \ll x_0 +$$
$$+ \int_{x_0}^{x} \frac{\pi(y,\mathfrak{H})}{y}\,dy + \pi(x,\mathfrak{H})\log x \ll x_0 + \pi(x,\mathfrak{H})\log x \ll x/h$$

and

$$\sum_{\substack{\mathfrak{p}^m\in\mathfrak{H}\\ N\mathfrak{p}^m\leqslant x,\, m\geqslant 2}} \log N\mathfrak{p} \ll \log x \cdot \sum_{p<\sqrt{x}} \log p \ll \sqrt{x}\log x \ll x/h$$

(since $h \ll D$), whence (44).

Lemma 7. *Let $V(x;\mathfrak{H},y)$ denote the number of ideals $\mathfrak{g}$ ($N\mathfrak{g} \leqslant x$) of the class $\mathfrak{H}$ which are divisible exclusively by the prime ideals $\mathfrak{p}$ with $y \leqslant N\mathfrak{p} \leqslant y^2$; then*

(46) $$V(x;\mathfrak{H},y) \ll x/h\log x \quad \textit{for} \quad x \geqslant D^{5k},\ k = \tfrac{1}{2}(n+3),\ D > D_0,$$
$$\textit{uniformly in} \quad 1 \leqslant y \leqslant \sqrt{x}.$$

Proof. Let in Lemma 3 (12) be the set of ideals $\mathfrak{a}$ of the class $\mathfrak{H}$ with $N\mathfrak{a} \leqslant x$ and let Q be the set of all the ideals $\mathfrak{q}$ in K (prime to $\mathfrak{f}$) such that $y \leqslant N\mathfrak{q} \leqslant y^2$. Then, by [2] (4),

$$\sum_{\mathfrak{q}} 1/N\mathfrak{q} \ll \sum_{y\leqslant p\leqslant y^2} 1/p \ll 1,$$

which is (31). Among the N_z ideals (18) there are all the ideals $\mathfrak{g} \in \mathfrak{H}$ with $N\mathfrak{g} \leqslant x$, whence (46) follows from Lemmas 4 and 5.

An estimate for the number of functions $\zeta(s,\chi)$ having a zero in the neighbourhood of $s = 1$

7. Lemma 8. *Let $\zeta(s,\chi)$ has a zero $\varrho_0 = \beta_0 + i\tau_0$ in the rectangle $R_0(1-\lambda/\log D \leqslant \sigma \leqslant 1,\ |t| \leqslant e^{\lambda}/\log D)$ $(0 < c \leqslant \lambda \leqslant \frac{2}{5}\log D)$ and χ be neither the principal nor the exceptional character ($\chi \neq \chi_0$, $\chi \neq \chi'$). Then there is a number $\tau_1 = \tau_1(\chi) \ll \lambda e^{\lambda}/\log D$ and there are two rectangles*

$$R_{\mathrm{I}}(1-\alpha \leqslant \sigma \leqslant 1+3\alpha,\ |t-\tau_1| \leqslant 4\alpha),$$
$$R_{\mathrm{II}}(1+\alpha/2 \leqslant \sigma \leqslant 1+3\alpha/2,\ |t-\tau_1| \leqslant \alpha),\ (\alpha = \lambda/\log D)$$

such that either (I) *for appropriate $s_1 \in R_{\mathrm{I}}$*

(47) $$|F_1(s_1)| > \log D, \quad \textit{where} \quad F_1(s) = \sum_{\substack{\mathfrak{p}\\ Z<N\mathfrak{p}=p<D^{6k}}} \frac{\chi(\mathfrak{p})\log N\mathfrak{p}}{N\mathfrak{p}^s},$$

$Z = D^{\{\log(1+\lambda)\}/(2+2\lambda)}$, $k = \frac{1}{2}(n+3)$, *or* (II) *for appropriate $s_2 \in R_{\mathrm{II}}$*

(48) $$|F_2(s_2)| > \exp(-c_0\lambda)\log D, \quad \textit{where} \quad F_2(s) = \sum_{\substack{\mathfrak{p}\\ N\mathfrak{p}=p\geqslant D^{6k}}} \frac{\chi(\mathfrak{p})\log N\mathfrak{p}}{N\mathfrak{p}^s}$$

and $c_0 > 0$ depends only on n.

This is a simple consequence of [2] Lemmas 1, 2 (with $\Lambda = \log D$, $\lambda \in [c, \frac{2}{5}\Lambda]$, $\gamma_0 = e^{\lambda}/\Lambda$, $M = D^{6k}$); cf. [2] § 6.

Lemma 9. *Let $Q = Q(D,\lambda)$ for $c \leqslant \lambda \leqslant 2\log D$, $D > D_0$ denote the number of the functions $\zeta(s,\chi)$ having at least one zero in the region*

$$R_0(1-\lambda/\log D \leqslant \sigma \leqslant 1,\ |t| \leqslant e^{\lambda}/\log D).$$

Then for appropriate constant $C_1 > 0$ (which depends only on n)

(49) $$Q < \exp(C_1\lambda).$$

The proof of this lemma is the object of the rest of this paper.

For $\lambda \geqslant \frac{2}{5}\log D$ (49) is evidently true, since then h (which is the total number of the functions $\zeta(s,\chi)$) does not exceed $e^{5\lambda}$. And if $h \ll 1$, then taking C_1 large enough (49) holds for all $\lambda \geqslant c$. Hence it remains to prove the lemma for the range $c \leqslant \lambda \leqslant \frac{2}{5}\log D$ supposing $h > 16$ (say) in which case the functions $\zeta(s,\chi_0)$, $\zeta(s,\chi')$ may be left out. By Q' we denote the number of the functions $\zeta(s,\chi)$ with $\chi \neq \chi_0, \neq \chi'$ having a zero in R_0. To these functions we may apply Lemma 8. By Q_1 and Q_2 we denote the numbers of the functions $\zeta(s,\chi)$ for which there holds respectively the conclusion (I) or (II) of that lemma.

8. Let us suppose first that

(50) $$Q_2 \geqslant \tfrac{1}{2}Q'.$$

The rectangles R_{II} (where τ_1 depends on χ) lie in the region

$$R_2(1+\tfrac{1}{2}a \leqslant \sigma \leqslant 1+\tfrac{3}{2}a, \quad |t| < a\lambda e^{\lambda}/\log D)$$

(for a suitable $a \ll 1$) which can be covered by a set of

(51) $$N \leqslant c_1\lambda^2\exp(1+2c_0)\lambda$$

(c_0 defined by (48)) congruent squares with the side of the length

$$\eta = 1/b\exp(c_0\lambda)\log D$$

running parallel to the axis. Then there is at least one square, R_3, say, in which there lie

(52) $$Q_3 \geqslant Q_2/N$$

of the points $s_2 = s_2(\chi)$. By (48), [1] (6), [2] (2), in R_2

$$|F_2'(s)| < c_2\sum_{p\geqslant D^{6k}}\frac{\log^2 p}{p^{1+a/2}} < c_3\int_{D^{6k}}^{\infty}\frac{x\log x}{x^{2+a/2}}dx$$
$$= c_4(\lambda^{-1}e^{-3k\lambda}\log^2 D+\lambda^{-2}e^{-3k\lambda}\log^2 D) < c_5\lambda^{-1}e^{-3k\lambda}\log^2 D.$$

Hence by s_0 denoting the centre of R_3, we have, by (48),

$$|F_2(s_0)| \geqslant |F_2(s_2)|-|F_2(s_2)-F_2(s_0)| \geqslant |F_2(s_2)|-\Big|\int_{s_2}^{s_0}F_2'(s)\,ds\Big|$$
$$\geqslant \exp(-c_0\lambda)\log D-\eta\max_{s\in R_3}|F_2'(s)| > \tfrac{1}{2}\exp(-c_0\lambda)\log D$$

(if $b \ll 1$ is large enough), whence

(53) $$|F_2(s_0)|^2 > \tfrac{1}{4}\exp(-2c_0\lambda)\log^2 D.$$

Now we add (53) over the Q_3 characters for which this inequality holds and compare with the corresponding sum over all χ in which we use (1). Dividing through by $\log^2 D$ and using (50), (51), (52), (48), (45), [1] (6) we get the inequalities

$$Q'c_6\lambda^{-2}\exp\{-(1+4c_0)\lambda\} < Q_3\cdot\tfrac{1}{4}\exp(-2c_0\lambda)$$
$$< \sum_{\chi}\Big|\frac{1}{\log D}\sum_{\substack{\mathfrak{p}\\ N\mathfrak{p}=p\geqslant D^{6k}}}\chi(\mathfrak{p})N\mathfrak{p}^{-s_0}\log N\mathfrak{p}\Big|^2$$
$$\leqslant \frac{h}{\log^2 D}\sum_{\substack{\mathfrak{p},\mathfrak{p}_1\\ \mathfrak{p}\sim\mathfrak{p}_1\\ N\mathfrak{p}=p\geqslant D^{6k}\\ N\mathfrak{p}_1=p_1\geqslant D^{6k}}}\frac{\log N\mathfrak{p}\cdot\log N\mathfrak{p}_1}{(N\mathfrak{p}\cdot N\mathfrak{p}_1)^{1+a/2}}$$
$$= \frac{h}{\log^2 D}\sum_{\nu=1}^{h}\Big(\sum_{\substack{\mathfrak{p}\in\mathfrak{H}_\nu\\ N\mathfrak{p}=p\geqslant D^{6k}}}\frac{\log N\mathfrak{p}}{N\mathfrak{p}^{1+a/2}}\Big)^2 < c_7\lambda^{-2}e^{-5k\lambda}$$

(since

$$\sum_{\substack{\mathfrak{p}\in\mathfrak{H}\\ N\mathfrak{p}\geqslant D^{6k}}}\frac{\log N\mathfrak{p}}{N\mathfrak{p}^{1+a/2}} \ll \int_{D^{6k}}^{\infty}\frac{x/h}{x^{2+a/2}}dx \ll \frac{1}{h}\lambda^{-1}e^{-3k\lambda}\log D),$$

whence $Q' < \exp(c_8\lambda)$. This proves the lemma in the case of (50).

9. Now we suppose that

(54) $$Q_1 \geqslant \tfrac{1}{2}Q'.$$

The Q_1 rectangles R_{I} (with τ_1 depending on χ) lie in the region

$$R_1(1-a \leqslant \sigma \leqslant 1+3a, \quad |t| < c_1\lambda e^{\lambda}/\log D)$$

which can be covered by a set of

(55) $$N_1 < c_2\lambda^2\exp(1+12k)\lambda$$

congruent squares having the length of the side

$$\eta = 1/b\exp(6k\lambda)\log D.$$

There is at least one square, R_4, say, in which there lie

(56) $$Q_4 \geqslant Q_1/N_1$$

of the points $s_1 = s_1(\chi)$. By (47), [2] (3), we have in R_1

$$|F_1'(s)| \leqslant c_3\sum_{p<D^{6k}}\frac{\log^2 p}{p^{1-a}} < c_4e^{6k\lambda}\sum_{p<D^{6k}}\frac{\log^2 p}{p} < c_5e^{6k\lambda}\log^2 D.$$

Hence, by s_0 denoting the centre of R_4, we have, by (47)

$$|F_1(s_0)| \geqslant |F_1(s_1)| - |F_1(s_1) - F_1(s_0)| > \log D - \Big|\int_{s_1}^{s_0} F_1'(s)\,ds\Big|$$
$$> \log D - \eta c_5 e^{6k\lambda} \log^2 D > \log D - \tfrac{1}{2}\log D = \tfrac{1}{2}\log D$$

(supposing $b \ll 1$ large enough).

Now let the ideals $\mathfrak{p}$ with $Z < N\mathfrak{p} = p < D^{6k} = M$, $\mathfrak{p} \nmid \mathfrak{f}$ be distributed into sets $S_1, \ldots, S_\nu$, the set S_j containing all the $\mathfrak{p}$ satisfying

(57) $$2^{-j}\log M < \log N\mathfrak{p} \leqslant 2^{1-j}\log M\,, \qquad \nu < c_6 \log(2+2\lambda)\,.$$

Then there is at least one set $S_j = S$, say, such that for at least

(58) $$Q_5 \geqslant Q_4/\nu$$

of the characters χ we have

(59) $$\Big|\sum_{\mathfrak{p}\in S} \chi(\mathfrak{p}) N\mathfrak{p}^{-s_0} \log N\mathfrak{p}\Big| > \frac{1}{2\nu}\log D\,.$$

Raising to the power 2^{j+1} we get the inequality

(60) $$\Big|\sum_{\substack{\mathfrak{c}\\ M<N\mathfrak{c}<M^2}} \chi(\mathfrak{c}) a_{\mathfrak{c}} N\mathfrak{c}^{-s_0}\Big|^2 > (\log D)^{2^{j+1}} \exp(-c_7\lambda)\,,$$

where

(61) $$0 \leqslant a_{\mathfrak{c}} \leqslant 2^j! \prod_{\substack{1\leqslant l\leqslant 2^j\\ \mathfrak{p}_l\in S}} \log N\mathfrak{p}_l \leqslant 2^j!\,(\log M)^{2^j} < (\log D)^{2^j} \exp(c_8\lambda)\,.$$

Let $\mathfrak{a}$ and $\mathfrak{b}$ denote ideals all the prime divisors of which are in the set S defined by (57), (58). Then we have, by (61), (1),

(62) $$\sum_{\chi}\Big|\sum_{\substack{\mathfrak{c}\\ M<N\mathfrak{c}<M^2}} \chi(\mathfrak{c}) a_{\mathfrak{c}} N\mathfrak{c}^{-s_0}\Big|^2$$
$$< (\log D)^{2^{j+1}} \exp(2c_8\lambda) M^{4a} h \sum_{M<N\mathfrak{a}<M^2} N\mathfrak{a}^{-1} \cdot \sum_{\substack{\mathfrak{b}\\ M<N\mathfrak{b}<M^2\\ \mathfrak{b}\sim\mathfrak{a}}}' N\mathfrak{b}^{-1} < (\log D)^{2^{j+1}} \exp(c_9\lambda)\,,$$

since, by (46), [1] (6),

$$\sum_{\substack{\mathfrak{b}\\ M<N\mathfrak{b}<M^2\\ \mathfrak{b}\in\mathfrak{H}}}' N\mathfrak{b}^{-1} \ll \int_M^{M^2} \frac{x}{h\log x}\,\frac{dx}{x^2} + \frac{M^2/h\log M^2}{M^2} \ll 1/h\,,$$

$$\sum_{\substack{\mathfrak{a}\\ M<N\mathfrak{a}<M^2}} N\mathfrak{a}^{-1} = \sum_{\nu=1}^{h}\ \sum_{\substack{\mathfrak{b}\in\mathfrak{H}_\nu\\ M<N\mathfrak{b}<M^2}} N\mathfrak{b}^{-1} \ll 1\,.$$

Summing (59) over the corresponding characters, comparing with (62) and dividing through by $(\log D)^{2^{j+1}}$ we obtain, by (54), (55), (56), (58), (60),

$$\frac{Q'\exp(-c_7\lambda)}{c_{10}\lambda^2\log(2+2\lambda)\exp(1+12k)\lambda} < Q_5\exp(-c_7\lambda) < \exp(c_9\lambda)$$

whence $Q' < \exp(c_{11}\lambda)$. This proves the lemma.

The theorem of § 1 is an immediate consequence of Lemma 9 and [1] (43).

Note. In the original method of Linnik and Rodosskiĭ for Dirichlet's L-functions (see [9] p. 339, [7] p. 173 or [8] X (2.66)) some numbers v, corresponding to $N\mathfrak{b}$ in (62) are treated as numbers of arithmetical progression $Du+l$ with $(D, l) = 1$ *having no prime divisor* $< Z$. An estimate

(63) $$\nu(x) \ll x/\varphi(D)\log Z \qquad (x \geqslant D^2)$$

for the number $\nu(x)$ of such numbers $v \leqslant x$ has been proved using the Brun's sieve method (see [7] p. 173). The corresponding result for an algebraic field K of degree $n \geqslant 2$ may not be true (1) and the method does not work. The principal difficulty to overcome in proving the theorem of this paper consisted in the construction of a suitable substitute for (63). (2)

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(1) Take, for example, $D = |\Delta|$, $Z = D^\varepsilon$ with $\varepsilon > 0$ arbitrarily small and suppose that all the prime ideals $\mathfrak{p} \in K$ with $N\mathfrak{p} < Z$ have the degrees $f \geqslant 2$ (this has never been disproved). Then the factor $\log Z$ in the denominator disappears, since in (36) we have

$$S_z \ll \sum_m' \frac{m^{1/2}}{m^2} \ll 1 \qquad \text{(cf. [2], § 13).}$$

(2) I found the way to (46) in March 1960 when there was not yet Turán's paper [11] which makes (46) superfluous for the proof of the present theorem (*Added in proof 8th Febraury* 1962).

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On polynomial transformations

by

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1. We shall say that subset X of a field R has property (P) if every polynomial $P(x)$ with coefficients from R such that $P(X) = X$ is linear. It is easy to see that any number field in which the "Irreduzibilitätssatz" of Hilbert is true has property (P). Consequently, any algebraic extension of the field of rational numbers has property (P) and any number field which is transcendental extension of some (its) infinite subfield also has this property. (E.g. see [1], [3]). On the other hand, it is trivial that no finite set has property (P). The problem can be posed, having a fixed number set Z, to characterize the subsets of Z with property (P). In this paper we solve this problem in the case where Z is an algebraic number field. (By an algebraic number field we always understand a finite algebraic extension of the field of rational numbers.) Indeed, we shall prove

Theorem I. *A subset X of an algebraic number field has property* (P) *if and only if it is infinite.*

We shall say that a set Z has property (P) hereditarily if every infinite subset of Z has property (P). Thus algebraic number fields have property (P) hereditarily. It turns out that also every finitely generated transcendental extension of an algebraic number field has property (P) hereditarily. This follows from

Theorem II. *Let K be a finitely generated transcendental extension of a field R. Then K has hereditarily property* (P) *if and only if R has this property*. (The "only if" parts of our theorems are of course trivial.)

2. For the proof of our theorems we need the following

Lemma 1. *Suppose that $T(x)$ is a transformation of the set X onto itself. Suppose that there exist two functions $f(x)$ and $g(x)$ defined on X, with values in the set of natural numbers, subject to the conditions:*

(a) *For every constant c the equation $f(x)+g(x) = c$ has only a finite number of solutions,*

(b) *There exists a constant C such that from $f(x) \geqslant C$ follows $f\big(T(x)\big) > f(x)$,*