is solvable in \( F \). Hence (27) reduces to
\[
f(x, \ldots, a, d) = \sum_{j=0}^{k} b_j x_j^j + c_n x_n^k + d,
\]
where clearly
\[
\psi(b_j) = \lambda_j \quad (j = 1, \ldots, k), \quad \psi(c_n) = \lambda_{k+1},
\]
and the induction is complete.

Reference


DUKE UNIVERSITY

Reçu par la Redaction le 9. 3. 1961

ACTA ARITHMETICA
VII (1962)

Congruence properties of certain linear homogeneous difference equations

by

L. CARLITZ (Durham, North Carolina)

1. Introduction. In a recent paper [1] the writer considered the recurrence

\[
u_{n+1} = f(n)u_n + g(n)u_{n-1},
\]

where \( f(n), g(n) \) are polynomials in \( n \) (and possibly some additional indeterminates) with integral coefficients. It was assumed that

\[
\begin{align*}
    u_0 &= 1, & u_1 &= f(0), & g(0) &= 0.
\end{align*}
\]

The main result of [1] is contained in the congruence

\[
\sum_{r=0}^{s} (-1)^r \binom{s}{r} u_{s+r} = 0 \pmod{m^r},
\]

for all \( s \geq 0, m \geq 1, r \geq 1 \) and where

\[
t_r = \lceil (r + 1)/2 \rceil,
\]

the greatest integer \( \leq (r + 1)/2 \). Indeed, to get (1.3) it is only necessary to assume that the coefficients of the polynomials \( f(n), g(n) \) are integral \( \pmod{m} \).

A number of applications of (1.2) were given, in particular to the polynomials of Hermite and Laguerre.

It seems natural to consider the recurrence

\[
u_{n+1}^{(k)} = a_0(n)u_n^{(k)} + a_1(n)u_{n-1}^{(k)} + \cdots + a_k(n)u_{n-2}^{(k)}
\]

of order \( k+1 \), where the \( a_j(n) \) are polynomials in \( n \) with integral coefficients. Corresponding to (1.2) we now assume that

\[
a_j(n) = 0 \quad (j = 0, 1, \ldots, f-1, j = 1, \ldots, k);
\]

also we suppose that (1.5) holds for all \( n \geq 0 \). In view of (1.6) it is not necessary to explicitly define \( u_{n+1}^{(k)}, \ldots, u_{n+k}^{(k)} \). We take \( u_0^{(k)} = 1 \) and it follows that

\[
u_0^{(k)} = a_0(0), \quad u_1^{(k)} - a_0(1)u_1^{(k)} + a_1(1), \quad etc.
\]
We shall show that \( u_n^{(q)} \) satisfies the congruence
\[
(1.7) \quad \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{u_j^{(q)}}{u_m^{(q)}(u_m^{(q)})^{j}} = 0 \quad (\text{mod } m^n),
\]
for all \( n \geq 0 \), where \( q \) is defined by (1.4). For a somewhat more general result see Theorem 4 below.

A number of applications of (1.7) are discussed. For example the sequence \( u_n \) defined by
\[
\exp \left( a_0 + a_1 \frac{T}{2} + \cdots + a_{k-1} \frac{T^{k-1}}{k-1} \right) = \sum_{n=0}^{\infty} u_n \frac{T^n}{n!},
\]
where the \( a_i \) are integers (or polynomials in several indeterminates with integral coefficients), is shown to satisfy a recurrence of the form (1.5). Moreover the final result is made more explicit by determining the residue of \( u_n \mod m \), when certain conditions are satisfied (see Theorems 8 and 9 below). In the second place the sequence \( (u_n) \) defined by means of
\[
(1 - at)^{-s}(1 - bt)^{-t}(1 - ct)^{-t} = \sum_{n=0}^{\infty} u_n \frac{T^n}{n!}
\]
also satisfies a recurrence of the form (1.5) of order three; the number of factors on the left side can be increased. Finally some more special applications are discussed in \$7\$.

2. As in [1] we consider in place of (1.5) the recurrence of order \( k + 1 \):
\[
(2.1) \quad u_{n+1}^{(q)}(x) = \left( x + a_{n+1}(x) \right) u_n^{(q)}(x) + \sum_{j=1}^{n} a_j(x) u_{n-j}^{(q)}(x),
\]
where as before the \( a_j(x) \) are polynomials in \( x \) with integral coefficients; the \( a_j(x) \) may contain additional indeterminates but are independent of \( x \). We assume that conditions (1.6) are satisfied and that (2.1) holds for all \( n \geq 0 \). We take \( u_0^{(q)}(x) = 1 \) and it follows that
\[
u_1^{(q)}(x) = x + a_0(x), \quad \nu_2^{(q)}(x) = \left( x + a_1(x) \right) u_1^{(q)}(x) + a_1(x), \quad \text{etc.}
\]
Clearly \( u_n^{(q)}(x) \) is a polynomial in \( x \) of degree \( n \) with highest coefficient equal to \( 1 \).

We show first that if \( m \) is an arbitrary integer \( \geq 1 \) then
\[
(2.2) \quad u_m(x) u_n(x) = u_{m+n}(x) \quad (\text{mod } m^n),
\]
for all \( n \geq 0 \), where for brevity we put
\[
(2.3) \quad u_n(x) = u_n^{(q)}(x).
\]
We may state

**Theorem 1.** Let \( u_s(x) \) be the sequence of polynomials defined by (2.1) and (1.6), where the \( a_s(n) \) are polynomials in \( n \) with integral coefficients. Then
\[
u_s(x) u_m(x) = u_{s+m}(x) \quad \text{(mod \( m \))}
\]
for all \( s \geq 0, m \geq 1 \).

A little more generally we have

**Theorem 2.** Let \( u_s(x) = u_s^{(0)}(x) \) be the sequence of polynomials defined by (2.1) and (1.6), where the \( a_s(n) \) are polynomials in \( n \) with coefficients integral (mod \( m \)). Then we have
\[
u_s(x) u_m(x) = u_s(x) u_m^t(x) \quad \text{(mod \( m \))}
\]
for all \( s \geq 0, t \geq 1, m \geq 1 \).

An immediate corollary is contained in

**Theorem 3.** Let \( u_s = u_s^{(1)} \) be the sequence defined by (1.5) and (1.6), where the \( a_s(n) \) are polynomials in \( n \) with coefficients integral (mod \( m \)). Then
\[
u_s(x) u_m(x) = u_s(x) u_m^{(1)}(x) \quad \text{(mod \( m \))}
\]
for all \( s \geq 0, t \geq 1, m \geq 1 \).

3. It follows from (2.1) that
\[
u_s(x) = u_s(x) - \sum_{i=0}^s a_i(n) u_{s-i}(x).
\]
Repeated application of (3.1) leads to
\[
u_s(x) = \sum_{i=0}^s A_s(n) u_{s+i}(x)
\]
where in the summation we may suppose \( j \geq -n \); the \( A_s(n) \) are polynomials in \( n \) with coefficients integral (mod \( m \)). It follows that
\[
u_s(x) u_m(x) = \sum_{i=0}^m B_s(n) u_{s+i}(x)
\]
where the \( B_s(n) = B_s(n; t, m) \) are also polynomials in \( n \) with coefficients integral (mod \( m \)). Also we may assume that in the summation \( j \geq -n \), or what is the same thing
\[
u_s(x) = \sum_{i=0}^m B_s(n) u_{s+i}(x)
\]

We shall require the

**Lemma.** Let \( u_s(x), u_s^{(0)}(x), u_s^{(1)}(x), \ldots, u_s(n) \) denote a set of polynomials in \( x \) with coefficients integral (mod \( m \)) and highest coefficients equal to 1. Also let
\[
\text{deg} u_s(x) = s \quad (0 \leq s \leq N).
\]

Let \( A_s, A_{s+1}, \ldots, A_N \) be integral (mod \( m \)) and such that
\[
\sum_{i=0}^m A_i u_s^{(i)} = 0 \quad \text{(mod \( m \))}.
\]

Then it follows that
\[
A_s = 0 \quad \text{(mod \( m \))} \quad (0 \leq s \leq N).
\]

For the proof see [1], p. 151.

By (2.3) and (3.1) we have
\[
\sum_{j=-m}^m B_j(n) u_{s+j}(x) = 0 \quad \text{(mod \( m \))}
\]
Applying the Lemma to (3.4) we get
\[
B_j(n) = 0 \quad \text{(mod \( m \))} \quad (-km \leq j \leq km).
\]

We now define the operator \( \Delta \) by means of
\[
\Delta u_s = u_s^{(0)}(x) \psi_n - u_{s+1}(x)
\]
where \( t \) and \( m \) are fixed integers \( \geq 1 \). More generally we define for \( r \geq 1 \)
\[
\Delta^r u_s = u_s^{(r)}(x) \psi_{n-r} - \Delta^{r-1} u_{s+1}(x)
\]
In (3.6) and (3.7) \( \psi_n \) is an arbitrary function of \( n \). It follows from (3.7) that
\[
\Delta^r u_s = \sum_{j=-m}^m (-1)^j \binom{r}{j} u_s^{\Delta^{r-j}}(x) \psi_{n-j}
\]

If we apply \( \Delta^{-1} \) to (3.9) we get
\[
\Delta u_s = \sum_{j=-m}^m \Delta^{-1}[B_s(n) u_{s+j}(x)]
\]

In addition to the operator \( \Delta \) we shall require also the operator \( \Delta' \) defined by
\[
\Delta' u_s = \sum_{j=-m}^m (-1)^j \binom{r}{j} \psi_{n+j}
\]
Cleary (3.10) is equivalent to
\[
\Delta' u_s = \sum_{j=-m}^m (-1)^j \binom{r}{j} \psi_{n+j}
\]
Returning to (3.9) we get
\[ d^{r-1} \{ B_{r}(n) u_{n+t+\Delta n}(x) \} = \cdots \]

\[ = \sum_{t=0}^{r-1} (-1)^{t} \binom{r-1}{t} u_{n+t-\Delta n}(x) B_{r-t-\Delta n}(x) \sum_{s=0}^{r-1} (-1)^{s} \binom{r-1}{s} \delta B_{s}(n) \]

\[ = \sum_{t=0}^{r-1} (-1)^{t} \binom{r-1}{t} \delta B_{r}(n) \sum_{s=0}^{r-1} (-1)^{s} \binom{r-1}{s} u_{n+t-\Delta n}(x) u_{n+t+\Delta n}(x) \]

\[ = \sum_{t=0}^{r-1} (-1)^{t} \binom{r-1}{t} \delta B_{r}(n) \sum_{s=0}^{r-1} (-1)^{s} \binom{r-1}{s} u_{n+t-\Delta n}(x) u_{n+t+\Delta n}(x) \]

Substituting in (3.9) we get

\[ d^{r} u_{n}(x) = \sum_{r=0}^{\infty} \sum_{t=0}^{r} \binom{r}{t} (-1)^{t} \delta B_{t}(n) \cdot d^{r-t} u_{n+t+\Delta n}(x) \]

We shall now prove by an induction on \( r \) that

\[ d^{r} u_{n}(x) = 0 \quad (\text{mod } m^{r}) \]

for all \( r \geq 1 \), where \( r \) is defined by (1.4). The case \( r = 1 \) is contained in (2.4). We accordingly assume that (3.13) holds and include the value \( r = 1 \). Since \( B_{r}(n) \) is a polynomial in \( n \) with coefficients integral (mod \( m \)), it follows from (3.10) and the elements of finite differences that

\[ \delta B_{r}(n) = 0 \quad (\text{mod } m^{r}) \]

Consider the typical product

\[ A_{r} = \delta B_{r}(n) \cdot d^{r-1} u_{n+t+\Delta n}(x) \]

occurring in the right member of (3.12). For \( i = 0 \) it follows from (3.5) and the inductive hypothesis that

\[ A_{0} = 0 \quad (\text{mod } m^{i+r(n)}) \]

For \( i \geq 1 \) it follows from (3.14) and the inductive hypothesis that

\[ A_{i} = 0 \quad (\text{mod } m^{i+r(n)}) \]

Since

\[ 1 + [r/2] \geq [(r+1)/2] \]

\[ i + [(r-1)/2] \geq [(r+1)/2] \quad (1 \leq i < r) , \]

it is evident from (3.12), (3.15), (3.16) and (3.17) that

\[ d^{r} u_{n}(x) = 0 \quad (\text{mod } m^{r}) \]

We may now state the main result.

**Theorem 4.** Let \( u_{n}(x) = u_{n}^{(0)}(x) \) be the sequence of polynomials defined by (2.1) and (1.6), where the \( a_{r}(n) \) are polynomials in \( n \) with coefficients integral (mod \( m \)). Then we have

\[ \sum_{t=0}^{r} (-1)^{t} \binom{r}{t} u_{m}^{(r-t)}(x) u_{n+t+\Delta n}(x) = 0 \quad (\text{mod } m^{r}) \]

for all \( n \geq 0 \), \( r \geq 1 \), \( t \geq 1 \), \( m \geq 1 \).

**Theorem 5.** Let \( u_{n} = u_{n}^{(0)} \) be the sequence defined by (1.5) and (1.6), where the \( a_{r}(n) \) are polynomials in \( n \) with coefficients integral (mod \( m \)). Then

\[ \sum_{t=0}^{r} (-1)^{t} \binom{r}{t} u_{m}^{(r-t)}(x) u_{n+t+\Delta n}(x) = 0 \quad (\text{mod } m^{r}) \]

for all \( n \geq 0 \), \( r \geq 1 \), \( t \geq 1 \), \( m \geq 1 \).

**Remark.** By making very slight changes in the proof we can replace (3.18) and (3.19) by

\[ \sum_{t=0}^{r} (-1)^{t} \binom{r}{t} u_{m}^{(r-t)}(x) u_{n+t+\Delta n}(x) = 0 \quad (\text{mod } m^{r}) \]

respectively.

**4. We state**

**Theorem 6.** If the hypothesis of Theorem 4 are satisfied then

\[ \sum_{t=0}^{r} (-1)^{t} \binom{r}{t} u_{m}^{(r-t)}(x) u_{n+t+\Delta n}(x) = 0 \quad (\text{mod } m^{r}) \]

for all \( n \geq 0 \), \( t \geq 1 \). In particular

\[ \sum_{t=0}^{r} (-1)^{t} \binom{r}{t} u_{m}^{(r-t)}(x) u_{n+t+\Delta n}(x) = 0 \quad (\text{mod } m^{r}) \]

For a more general result we put

\[ u_{m}^{(r)}(x) = u_{m}^{(0)} u_{m}^{(1)} \ldots u_{m}^{(r)}(x) \]

where it is understood that after expanding the right member by the multinomial theorem, each \( u_{m}^{(r)}(x) \) is replaced by \( u_{m+t+\Delta n}(x) \).
Theorem 7. Let $\lambda_1, \ldots, \lambda_s$ be integral (mod $m$) and such that
\begin{equation}
\lambda_1 + \cdots + \lambda_s = 0 \pmod{m}.
\end{equation}
Then if the hypothesis of Theorem 4 are satisfied it follows that
\begin{equation}
\sum_{n=0}^{m-1} \chi_{\lambda_1 m, \ldots, \lambda_s m} = 0 \pmod{m^r},
\end{equation}
for all $n_1 \geq 0, \ldots, n_r \geq 0, r \geq 1, t \geq 1, m \geq 1$.

The proof of this result is exactly the same as the proof of Theorem 2
of [1] and will be omitted.

5. We now discuss several applications of the above results. To
begin with, consider the sequence $(u_n)$ defined by
\begin{equation}
\exp \left( a_0 + a_1 \frac{t}{2} + \cdots + a_k \frac{t^{k+1}}{k+1} \right) = \sum_{n=0}^{\infty} u_n \frac{t^n}{n!}.
\end{equation}
Differentiation with respect to $y$ yields
\begin{equation}
(a_0 + a_1 \frac{t}{2} + \cdots + a_k \frac{t^{k+1}}{k+1}) \sum_{n=0}^{\infty} u_n \frac{t^n}{n!} = \sum_{n=0}^{\infty} u_{n+1} \frac{t^n}{n!},
\end{equation}
so that
\begin{equation}
u_{n+1} = a_0 u_n + a_1 n u_{n-1} + a_2 n(n-1) u_{n-2} + \cdots + a_k n(n-1)\cdots(n-k+1) u_{n-k}.
\end{equation}

In particular when $k = 1$, (5.1) includes the familiar generating function for the Hermite polynomials $\exp \left( 2at - t^2 \right)$.

In the general case we may put
\begin{equation}
u_n = C_\lambda(t_0, a_0, \ldots, a_k),
\end{equation}
where $C_\lambda$ is the cycle indicator of the symmetric group ([8], p. 68); note however that in the general definition of $C_\lambda$ the number of indeterminates is not limited.

If the coefficients $a_0, a_1, \ldots, a_k$ are integers or rational numbers integral (mod $m$), it is clear that Theorem 5 applies and we get
\begin{equation}
\sum_{n=0}^{m-1} (-1)^n \left( \frac{m}{n} \right)^{k+1} \chi_{\lambda_1 m, \ldots, \lambda_s m} = 0 \pmod{m^{r+1}},
\end{equation}
for all $n \geq 0$.

However in the present situation we can make (5.4) more explicit by determining the residue (mod $m$) of $u_m$. Indeed it follows from (5.1) that
\begin{equation}
u_m = \sum_{n \geq 0} \frac{m! a_0 a_1 \cdots a_k}{n_1! n_2! \cdots n_{k+1}!} (k+1)^{n_{k+1}}.
\end{equation}

where the summation is extended over all non-negative $n_i$ such that
\begin{equation}n_1 + 2n_2 + \cdots + (k+1)n_{k+1} = m.
\end{equation}

If
\begin{equation}(m, (k+1)! \mid 1
\end{equation}
it follows easily from (5.5) and (5.6) that
\begin{equation}u_m = a_0^m \pmod{m}.
\end{equation}
Therefore (5.4) reduces to
\begin{equation}\sum_{n=0}^{m-1} (-1)^n \left( \frac{m}{n} \right)^{k+1} \chi_{\lambda_1 m, \ldots, \lambda_s m} = 0 \pmod{m^{r+1}}.
\end{equation}

In passing from (5.4) to (5.9) we have made use of the easily proved lemma that if
\begin{equation}a_m = \lambda \pmod{m}
\end{equation}
then (5.5) is equivalent to
\begin{equation}\sum_{n=0}^{m-1} (-1)^n \lambda^{m(n-\chi_{\lambda_1 m, \ldots, \lambda_s m}} = 0 \pmod{m^{r+1}}.
\end{equation}

We may state

Theorem 8. If $m$ satisfies (5.7) then the sequence $(u_n)$ defined by (5.1) satisfies (5.9) for all $m \geq 0$. The coefficients $a_0, a_1, \ldots, a_k$ are either integral (mod $m$) or polynomials in an arbitrary number of indeterminates with coefficients integral (mod $m$).

We remark that if $a_k$ is divisible by $j+1$ for $j = 0, 1, \ldots, k$ then Theorem 8 applies without any restriction on $m$. In other words the sequence $(u_n)$ defined by
\begin{equation}u_{n+1} = b_0 u_n + b_1 n u_{n-1} + b_2 n(n-1) u_{n-2} + \cdots + (k+1) b_{k-1} n(n-1)\cdots(n-k+1) u_{n-k},
\end{equation}
where the $b_j$ are integral (or polynomials with integral coefficients), satisfies (5.9) for all $m \geq 1$.

When (5.7) is not satisfied we can no longer assert (5.8). However in some cases it is still possible to obtain explicit results for the residue of $u_m$. For example if
\begin{equation}m = p = k+1,
\end{equation}
where $p$ is a prime, we find that (5.5) reduces to
\begin{equation}u_m = a_0^{p} - a_{p-1} \pmod{p}.
\end{equation}

To extend this result we put
\begin{equation}u_n = C_\lambda(t_0, a_0, \ldots, a_{p-1}),
\end{equation}
\begin{equation}v_n = C_\lambda(t_0, a_0, \ldots, a_{p-2}),
\end{equation}
\begin{equation}v_n = C_\lambda(t_0, a_0, \ldots, a_{p-3}).
\end{equation}
the notation being that of (5.3). Then by (5.1)
\begin{equation}
(5.12)
\sum_{m=0}^{1} \frac{m!}{s!((m-s)p)!} p^{r} r_{p-1} m_{m-r} .
\end{equation}
We take \( m = p^e \), where \( e \geq 1 \). Let \( p^f \) be the highest power of \( p \) dividing \( s \); then it is easily verified that the coefficient
\begin{equation}
(5.13)
A_e = \frac{m!}{s!((m-s)p)!} p^{f-1} .
\end{equation}
is divisible by exactly \( p^{f-1-1} \). Since \( m-sp \) is divisible by \( p^{f-1} \), it follows from (2.5) and (5.8) that
\begin{equation}
(5.14)
v_{m-s} = a_{e-s}^{m-s} \pmod{p^{f+1}},
\end{equation}
so that
\begin{equation}
(5.15)
A_r \equiv \frac{m!}{s!((m-s)p)!} p^{f+1} \pmod{p^f} .
\end{equation}

In the next place, since
\begin{equation}
(5.16)
\frac{m!}{s!((m-s)p)!} p^{f+1} = \left( \frac{p^{f+1}}{e} \right) \pmod{p^f}
\end{equation}
this follows from \((x+y)^{p^f} = \frac{(x+y)^p}{p^f} \pmod{p^f} \),
\begin{equation}
(5.17)
\frac{m!}{s!((m-s)p)!} = \sum_{i+j+k=n} h = (-1)^j \pmod{p^f} .
\end{equation}

for \( p \geq 2 \) (by the generalized Wilson theorem), we get
\begin{equation}
(5.18)
A_e = \left( \frac{p^{f+1}}{e} \right) \pmod{p^f} .
\end{equation}
Thus by (5.12), (5.13) and (5.14) it follows that
\begin{equation}
(5.19)
u_{p^e} = (a_{e}^{p-e})^{p^{e}} \pmod{p^{e}}
\end{equation}
provided \( k = p-1 \), \( p > 2 \).

We may state
\begin{equation}
\text{Theorem 9. If } \, k = p-1, \, p|m, \, e \geq 1, \, p > 2, \text{ then}
\end{equation}
\begin{equation}
\sum_{e=0}^{r} (-1)^{r} \frac{m!}{s!((m-s)p)!} p^{r-1} r_{p-1} m_{m-r} = 0 \pmod{p^{m}}
\end{equation}
for all \( n \geq 0 \). When \( e = 1, \) (5.16) holds for all \( p \). The coefficients \( a_0, a_1, ..., a_{p-1} \) are either integral \( \pmod{p} \) or polynomials in an arbitrary number of indeterminates with coefficients integral \( \pmod{p} \).

6. As a second application we consider the sequence defined by means of
\begin{equation}
(6.1)
(1-az)^{s1}(1-bz)^{s2}(1-cz)^{s3} = \sum_{n=0}^{\infty} u_{n} p^{\frac{n}{m}}
\end{equation}
where
\begin{equation}
(6.2)
u_{n} = v_{n}(a, b, c; z)
\end{equation}
It follows easily from (6.1) that
\begin{equation}
(6.3)
u_{n} = \sum_{i+j+k=n} \frac{m!}{i!j!k!} a^{i} b^{j} c^{k} z^{i+j+k}
\end{equation}
where
\begin{equation}
(6.4)(a_i) = a(a+1) ... (a+i-1), \, \, (a)_0 = 1
\end{equation}
and the summation is over all non-negative integers \( i, j, k \) such that \( i+j+k = n \). Thus \( u_{n} \) is a polynomial in the six variables \( a, b, c, \), with integral coefficients.

The generating function
\begin{equation}
(1-az)^{s1}(1-bz)^{s2}(1-cz)^{s3}
\end{equation}
has received some attention (see for example [4] and [5], vol. 5, p. 248).

For simplicity we confine ourselves to the case of three factors; the more general case can be handled without difficulty.

Differentiating (6.1) with respect to \( z \) we get
\begin{equation}
\sum_{n=0}^{\infty} u_{n+1} p^{\frac{n}{m}} = \left( \frac{az}{1-at} + \frac{by}{1-bt} + \frac{cz}{1-ct} \right) \sum_{n=0}^{\infty} u_{n} p^{\frac{n}{m}}
\end{equation}
If we put
\begin{equation}
(6.5)
A = \frac{az}{1-at} + \frac{by}{1-bt} + \frac{cz}{1-ct} = \frac{A_0 + Bt + Cz}{1-at + bt + cz}
\end{equation}
it follows that
\begin{equation}
(6.6)
u_{n+1} = (a+b+c)u_{n} + (a+c+ab)u_{n-1} - abcn(n-1)(n-2)u_{n-3}
\end{equation}
This is evidently of the form (1.5). Hence if the parameters \( a, b, c, \) \( a, b, c \) are integral \( \pmod{m} \) or indeterminates (or polynomials with integral coefficients) Theorem 5 applies. Moreover \( u_{n} \) is explicitly determined by (6.3). In certain cases (6.3) can be simplified considerably. If first \( a, b, c, \), are integral \( \pmod{m} \), we have
\begin{equation}
(6.7)
u_{n} = m! \sum_{i+j+k=n} \frac{(a_i)(b_j)(c_k)}{i!j!k!} a^{i} b^{j} c^{k}
\end{equation}
so that in this case

\[(6.5) \quad u_m = 0 \pmod{m} .\]

Using (6.5), (3.19) reduces to

\[(6.6) \quad u_m = 0 \pmod{m^n} \quad (n \geq rm) .\]

On the other hand if \(x, y, z\) are indeterminates and \(m = p\), then since

\[(6.7) \quad \sigma(x) = \sigma(x) - x \pmod{p} ,\]

(6.5) implies

\[\sigma(y^p - y) + \sigma(z^p - z) \pmod{p} .\]

For general \(m\), if \(x\) is integral then (6.3) yields

\[(6.8) \quad u_m = \sum_{\alpha \equiv \beta (\text{mod } m)} a_{\alpha} b_{\beta}(x)(y)(z) \pmod{m} ,\]

while if both \(y\) and \(z\) are integral we get

\[(6.9) \quad u_m = \sigma^m(x, y, z) \pmod{m} .\]

If we specialize the parameters the recurrence (6.4) simplifies considerably. For example if \(a = b = \omega = c = \sigma, \quad \sigma + \omega + 1 = 0\) and \(y = \sigma x, \quad z = \omega x\), (6.4) reduces to

\[(6.10) \quad u_{n+1} = 3xu_n + n(n+1)(n-2)u_{n-1} .\]

From (6.10) it follows that \(u_n\) is a polynomial in \(3x\) with integral coefficients, which is not obvious from (6.3). It follows from (6.10) that

\[(6.11) \quad u_n = \sum_{\sigma \leq \alpha} c_{\sigma, \alpha}(3x)^{n-\ell},\]

where the coefficients \(c_{\sigma, \alpha}\) are integers that satisfy the mixed recurrence

\[(6.12) \quad c_{n+1, \alpha} = c_{n, \alpha + n(n+1)(n-2)} c_{n-1, \alpha + 1},\]

together with

\[c_{n, 0} = 1 \quad (n = 0, 1, 2, \ldots) .\]

Another way of determining \(u_n\) is by means of

\[(6.13) \quad \sum_{n=0}^{\infty} \frac{u_n}{n!} = \exp(3x \mathcal{F}(t)) ,\]

where

\[\mathcal{F}(t) = \sum_{r=0}^{\infty} \frac{t^{r+1}}{(r+1)^{3r+1}} .\]

It is easily verified that (6.13) and (6.10) are equivalent.

It follows from (6.10) that

\[(6.14) \quad u_n = \begin{cases} 3(x^p - x) \pmod{p} & (p = 1 \pmod{3}) , \\ 3x^p \pmod{p} & \text{otherwise} .\end{cases}\]

7. Let \(g_n\) denote the number of polygons of \(n\) sides (including degenerate cases) formed by a network of \(n\) lines. Robinson [10] showed that \(g_n\) satisfies the recurrence

\[(7.1) \quad g_{n+1} = ng_n + \frac{1}{2} n(n-1)g_{n-1} \quad (n \geq 2) ,\]

where \(g_1 = g_2 = 0, g_3 = 1\), it is convenient to define \(g_0 = 1\). Thus Theorem 5 applies to (7.1) and we get

\[(7.2) \quad \sum_{r=0}^{\infty} (-1)^r r! g_{n-r} x^{n+r} = 0 \pmod{m^n} ,\]

for all \(n \geq 0\), provided \(m \geq 1\) and odd. The writer [3] has given a direct proof of (7.2). Moreover

\[(7.3) \quad \sum_{r=0}^{\infty} (-1)^r r! x^{n+r} = 0 \pmod{m^n} ,\]

In the next place let \(K_n = K(3, n)\) denote the number of reduced three-line latin rectangles. Riordan [9] (see also [8], pp. 204-210) showed that \(K_n\) satisfies

\[(7.4) \quad K_{n+1} = (n+1)^2 K_n + \frac{1}{2} n(n+1)K_{n-1} + 3n(n^2-1)K_{n-2} + K_{n+1} ,\]

where

\[(7.5) \quad K_{n+1} + (n+1)K_n = -n \cdot 2^{n+1} .\]

The writer showed that \(K_n\) satisfies the congruence

\[(7.6) \quad \sum_{r=0}^{\infty} (-1)^r r! x^{n+r} K_{n+r} = 0 \pmod{m^n} ,\]

for all \(n \geq 0\), \(m \geq 1\). Now Karawals [8] had earlier found that \(K_n\) satisfies a certain recurrence of the fifth order, which indeed can be obtained by eliminating \(k_n\) from (7.4) and (7.5). However this recurrence is not of the form (1.5) and therefore the general theorems of this paper are not immediately applicable even though (7.6) is of the same form as (3.19). Notice also that the modulus in (7.6) is \(m^n\) rather than \(m^r\).

In conclusion we mention that in certain cases recurrences of the third order have been found for hypergeometric polynomials (7), Chapter 14. For example, for the polynomial

\[f_n(x) = 3F(-n, n+1; 1; x)\]
Sur le problème de M. Werner Münch
par
G. Sansone (Firenze) et J. W. S. Cassels (Cambridge)

Un de nous a donné récemment [2] une réponse négative au problème de M. Münch (*) : existent trois nombresrationnels $u, v, w$, tels que

\[ u + v + w = uwv = 1. \]


Comme on vérifie sans peine (voir [1], [2]), la réponse négative au problème de M. Münch équivaut à l’énoncé suivant :

**Théorème.** Les seules solutions de l’équation

\[ x^3 + y^3 + z^3 = xyz \]

en nombresrationnels sont les solutions banales, c’est-à-dire les solutions où $xyz = 0$.

Sans nuire à la généralité, nous supposons par ailleurs qu’il existe des entiers $(x, y, z)$ tels que (2) tienne, et tels que

\[ xyz \neq 0, \quad 3xyz. \]

Nous supposons aussi que $|xyz|$ est le plus petit possible, c’est-à-dire que

\[ |xyz| \geq |xyz| \]

pour toute solution entière $(a_1, b_1, c_1)$ non banale de l’équation (2).

Posons

\[ \gamma = 3x + 3y + z, \]
\[ \sigma = 3x + 3y + z, \]
\[ \delta = 3x + 3y + z, \]

(*) Pour l’histoire de ce problème, voir [1].