On the zeros of Hecke's L-functions III

by

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Introduction

1. In the two previous papers (see [1], [2]) some results were proved about the location of the zeros near the line \( \sigma = 1 \) of the Hecke-Landau function \( \langle i, \chi \rangle \) (corresponding to a field \( K \) of degree \( n \geq 1 \)) with characters \( \chi \) modulo \( D = |D| \cdot N \rightarrow \infty \) (where \( D \) denotes the discriminant of the field) and \( n \ll 1 \). In the present paper we will prove the following

**Theorem.** Let \( N(a, \chi) \) denote the number of the zeros of the function \( Z(s) = \prod_{\chi} \zeta(s, \chi) \) in the rectangle \( 1 - a \leq \sigma \leq 1, \ |t| \leq T \). Then there is a constant \( C > 0 \) (which depends only on \( n \)) such that for any \( \lambda \in [0, 2 \log D], \) \( \lambda > D^2 > 1 \)

\[
N(a, \chi) < C \lambda \exp(C \lambda) \quad \text{if} \quad a \leq C \lambda \exp(C \lambda).
\]

The method used in this paper is in the outline the same as that employed by Rokoskifi for Dirichlet's L-functions (see [9], pp. 332-341 or [8] X § 10). There is however an essential digression from his final arguments as will be shown at the end of the paper.

The notation used here is the same as in [1], [2].

We shall need the known estimate \( \chi \ll D \) (see [4] § 3 and [5] p. 66) for the number of classes \( \mathfrak{p} \), the proof of which must be postponed to the next paper of this series, where it fits in more conveniently.

**Estimates for the number of ideals with the norm \( \ll \)

2. By the fundamental properties of group characters (see, for example, [12] § 13)

\[
\sum_{\mathfrak{p}} \chi(\mathfrak{p}) = \begin{cases} 1 & \text{if } \mathfrak{p} \text{ is the principal class,} \\ 0 & \text{otherwise}, \end{cases}
\]

whence

\[
\sum_{\mathfrak{p}} \chi(a) \chi(b) = \begin{cases} 1 & \text{if } a \sim b \text{ (i.e. } a \text{ and } b \text{ are in the same class } \mathfrak{p} \text{),} \\ 0 & \text{otherwise}. \end{cases}
\]
Let us introduce the function
\[\zeta(s, \mathfrak{S}) = \sum_{n \in \mathfrak{S}} n^{-s}(s > 1).\]
For a fixed \(e \in \mathfrak{S}\) we have, by (1), (2), (1) (8)
\[\zeta(s, \mathfrak{S}) = e^{-s} \sum_{n \in \mathfrak{S}} \zeta(n) \zeta(s, \mathfrak{S}).\]
Hence \(\zeta(s, \mathfrak{S})\) is regular in the whole plane, except for a simple pole at \(s = 1\) with the residue
\[\text{Res}_{s=1} \zeta(s, \mathfrak{S}) = h^{-1} \text{Res}_{s=1} \zeta(s, \mathfrak{S}),\]
which does not depend on \(\mathfrak{S}\).

(3) and [1] (32) implies that for \(|n-1| > \frac{1}{2}\) we have uniformly in \(s\)
\[\zeta(s, \mathfrak{S}) < \delta^{-s} D^2 \log D \log(1+|n|)^{2+\varepsilon} \quad (-\delta < \sigma < 1+\delta, \ 0 < \delta < 1/(\log D)).\]

The results of this paragraph are based on the following lemma. Let \(a(x)\) for \(x > 0\) be a positive non-decreasing function of \(x\) such that \(a(2x) < a(x)\) for all \(x > 0\)
\[a_m < a(m).\]
Further suppose the series
\[f(s) = \sum_{m=1}^{\infty} a_m m^{-s}\]
is convergent for \(s > 1\), and for some fixed \(l\)
\[(\eta-1) \sum_{m} |a_m| m^{-\eta} < 1 \quad \text{if} \quad 1 < \eta < 2.\]
Then we have uniformly for \(1 < \eta < 2, \ T > 0, \ u > 2\)
\[\sum_{a=x}^{x+a} \frac{1}{2\pi i} \int_{u-iT}^{u+iT} \frac{f(s)ds}{s} \ll \frac{1}{T} \left[ (\eta-1)^{-1} + a \log a \right] + a(u).\]

For \(l = 1\) and \(a(x) = \log x\) this lemma is proved by Landau (3), Hlavatsky (3) and the proof still holds under the generalised conditions.

**Lemma 1.** Let \(\nu(x, \mathfrak{S})\) denote the number of ideals of the class \(\mathfrak{S}\) having the norm \(\leq x, \ \sigma \geq 1\), and let \(\nu(x) = \sum_{\mathfrak{S}} \nu(x, \mathfrak{S}).\) Then
\[\nu(x, \mathfrak{S}) = \mu x + O(D^3 x^{1+\varepsilon})\]
and
\[\nu(x) = \mu x + O(D^3 x^{1+\varepsilon}),\]
where \(\mu = h^{-1} \text{Res}_{s=1} \zeta(s, \mathfrak{S}), \ k = \frac{1}{2}(n+3).\)

Proof. Since there are only \(\approx 1\) ideals of the field having the norm \(\leq 3, \ \sigma < 3\) the lemma is evidently true. In what follows we suppose \(\sigma > 3\).

For any of the functions \(f(x) = \zeta(x, \mathfrak{S}), \zeta(x, \mathfrak{S})\) the coefficients \(a_m\) in (5) are \(\approx d_4(m)\) (in the notation of [2] § 13). By [1] (3) and induction (with respect to \(n\)) \(a_4(m) \ll e^{c(m)}\) for any \(e > 0\). By (3), (1) (8), (1) (14), we have, for \(\sigma > 1, \zeta(x, \mathfrak{S}) \ll \nu(x, \mathfrak{S}) \ll (x, \mathfrak{S}) \ll (x, \mathfrak{S})^{1+\varepsilon}.\) Hence (6) holds for \(l = n\).

Taking \(\eta = 1+1/(\log x, T = x^{\varepsilon+\varepsilon})\) we have, by (7),
\[\nu(x, \mathfrak{S}) = \frac{1}{2\pi i} \int_{u-iT}^{u+iT} \frac{x}{s} \zeta(s, \mathfrak{S}) ds + O(x^{1+\varepsilon+\varepsilon}) + \log x\]
(the constant in \(O\) depending on \(e\)). For \(x = 1/(n+3)(n+3)\) the remaining term in (10) is \(\ll x^{-2} \log x \ll x^{-1+\varepsilon}.

Passing over a pole of the integrand at \(s = 1\) with the residue \(\mu\) we replace the path of integration in (10) by \(I_3 + I_4 + I_5\), where \(I_3, I_4, I_5\) denote straight lines joining the points \(\eta-iT, 1/\log Dx-iT, 1/\log Dx+iT\), \(1/\log Dx+iT, \eta+iT\), respectively. By (4) (with \(\delta = 1/\log D\))
\[\int_{I_3} \frac{x}{s} \zeta(s, \mathfrak{S}) ds \ll D^{3+\varepsilon} \log D x \ll D^{3+\varepsilon} x^{-1+\varepsilon},\]
\[\int_{I_4} \frac{x}{s} \zeta(s, \mathfrak{S}) ds \ll x^{3+\varepsilon} \log D x + D^{3+\varepsilon} x^{-1+\varepsilon} \ll D^{3+\varepsilon} x^{-1+\varepsilon}.

From this and (10) we deduce (8). Using (1) (32) instead of (4) by the same argument we get (9).

**Lemma 2.** If \(\nu(x, \mathfrak{S}, b)\) denotes the number of such ideals of the class \(\mathfrak{S}\) which are divisible by \(b\) and have the norm \(\leq x\) \((x \geq 1), \ b = (f, s) = a, \ N_b = x\), then
\[\nu(x; \mathfrak{S}, b) = \mu x N_b + O(D^3 x^{1+\varepsilon}),\]
where \(\mu, k\) are defined by (9).

Proof. Write \(a = bk\). If \(a\) runs through the ideals of the lemma and \(b \in \mathfrak{S}^*\) (say), then \(c\) runs through the class of the ideal \(\mathfrak{S}^*/b\) having the norms \(\leq x/N_b\) and conversely. Hence (11) follows from (8).

**An application of Selberg’s sieve method**

**Lemma 3.** Let \(a_n \ (m = 1, \ldots, X)\)
be a set of ideals in \(K\) such that for any \(b\)
\[\sum_{b \mid n} 1 = X/(f(b) + R_b),\]
where
where $f(b)$ is a multiplicative function $\neq 0$, i.e.

$$(14) \quad f(ab) = f(a)f(b) \quad \text{if} \quad (a, b) = 1. $$

Further let $Q$ be a fixed set (empty or not) of prime ideals, generally denoted by $q$, and let for $s > 1$ $N_s$ denote the number of ideals (12) not divisible by any prime ideal $p \neq q$ having the norm $N_p < s$. Writing

$$(15) \quad F(b) = \sum_{b|b} \mu(b)f(b) \frac{\mu(b)}{b},$$

$$(16) \quad S_c = \sum_{b|-b} \mu(b)F(b), \quad S_d(c) = \sum_{\left(b, b', b''\right) = 1} \frac{\mu(b)}{F(b)},$$

$$(17) \quad \lambda_e = \mu(c)\prod_{p | q^e} (1 - 1/f(p))^{-1}S_d(c)/S_c \quad \text{if} \; q \nmid c, \; N_c \leq s,$$

$0 \quad \text{otherwise},$

we have

$$(18) \quad N_s \leq X/S_c + \sum_{N_c \leq s} |\lambda_e|_e R_{\phi(n)(n, a)}.$$

It is understood that all the ideals used in this lemma and further on are prime to $f$.

Proof. (The method used in this proof is borrowed from Selberg's paper [10] where an analogous result is obtained for a set of natural numbers. But there are no exceptional primes $q$ in his paper corresponding to the ideals $q$ of the present lemma.)

We begin the proof by taking $\lambda_e = 1$ and $\lambda_e = 0$ whenever $q | c$ or $N_c > s$ (or both). For all other $c$ let for the present $\lambda_e$ be any real numbers. Then

$$(19) \quad N_s \leq \sum_a \left( \sum_{\lambda_e} \lambda_e \right)^2,$$

since the inner sum is 1 if $a_m$ has no other prime divisors than $q$ or $p \neq q$ with $Np > s$ (or both), and 0 otherwise. By $b_1, ..., b_r$ denoting all the divisors of $a_m$ we have

$$\left( \sum_{\lambda_e} \lambda_e \right)^2 = (\lambda_{b_1} + ... + \lambda_{b_r})^2 = \lambda_{b_1} \lambda_{b_2} \left( \sum_{\lambda_e} \lambda_e \right),$$

since '$b_1|a_m$ and $b_2|a_m$ is equivalent to

$$\frac{b_1b_2}{\text{lcm}(b_1, b_2)}|a_m.$$

Hence, by (19)

$$(20) \quad N_s \leq \sum_{N_c \leq s} \lambda_e \lambda_a \sum_{\left(\frac{c}{a}ight)} 1.$$

By (14) $f(a) = 1$. It may be proved by induction with respect to the number of different prime factors of $b = (a, b)$ that

$$(21) \quad f(ab) = f(a)f(b)/f(b).$$

This is evident for $b = a$. Let $a = a'p^n, b = b'p^m$ and let $a'$ (for example) be prime to $p$; then $(a, b) = (a'p^n, b'p^m) = p^m(a', b') = p^m(a', b)$. Supposing (21) is true for $a'$, $b$ we have, by (14),

$$f\left(\frac{a}{a', b}\right)f\left(\frac{b}{a', b}\right) = f\left(\frac{a'}{a', b}\right)f\left(\frac{b'}{a', b}\right),$$

whence (21).

By (21), (13),

$$\sum_{\left(\frac{c}{a}ight)} \frac{1 - Xf\left(\frac{c}{a}\right)}{f\left(\frac{c}{a}\right)} + R_{\phi(n)(n, a)} = Xf\left(\frac{c}{a}\right) + \frac{R_{\phi(n)(n, a)}}{f\left(\frac{c}{a}\right)}.$$}

Hence, by (20),

$$(22) \quad N_s \leq XU + \sum_{N_c \leq s} |\lambda_e|_e R_{\phi(n)(n, a)}.$$

if we write

$$(23) \quad U = \sum_{N_c \leq s} \lambda_e \lambda_a f\left(\frac{c}{a}\right),$$

From (15) by inversion (cf. [6] Satz 38),

$$f(b) = \sum_{\frac{b}{a}} F(b)$$

whence

$$f\left(\frac{c}{a}\right) = \sum_{\frac{b}{a}} F(b)$$

and, by (23),

$$(24) \quad U = \sum_{\frac{b}{a}} F(b) \left( \sum_{\frac{c}{a}} \lambda_a \lambda_a \right)^{1/2} = \sum_{\lambda_a \lambda_a} F(b) b^2,$$
if we write

\[ y_b = \sum_{N \neq \sigma} \lambda_b f(c). \]

Hence (since \( \sum \mu(b) = 1 \) if \( a = \sigma \), and \( = 0 \) otherwise)

\[
\sum_{N \neq \sigma} \mu(b) y_b = \sum_{N \neq \sigma} \mu(b) \sum_{\sigma \neq N} \lambda_b f(c)
= \sum_{N \neq \sigma} \mu(b) \sum_{\sigma \neq N} \lambda_b f(bc) = \sum_{N \neq \sigma} \mu(b) \sum_{\sigma \neq N} \lambda_b f(b) = \frac{\lambda_b}{f(c)}
\]

or

\[ \frac{\lambda_b}{f(c)} = \sum_{N \neq \sigma} \mu(b) y_b, \]

whence, putting \( \zeta = 0 \),

\[ 1 = \sum_{N \neq \sigma} \mu(b) y_b. \]

Now we choose the numbers \( \lambda_b \) (1 \( < N \sigma \leq \varepsilon \), \( \varepsilon \) constant) satisfying

\[ \frac{\mu(b)}{f(b)} S_b^{-1} = y_b, \]

i.e.

\[
\sum_{N \neq \sigma} \lambda_b \frac{\mu(b)}{f(b)} S_b^{-1} = y_b,
\]

whence, by the argument used in proving (25)

\[ \lambda_b = f(c) \sum_{N \neq \sigma} \mu(b) \frac{\mu(b)}{f(b)} S_b^{-1}. \]

If \( \sigma \) is a quadratfrei ideal (i.e. not divisible by a square \( \neq \sigma \)), then by (15), (14),

\[ F(c) = f(c) \prod_{p | \sigma} \left(1 - \frac{1}{f(p)}\right). \]

Owing to the factors \( \mu(c), \mu(b) \) we may suppose that the ideals \( c, b \) in (28) are quadratfrei and prime to each other. Then, by (29), \( F(c)F(b) = F(c)F(b) \) and we have

\[
\lambda_b = \mu(c) \left(\frac{1}{f(p)}\right) \sum_{N \neq \sigma} \mu(b) S_b^{-1} = \mu(c) \left(\frac{1}{f(p)}\right) S_b(c)/S_b
\]

(cf. (16)), which is (17). For \( \lambda_b \) satisfying this equation we have, by (24), (27), (28)

\[ U = \sum_{N \neq \sigma} F(b) y_b \frac{\mu(b)}{f(b)} S_b^{-1} = S_b^{-1} \sum_{N \neq \sigma} \mu(b) y_b = S_b^{-1}. \]

From this and (22) we obtain (18).

4. LEMMA 4. Let (12) be the set of ideals \( a \) of a class \( \mathfrak{S} \) having the norms \( Na \leq \varepsilon \), \( \varepsilon > D^a \), \( k = \frac{1}{2}(n+3) \), and let

\[ s \geq \max\{a^a, D^a\} \]

(30)

\[ \sum_{q \leq q} 1/N_q \leq 1 \]

(31)

for \( q \) running through all the prime ideals \( q \) of \( \mathfrak{S} \). Then the main term in (18) satisfies

\[ X/S_b < \varepsilon \log^a \varepsilon. \]

PROOF. Since \( \varepsilon < D \), we have, by (8), (1) (13),

\[ \nu(y, \mathfrak{S}) = \mu(y) + O(D^a (\varepsilon y)^{1-\varepsilon}) \quad \mu = \varepsilon^{-1} (\xi + z) > D^{-a} \]

if \( D > D_a \). Hence for

\[ y \geq y_b = D^a \]

the order of magnitude of the principal term in (32) is larger than that of the remaining term as \( D \to \infty \). By (32)

\[ X = \mu y + O(D^a (\varepsilon y)^{1-\varepsilon}). \]

Hence, by (11),

\[ \nu(z; \mathfrak{S}, b) = \mu z/N_b + O\left(D^a (\varepsilon y_b)^{1-\varepsilon} / N_b\right) \]

\[ = X/N_b + O\left(D^a (\varepsilon y_b)^{1-\varepsilon} / N_b\right). \]

Comparing with (13) we deduce

\[ f(b) = N_b, \quad B_b \\leq D^a (\varepsilon y_b)^{1-\varepsilon}. \]
Let \( \nu(y) \) denote (as in § 2) the number of all the ideals of the field \( K \)
prime to \( \mathfrak{f} \) and having the norm \( \leq y \). We have, by (32),
\[
(35) \quad \nu(y) > \frac{1}{2} h \mu y \quad \text{for} \quad y \geq y_0.
\]
By (16), (29), (34),
\[
(36) \quad S_2 = \sum_{\substack{N \leq y \leq \infty}} \sum_{\substack{N \leq k \leq \infty}} \frac{\mu(b)}{N} \left( 1 + \frac{1}{Np} + \frac{1}{Np^2} + \ldots \right)
= \sum_{\substack{N \leq y \leq \infty}} \frac{1}{N}.
\]
where \( (a) \) denotes the set of ideals \( b \) (prime to \( \mathfrak{f} \)) in \( K \) such that the product of all different prime factors of \( b \) is in norm \( \leq x \). Let \( g \) denote a typical ideal in \( K \) which is divisible by no other prime ideals than the \( q \) \( \mathfrak{f} \); then
\[
1 \leq \prod_{q \not| \mathfrak{f}} \left( 1 + \frac{1}{Nq} + \frac{1}{Nq^2} + \ldots \right) = \sum_{\substack{N \leq y \leq \infty}} \frac{1}{N} = V,
\]
say. Hence, by (31), \( V < 1 \). (If \( \mathfrak{f} \) is an empty set, then, by definition, \( V = 1 \)).
By (35), (36), (30), (1) (6),
\[
S_2 V > \sum_{\substack{N \leq y \leq \infty}} \frac{1}{N} \geq \frac{1}{N} = \int_{y_0}^{y} \frac{\nu(y) dy}{y^2} > \frac{1}{2} h \mu \log x
\]
(since \( z/y_0 > z/\log x \)). Considering that \( \log x > \frac{1}{2} \log x \) we deduce
\[
S_2 V > h \mu \log x,
\]
whence
\[
S_2 > h \mu \log x,
\]
(since \( V < 1 \)). Combining this with (33) (where the remaining term may be dropped, since \( x \geq y_0 \)) we obtain
\[
X/S_2 \ll \mu \nu h \mu \log x = \frac{z}{h \mu \log x},
\]
the desired result.

5. Lemma 5. Let \( W \) denote the remaining term in (18) and let in Lemma 4
\[
(37) \quad z^{1/k} = \frac{z^{1/k}}{\log x} \ll p^1;
\]
then
\[
W \ll \frac{z}{h \mu \log x}.
\]

On the series of Hecke’s L-functions III
Denoting the first term in this sequence of equations by \( S_1(x) \), we have

\[
\sum_{N_0, N_1 \in \mathcal{O}_x} \left( \frac{N_1(x, N_0)}{N_0} \right)^{1/\omega} \leq \sum_{N_0, N_1 \in \mathcal{O}_x} S_1(x) \leq (h \mu)^{2\omega_1} \sum_{N_0, N_1 \in \mathcal{O}_x} N_1^{1-1/\omega_2} \leq D^{1/2}(h \mu)^{2\omega_2},
\]

by (41). Hence, by (39), (37),

\[ W \leq D^{1/2}(h \mu)^{2\omega_2 - 1/k} e^{2h D} = x/h \log x. \]

This is the desired result.

Taking \( x > D^k \) we have (since \( k < D \), \( h \mu < D^{1/2} \))

\[ D^{1/2}(h \mu)^{2\omega_2} x^{1/k} \log x \leq x^{1/k} \]

whence, by (37),

\[ z \geq x^{1/k}, \quad z > D^k \]

which is (30).

6. **Lemma 6.** If \( \pi(x, \mathfrak{S}) \) denotes the number of prime ideals of the class \( \mathfrak{S} \) having the norm \( \leq x \), then

\[ \pi(x, \mathfrak{S}) \sim x/h \log x \quad \text{for} \quad x \gg D^k, \quad k = \frac{1}{4}(n + 3). \]

Proof. Let in Lemma 3 (12) be the set of ideals \( \mathfrak{a} \) of the class \( \mathfrak{S} \) with \( N_\mathfrak{a} \leq x \) and let \( \mathfrak{Q} \) be the set of all the ideals \( \mathfrak{a} \) in \( \mathfrak{E} \) (prime to \( f \)) such that \( y \leq N_\mathfrak{a} \leq y^2 \). Then, by [2] (4),

\[ \sum_{n} 1/N_\mathfrak{a} \leq \sum_{n \leq y^2} 1/p \leq 1, \]

which is (31). Among the \( N_\mathfrak{a} \) ideals (18) there are all the ideals \( \mathfrak{g} \in \mathfrak{S} \) with \( N_\mathfrak{g} \leq x \), whence (46) follows from Lemmas 4 and 5.

7. **Lemma 7.** Let \( V(x; \mathfrak{S}, y) \) denote the number of ideals \( \mathfrak{a} \) \( (N_\mathfrak{a} \leq x) \) of the class \( \mathfrak{S} \) which are divisible exclusively by the prime ideals \( \mathfrak{P} \) with \( y \leq N_\mathfrak{P} \leq y^2 \); then

\[ V(x; \mathfrak{S}, y) \ll x/h \log x \quad \text{for} \quad x \gg D^k, \quad k = \frac{1}{4}(n + 3), \quad D > D_0, \]

uniformly in \( 1 \leq y \leq y^2 \).

Proof. Let in Lemma 3 (12) be the set of ideals \( \mathfrak{a} \) of the class \( \mathfrak{S} \) with \( N_\mathfrak{a} \leq x \) and let \( \mathfrak{Q} \) be the set of all the ideals \( \mathfrak{a} \) in \( \mathfrak{E} \) (prime to \( f \)) such that \( y \leq N_\mathfrak{a} \leq y^2 \). Then, by [2] (4),

\[ \sum_{n} 1/N_\mathfrak{a} < \sum_{n \leq y^2} 1/p < 1, \]

An estimate for the number of functions \( \zeta(s, \chi) \) having a zero in the neighbourhood of \( s = 1 \)

\[ \frac{\log N_\mathfrak{a}}{N_\mathfrak{a}^{1/2}} \ll \log x \quad \text{for} \quad x > D^{1/2}, \quad D > D_0. \]

Proof. Writing \( x = D^{1/2} \) we have \( x > a_0 D \). By (43), [2] (2), [1] (6)

\[ \sum_{n \leq x} \log N_\mathfrak{a} < \sum_{n \leq x} \log p + \sum_{n \leq x} \log N_\mathfrak{a} \ll a_0 + \]

\[ + \sum_{n \leq x} \frac{\pi(n, \mathfrak{S})}{y} \log x \ll a_0 + \pi(x, \mathfrak{S}) \log x \ll x/h \]

and

\[ \sum_{n \leq x} \log N_\mathfrak{a} \ll x/h \sum_{n \leq x} \log p \ll y^2 \log x \ll x/h \]

(since \( h < D \)), whence (44).
Then for appropriate constant \( C_1 > 0 \) (which depends only on \( n \))

\[
Q < \exp(C_1 \lambda).
\]

The proof of this lemma is the object of the rest of this paper.

For \( \lambda \gg \log D \) (49) is evidently true, since then \( \lambda \) (which is the total number of the functions \( \zeta(\sigma, \chi) \)) does not exceed \( \exp(\lambda) \). And if \( \lambda < 1 \), then taking \( C_1 \) large enough (49) holds for all \( \lambda > c \). Hence it remains to prove the lemma for the range \( \lambda = \log D \) assuming \( \lambda > 16 \) (say) in which case the functions \( \zeta(\sigma, \chi_0, \zeta(\sigma, \chi') \) may be left out. By \( Q' \) we denote the number of the functions \( \zeta(\sigma, \chi) \) with \( \chi \neq \chi_0 \), \( \chi' \) having a zero in \( B_\sigma \). To these functions we may apply Lemma 8. By \( Q \), and \( Q_1 \) we denote the numbers of the functions \( \zeta(\sigma, \chi) \) for which there holds respectively the conclusion (I) or (II) of that lemma.

8. Let us suppose first that

\[
Q_2 \geq \frac{1}{4} Q'.
\]

The rectangles \( R_2 \) (where \( r \) depends on \( \chi \)) lie in the region

\[
R_2(1 + \frac{1}{2} a \leq \sigma \leq 1 + \frac{1}{2} a, \quad |\chi| < a e^{\log D})
\]

(for a suitable \( a < 1 \)) which can be covered by a set of

\[
N \leq \alpha_2 \exp(1 + 2a_1) \lambda
\]

\( \alpha_2 \) defined by (48) \) congruent squares with the side of the length

\[
\eta = 1/b \exp(\alpha_1) \log D
\]

running parallel to the axis. Then there is at least one square, \( R_2 \), say, in which there lie

\[
Q_1 \geq Q_2 \eta N
\]

of the points \( s = s_0(\chi) \). By (48), (1), (6), (2), in \( R_2 \)

\[
|F'(s)| \leq \alpha_1 \sum_{p \leq \log D} \frac{\log p}{p^{1+\epsilon}} \leq \alpha_1 \int_{\log D}^{\infty} \frac{x \log x}{x^{1+\epsilon}} \, dx
\]

\[
= \alpha_1 (1 - e^{-\epsilon}) \log^2 D + \frac{x^2 e^{-\epsilon}}{\epsilon} \log D < \alpha_1 (1 - e^{-\epsilon}) \log^2 D.
\]

Hence by \( \delta_2 \) denoting the centre of \( R_2 \), we have, by (48),

\[
|F'_2(s_0)| \leq |F'(s_0)| - |F'(s_0) - F'(s)| \geq |F'(s_0)| - \int_{s_0}^{s} |F'(s)| \, ds
\]

\[
\geq \exp(-\alpha_1 \log D - \epsilon \text{max}_{s \in R_2} |F'(s)|) > \frac{1}{2} \exp(-\alpha_1 \log D).
\]

If \( \lambda < 1 \) is large enough), whence

\[
|F'(s_0)| > \frac{1}{2} \exp(-\alpha_1 \log D).
\]

Now we add (53) over the \( Q_0 \) characters for which this inequality holds and compare with the corresponding sum over all \( \chi \) in which we use (1). Dividing through by \( \log D \) and using (50), (51), (52), (48), (45), (1) (6) we get the inequalities

\[
Q_1 \lambda \exp(-1 + 4a_1) \lambda < Q_1 \lambda \exp(-2a_1) \lambda
\]

\[
< \sum_{p} \frac{1}{\log D} \sum_{\lambda \chi p = \lambda \chi^*} \chi(p) N_p^{-a_0} \log N_p
\]

\[
\leq \frac{h}{\log D} \sum_{\lambda \chi p = \lambda \chi^*} \log N_p \cdot \log N_p
\]

\[
\left( N_p \cdot \log N_p \right)^{1+\epsilon}
\]

\[
\leq \frac{h}{\log D} \sum_{\lambda \chi p = \lambda \chi^*} \left( \sum_{\lambda \chi p = \lambda \chi^*} \log N_p \right)^2 < \alpha_1 (1 - e^{-\epsilon}) \log D
\]

(since

\[
\sum_{\lambda \chi p = \lambda \chi^*} \log N_p
\]

\[
\leq \int_{\frac{h}{\log^2 D}}^{\infty} \frac{x \log x}{x^{1+\epsilon}} \, dx \leq \frac{1}{2} \log D
\]

whence \( Q' < \exp(\alpha_1) \). This proves the lemma in the case of (50).

9. Now we suppose that

\[
Q_1 \geq \frac{1}{4} Q'.
\]

The \( Q_1 \) rectangles \( R_1 \) (with \( r_1 \) depending on \( \chi \)) lie in the region

\[
R_1(1 - a < \sigma < 1 + 3a, \quad |\chi| < a e^{\log D})
\]

which can be covered by a set of

\[
N_1 < \alpha_1 \exp(1 + 12a_1) \lambda
\]

congruent squares having the length of the side

\[
\eta = 1/b \exp(6a_1 \lambda) \log D.
\]

There is at least one square, \( R_1 \), say, in which there lie

\[
Q_1 \geq Q_1 N_1
\]

of the points \( s = s_1(\chi) \). By (47), (2) (3), we have in \( R_1 \)

\[
|F'(s)| \leq \alpha_1 \sum_{p \leq \log D} \frac{\log p}{p^{1+\epsilon}} < \alpha_1 e^{\epsilon_1}
\]

\[
\sum_{p \leq \log D} \frac{\log^2 p}{p} < \alpha_1 e^{\epsilon_1} \log D.
\]
Hence, by \( s_n \) denoting the centre of \( R_n \), we have, by (47)

\[
|F_1(s_n)| \geq |F_1(s_n)| - |F_1(s_n) - F_2(s_n)| > \log D - \int F_1(s) ds
\]

\[
> \log D - \frac{\gamma_0^2 e^{\theta_1 s}}{\log D} \log D - \frac{1}{2} \log D = \frac{1}{2} \log D
\]

(supposing \( b < 1 \) large enough).

Now let the ideals \( p \) with \( Z < N = p < D^a = M \), \( p \) be distributed into sets \( S_1, \ldots, S_r \), the set \( S_r \) containing all the \( p \) satisfying

\[(57) \quad 2^{-r} \log M < \log N \leq 2^{-r+1} \log M, \quad r < \epsilon_0 \log (2 + 2) \]

Then there is at least one set \( S_j = S \), say, such that for at least

\[(58) \quad Q_{nl} > \epsilon_0 Q
\]

of the characters \( \chi \) we have

\[(59) \quad \left| \sum_{n \leq x} x(p) \left( Np - \frac{n}{\log Np} \right) \right| > \frac{1}{2} \log D.
\]

Raising to the power \( 2^j+1 \) we get the inequality

\[(60) \quad \left| \sum_{M < N < M^j} \chi(\epsilon) a_N \right| > (\log D)^{j+1} \exp(-\epsilon_0)\]

where

\[
(61) \quad 0 < \epsilon_0 < 2^j \int \sum_{p | N} \left( \frac{N}{p} \right) \log Np < 2^j (\log M)^j \log D < (\log D)^{j+1} \exp(\epsilon_0).
\]

Let \( a \) and \( b \) denote ideals all the prime divisors of which are in the set \( S \) defined by (57), (58). Then we have, by (61), (1),

\[
(62) \quad \sum_{M < N < M^j} \left| \sum_{M < N < M^j} \chi(\epsilon) a_N \right| < (\log D)^{j+1} \exp(3\epsilon_0) \sum_{M < N < M^j} Na^{-1} \sum_{3 < b < N} Nb^{-1} < (\log D)^{j+1} \exp(\epsilon_0),
\]

since, by (46), [1] (6),

\[
\sum_{M < N < M^j} Nb^{-1} < \int_{k=1}^{M^j} \frac{x}{k} \log x^2 + \frac{M^j \log M^j}{M^j} < 1/k,
\]

\[
\sum_{M < N < M^j} Na^{-1} = h \sum_{M < N < M^j} Nb^{-1} < 1.
\]

Summing (59) over the corresponding characters, comparing with (62) and dividing through by \( (\log D)^{j+1} \) we obtain, by (54), (55), (56), (60),

\[
Q_{n} \exp(-\epsilon_0) \sum_{n < \log (2 + 2) \log (1 + 12\lambda n)} < Q_{2} \exp(-\epsilon_0) < \exp(\epsilon_0)
\]

whence \( Q < \exp(\epsilon_0) \). This proves the lemma.

The theorem of §1 is an immediate consequence of Lemma 9 and [1] (43).

Note. In the original method of Lindelöf and Rosowskii for Dirichlet's L-functions (see [9] p. 339, [7] p. 173 or [8] X (2.66)) some numbers \( \nu \), corresponding to \( N \) in (62) are treated as numbers of arithmetical progression \( Du + 1 \) with \( (D, i) = 1 \) having no prime divisor \( < Z \). An estimate

\[
(63) \quad \nu(x) < x(\log D)^{2} (x > D^2)
\]

for the number \( \nu(x) \) of such numbers \( \nu \leq x \) has been proved using the Brun's sieve method (see [7] p. 173). The corresponding result for an algebraic field \( K \) of degree \( n \geq 3 \) may not be true (7) and the method does not work. The principal difficulty to overcome in proving the theorem of this paper consisted in the construction of a suitable substitute for (63) (7).

References


(7) Take, for example, \( D = |D|, Z = D^2 \) with \( \varepsilon > 0 \) arbitrarily small and suppose that all the prime ideals \( p < K \) with \( Np < Z \) have the degrees \( f \geq 2 \) (this has never been disproved). Then the factor \( \log D \) in the denominator disappears, since in (58) we have

\[
S_{n} < \sum_{n \leq x} \frac{1}{n} < 1 \quad (\text{cf. (2), (19)}).
\]

(7) I found the way to (60) in March 1940 when there was not yet Turin's paper [11] which makes (60) superfluous for the proof of the present theorem. (Added in proof 8th February 1940.)
On polynomial transformations

by

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1. We shall say that subset $X$ of a field $R$ has property (P) if every polynomial $P(x)$ with coefficients from $R$ such that $P(X) = X$ is linear. It is easy to see that any number field in which the "Irreduzibilitätssatz" of Hilbert is true has property (P). Consequently, any algebraic extension of the field of rational numbers has property (P) and any number field which is transcendental extension of some (its) infinite subfield also has this property. (E.g. see [1], [3]). On the other hand, it is trivial that no finite set has property (P). The problem can be posed, having a fixed number set $Z$, to characterize the subsets of $Z$ with property (P). In this paper we solve this problem in the case where $Z$ is an algebraic number field. (By an algebraic number field we always understand a finite algebraic extension of the field of rational numbers.) Indeed, we shall prove

**Theorem 1.** A subset $X$ of an algebraic number field has property (P) if and only if it is finite.

We shall say that a set $Z$ has property (P) hereditarily if every infinite subset of $Z$ has property (P). Thus algebraic number fields have property (P) hereditarily. It turns out that also every finitely generated transcendental extension of an algebraic number field has property (P) hereditarily. This follows from

**Theorem II.** Let $K$ be a finitely generated transcendental extension of a field $R$. Then $K$ has hereditarily property (P) if and only if $R$ has this property. (The "only if" parts of our theorems are of course trivial.)

2. For the proof of our theorems we need the following

**Lemma 1.** Suppose that $T(x)$ is a transformation of the set $X$ onto itself. Suppose that there exist two functions $f(x)$ and $g(x)$ defined on $X$, with values in the set of natural numbers, subject to the conditions:

(a) For every constant $c$ the equation $f(x) + g(x) = c$ has only a finite number of solutions,

(b) There exists a constant $C$ such that from $f(\alpha) > C$ follows $|T(x)| > f(\alpha)$.

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