Beweis. Folgt aus (3.2) durch wörtliche Wiederholung des Khintchineschen Beweises [3] (wiedergegeben auch im Büchlein [4], S. 89-95).

Aus (3.2) können verschiedene Verallgemeinerungen von Khintchineschen [3] und P. Lévyschen [2] Resultaten hergeleitet werden; es sei jedoch nur noch ein einziger Spezialfall hervorgehoben, der mir besonders interessant zu sein scheint und dereinen Satz von S. Hartman und mir [6] verschärft.

Man setze

$$(3.5)\qquad f(k_1, k_2, l) = \begin{cases} 1 & \text{falls } k_1, k_2 \equiv u, v \ (\mathrm{mod}\, r) \text{ mit einem} \\ & \text{festgelegten Zahlenpaar } u, v\,((u, v, r) = 1), \\ 0 & \text{sonst}. \end{cases}$$

Dann schließt man aus (3.4) daß die Dichte der Zahlen l mit

$$B_{l-1} \equiv u\,, \qquad B_l \equiv v\ (\mathrm{mod}\, r)\,, \qquad (u, v, r) = 1$$

fürst fast alle α gleich $c(r)^{-1}$ ist. Summiert man noch über u (mit $(u, v, r) = 1$), so erhält man

Satz 3.3. *Es gilt für fast alle* α

$$(3.6)\qquad \lim_{n\to\infty} \frac{1}{n} \sum_{\substack{l\leqslant n \\ B_l \equiv v (\mathrm{mod}\, r)}} 1 = c(r)^{-1} \sum_{\substack{0\leqslant u<r \\ (u,v,r)=1}} 1 = \frac{r}{c(r)} \cdot \frac{\varphi\big((v, r)\big)}{(v, r)}.$$

Mit (3.6) wird die Dichte der l mit $B_l \equiv v\ (\mathrm{mod}\, r)$ in der Kettenbruchentwicklung fast aller α bestimmt.

Literaturverzeichnis

[1] R. O. Kusmin, *Sur un problème de Gauss*, Atti Congr. Intern. Bologne 6 (1928), S. 83-89.

[2] P. Lévy, *Sur les lois de probabilité dont dépendent les quotients complets et incomplets d'une fraction continue*, Bull. Soc. Math. France 57 (1929), S. 178-194.

[3] A. Khintchine, *Metrische Kettenbruchprobleme*, Comp. Math. 1 (1935), S. 361-382.

[4] A. Khnitchine, *Kettenbrüche*, Leipzig 1956.

[5] P. Szüsz, *Über einen Kusminschen Satz*, Erscheint in den Acta Math. Acad. Sci. Hung.

[6] S. Hartman, P. Szüsz, *On congruence classes of denominators of convergents*, Acta Arithm. 6 (1960), S. 179-184.

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A note on a theorem of Hardy and Littlewood

by

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1. In this paper we shall be concerned with the behaviour of the function

$$F(y) = \sum_{n=1}^{\infty} \{\Lambda(n) - 1\} e^{-ny}, \qquad y > 0\,,$$

which is, in essentials, Abel-mean formed from series $\sum_n \{\Lambda(n)-1\}$.

Hardy and Littlewood proved [3], on starting from the formula of Cahen [1] and Mellin [5]

$$(1.1)\qquad \frac{1}{2\pi i} \int_{\varkappa-i\infty}^{\varkappa+i\infty} \Gamma(s) y^{-s} ds = e^{-y} \qquad (\varkappa > 0\,,\ \mathrm{re}\, y > 0)\,,$$

that, on the Riemann hypothesis,

$$F(y) = O\left(\frac{1}{y^{1/2}}\right), \qquad F(y) = \Omega_{\pm}\left(\frac{1}{y^{1/2}}\right)$$

as $y \to 0$ through positive values.

In the present paper we are going to show that one may dispense with the Riemann hypothesis in the, at least slightly weaker, Ω-estimation of $F(y)$ and, what is more, in place of the above ineffective estimation, will supply an explicit one.

The result is

Theorem. *For* $0 < \delta < c_1$ (¹) *we have*

$$(1.2)\qquad \max_{\delta\leqslant y\leqslant 1} |F(y)| > \frac{1}{\delta^{1/2}} \cdot \exp\left(-4 \frac{\log \frac{1}{\delta}}{\log\log \frac{1}{\delta}} \log\log\log \frac{1}{\delta}\right).$$

In the proof we shall apply the method of Turán, namely will use the following modification [4] of Turán's Satz X [8]:

(¹) $c_1, c_2, \ldots$ denote positive, numerically calculable constants.

LEMMA. *Suppose* $m \geqslant 0, z_1, z_2, \ldots, z_N$ *are complex numbers with*

$$1 = |z_1| \geqslant |z_2| \geqslant \ldots \geqslant |z_h| \geqslant \ldots \geqslant |z_N|$$

and

(1.3) $$|z_h| > 2\frac{N}{m+N}.$$

Then there exists an integer ν *with*

(1.4) $$m \leqslant \nu \leqslant m+N,$$

such that

(1.5) $$|b_1 z_1^\nu + b_2 z_2^\nu + \ldots + b_N z_N^\nu| \geqslant \left(\frac{|z_h|}{2}\right)^\nu \min_{h \leqslant j \leqslant N} |b_1 + b_2 + \ldots + b_j| \left(\frac{N}{24e(2N+m)}\right)^N.$$

2. Before we turn to the proof we shall list some know properties of the functions $\Gamma(s)$ and $\zeta(s)$ which will be used in the following:

(2.1) $$\log\Gamma(s) = s\log s - s - \tfrac{1}{2}\log s + \tfrac{1}{2}\log 2\pi + \frac{\theta}{r}$$

for $\operatorname{re} s \geqslant 0$, $|s| \geqslant 14$, $r = |s|$, $|\theta| \leqslant 1$ (²),

(2.2) $$|\Gamma(\sigma+it)| = \frac{k\Gamma(1+\sigma)}{\sqrt{\sigma^2+t^2}}\sqrt{\frac{2\pi t}{e^{\pi t}-e^{-\pi t}}},$$

where $1 \leqslant k \leqslant \sqrt{1+t^2}$, $0 < \sigma < 1$ (see [6], p. 25).

Further, by the formula of Legendre,

(2.3) $$|\Gamma(\tfrac{1}{2}+it)| = \sqrt{\frac{2\pi}{e^{\pi t}+e^{-\pi t}}}, \quad -\infty < t < +\infty.$$

Now let $N(T)$ stand for the number of zeros of $\zeta(s)$ in $0 < \sigma < 1$, $0 \leqslant t \leqslant T$. We have

(2.4) $$N(T) < c_2 T\log T, \quad \text{for} \quad T \geqslant 2.$$

Further, for $\sigma \geqslant -\frac{3}{2}$

(2.5) $$\left|\zeta(s) - \frac{1}{s-1}\right| \leqslant c_3\{1+(|t|+2)^{(1-\sigma)/2}\}\log(|t|+2).$$

For every $v \geqslant 2$ there is a T_v with

(2.6) $$v \leqslant T_v \leqslant v+1$$

such that

(2.7) $$\left|\frac{\zeta'}{\zeta}(\sigma+iT_v)\right| \leqslant c_4\log^2 T_v, \quad -\tfrac{3}{2} \leqslant \sigma \leqslant 2.$$

The first complex zeros of $\zeta(s)$ are approximately (see [7], p. 320)

(2.8) $$\tfrac{1}{2} \pm i\,14.13, \ \tfrac{1}{2} \pm i\,21.02, \ \tfrac{1}{2} \pm i\,25.01, \ \ldots$$

(²) See e.g. [2], p. 138. (2.1) has been slightly modified.

3. We turn to the proof of the theorem. Let us write

(3.1) $$\varepsilon = \tfrac{1}{3}\left(\log\frac{1}{\delta}\right)^{-2/3},$$

(3.2) $$A = \tfrac{1}{2}\log\log\frac{1}{\delta}.$$

Integer ν will be supposed to satisfy the inequality

(3.3) $$(1-\varepsilon)\log\frac{1}{\delta} \leqslant \nu A \leqslant \log\frac{1}{\delta}.$$

Further, put

(3.4) $$m = (1-\varepsilon)\frac{\log\frac{1}{\delta}}{A},$$

(3.5) $$N = \frac{1}{3}\frac{\log^{1/3}\frac{1}{\delta}}{\log\log\frac{1}{\delta}},$$

(3.6) $$l = \frac{\log^{1/3}\frac{1}{\delta}}{\left(\log\log\frac{1}{\delta}\right)^3}.$$

We start from the formula

(3.7) $$F(y) = -\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} y^{-s}\Gamma(s)\left(\frac{\zeta'}{\zeta}(s)+\zeta(s)\right)ds,$$

which follows by (1.1).

Substituting

$$y = e^{-\omega_1},$$

we integrate (3.7) ν times from 0 to $\omega_2, \omega_3, \ldots, \omega_\nu, \omega$ respectively. On the one hand we have then

(3.8) $$I_\omega = \int_0^\omega\int_0^{\omega_\nu}\ldots\int_0^{\omega_3}\int_0^{\omega_2} F(e^{-\omega_1})\,d\omega_1 d\omega_2\ldots d\omega_\nu$$

and on the other

(3.9) $$I_\omega = -\frac{1}{2\pi i}\int_{(2)}\frac{\Gamma(s)}{s^\nu}\left(\frac{\zeta'}{\zeta}(s)+\zeta(s)\right)e^{\omega s}ds + \frac{1}{2\pi i}\sum_{j=1}^{\nu}\frac{\omega^{\nu-j}}{(\nu-j)!}\int_{(2)}\frac{\Gamma(s)}{s^j}\left(\frac{\zeta'}{\zeta}(s)+\zeta(s)\right)ds.$$

(3.8) gives clearly

$$(3.10)\qquad |I_\omega| \leqslant \frac{\omega^\nu}{\nu!} \max_{e^{-\omega} \leqslant y \leqslant 1} |F(y)|.$$

Further we have

$$(3.11)\qquad \left|\int_{(2)} \frac{\Gamma(s)}{s^j}\left(\frac{\zeta'}{\zeta}(s)+\zeta(s)\right)ds\right| \leqslant c_5, \qquad j = 1, 2, \ldots$$

We apply Cauchy's theorem of residues to the first integral in (3.9). Let $\mathcal{L}$ be a connected broken line with segments alternately parallel to the axes, all lying in the strip $\frac{1}{5} < \sigma < \frac{2}{5}$, and such that

$$(3.12)\qquad \left|\frac{\zeta'}{\zeta}(s)\right| \leqslant c_6 \log^2(|t|+2) \quad \text{on} \quad \mathcal{L}.$$

The existence of $\mathcal{L}$ follows easily as e.g. in [8] (p. 186). We have then by (3.11)

$$(3.13)\qquad I_\omega = \sum_{\varrho > (\mathcal{L})} \Gamma(\varrho)\frac{e^{\varrho\omega}}{\varrho^\nu} - \frac{1}{2\pi i}\int_{(\mathcal{L})} \frac{e^{s\omega}}{s^\nu}\Gamma(s)\left(\frac{\zeta'}{\zeta}(s)+\zeta(s)\right)ds + O\left(\sum_{j=1}^{\nu} \frac{\omega^{\nu-j}}{(\nu-j)!}\right)$$

where $\varrho = \beta + i\gamma$ runs through the zeros of $\zeta(s)$.

Using (3.12) and (2.5) we get

$$\left|\frac{1}{2\pi i}\int_{(\mathcal{L})} \frac{e^{s\omega}}{s^\nu}\Gamma(s)\left(\frac{\zeta'}{\zeta}(s)+\zeta(s)\right)ds\right| \leqslant c_7(e^{2/5\,\omega}\cdot 5^\nu).$$

This, (3.10) and (3.13) give

$$(3.14)\qquad \left|\sum_{\varrho > (\mathcal{L})} \Gamma(\varrho)\cdot\frac{e^{\varrho\omega}}{\varrho^\nu}\right| \leqslant \frac{\omega^\nu}{\nu!}\max_{e^{-\omega} \leqslant y \leqslant 1}|F(y)| + c_7 e^{2/5\,\omega}\cdot 5^\nu + c_8 \sum_{j=1}^{\nu}\frac{\omega^{\nu-j}}{(\nu-j)!}.$$

4. In this section we shall estimate the sum $\sum_{\varrho > (\mathcal{L})}$ in (3.14) from below.

We put

$$(4.1)\qquad \omega = \nu A,$$

and split $\sum_{\varrho > (\mathcal{L})}$ into two with $|\gamma| < l$ or else $|\gamma| \geqslant l$. We obtain easily for sufficiently small δ

$$(4.2)\qquad \left|\sum_{|\gamma| \geqslant l} \Gamma(\varrho)\cdot\frac{e^{\omega\varrho}}{\varrho^\nu}\right| \leqslant c_9 \frac{e^\omega}{l^\nu} \leqslant \left(\frac{1}{\delta}\right)^{0.4}.$$

In order to estimate the other sum from below we shall use the Lemma.

Let $\varrho_1 = \beta_1 + i\gamma_1$, $\varrho_1 > (\mathcal{L})$ be that zero of $\zeta(s)$ with $|\gamma_1| < l$ for which $\left|\frac{e^{A\varrho}}{\varrho}\right|$ is maximal. We put the sum in question in the form

$$S = \left(\frac{e^{A\varrho_1}}{\varrho_1}\right)^\nu \sum_{\substack{|\gamma_j| < l \\ \varrho_j > (\mathcal{L})}} \Gamma(\varrho_j)\cdot\left(\frac{e^{A(\varrho_j - \varrho_1)}}{\varrho_j/\varrho_1}\right)^\nu$$

and define

$$b_j = \Gamma(\varrho_j), \qquad z_j = \frac{e^{A(\varrho_j-\varrho_1)}}{\varrho_j/\varrho_1}, \qquad z_h = \frac{e^{A(1/2+i\,14.13\ldots-\varrho_1)}}{\dfrac{1/2+i\,14.13\ldots}{\varrho_1}}.$$

Next apply Lemma with m, N given by (3,4), (3.5). This is permissible in view of (2.4). If N is greater than the actual number of the ϱ's in the considered domain, we put $z_j = b_j = 0$ for the missing ones.

Let us check (1.3):

$$|z_h| \geqslant \frac{1}{e^{1/2\,A}} = \frac{1}{\left(\log\frac{1}{\delta}\right)^{1/4}} > \frac{2N}{m+N}.$$

Choosing ν as required in (1.5) we obtain

$$(4.3)\qquad |S| \geqslant \left(\frac{e^{A\beta_1}}{|\varrho_1|}\right)^\nu\cdot\left(\frac{e^{A(1/2-\beta_1)}}{30/|\varrho_1|}\right)^\nu\cdot\left(\frac{N}{24e(2N+m)}\right)^N \times \min_{h \leqslant j \leqslant N}|\Gamma(\varrho_1)+\Gamma(\varrho_2)+\ldots+\Gamma(\varrho_j)|.$$

It is easy to see that

$$(4.4)\qquad \min_{h \leqslant j \leqslant N}|\Gamma(\varrho_1)+\Gamma(\varrho_2)+\ldots+\Gamma(\varrho_j)| \geqslant |\Gamma(\tfrac{1}{2}+i\,14.13\ldots)+\Gamma(\tfrac{1}{2}-i\,14.13\ldots)| - \sum_{|\gamma| \geqslant 21}|\Gamma(\varrho)|.$$

But (2.2) furnishes

$$(4.5)\qquad \sum_{|\gamma| \geqslant 21}|\Gamma(\varrho)| \leqslant \frac{\sqrt{2\pi}}{e^{27}},$$

and by (2.1) we get

$$247^\circ \leqslant \arg\Gamma(\tfrac{1}{2}+i\,14.13\ldots) \leqslant 262^\circ,$$

which means

$$|\Gamma(\tfrac{1}{2}+i\,14.13\ldots)+\Gamma(\tfrac{1}{2}-i\,14.13\ldots)| \geqslant 0.278\,|\Gamma(\tfrac{1}{2}+i\,14.13\ldots)|.$$

Using (2.3) we have together with (4.4) and (4.5)

$$(4.6)\qquad \min_{h \leqslant j \leqslant N}|\Gamma(\varrho_1)+\Gamma(\varrho_2)+\ldots+\Gamma(\varrho_j)| \geqslant \frac{\sqrt{2\pi}}{e^{24}} - \frac{\sqrt{2\pi}}{e^{27}} > e^{-24}.$$

Next we have

$$\left(\frac{N}{24e(2N+m)}\right)^N \geqslant \left(\frac{N}{216m}\right)^N > e^{-1/8\log^{1/8}\frac{1}{\delta}}.$$

This, (4.3) and (4.6) give

$$|S| \geqslant \frac{e^{1/2\,\Delta\nu}}{30^{\nu}} \cdot e^{-\log^{1/3}\frac{1}{\delta}} e^{-24}$$

$$> \left(\frac{1}{\delta}\right)^{1/2} \exp\left\{-\log^{1/3}\frac{1}{\delta} - 2\log 30 \cdot \frac{\log\frac{1}{\delta}}{\log\log\frac{1}{\delta}} - \frac{1}{6}\left(\log\frac{1}{\delta}\right)^{1/3} - 24\right\}$$

$$> 2\left(\frac{1}{\delta}\right)^{1/2} \exp\left(-8\,\frac{\log\frac{1}{\delta}}{\log\log\frac{1}{\delta}}\right).$$

Using now (3.14) and (4.2) we obtain

$$\frac{\omega^{\nu}}{\nu!} \max_{e^{-\omega} \leqslant y \leqslant 1} |F(y)| \geqslant \left(\frac{1}{\delta}\right)^{1/2} \exp\left(-8\,\frac{\log\frac{1}{\delta}}{\log\log\frac{1}{\delta}}\right) - c_8 \sum_{j=1}^{\nu} \frac{\omega^{\nu-j}}{(\nu-j)!}. \tag{4.7}$$

But

$$\frac{\omega^{\nu}}{\nu!} \leqslant \left(\frac{e\log\frac{1}{\delta}}{\nu}\right)^{\nu} < e^{8\frac{\log 1/\delta}{\log\log 1/\delta}\log\log\log 1/\delta}$$

and also

$$\sum_{j=1}^{\nu} \frac{\omega^{\nu-j}}{(\nu-j)!} \leqslant \nu\,\frac{\omega^{\nu}}{\nu!},$$

whence by (4.7)

$$\max_{\delta \leqslant y \leqslant 1} |F(y)| > \left(\frac{1}{\delta}\right)^{1/2} \exp\left(-4\,\frac{\log\frac{1}{\delta}}{\log\log\frac{1}{\delta}} \log\log\log\frac{1}{\delta}\right),$$

Q.E.D.

References

[1] Cahen, *Thèse*, Paris 1894.

[2] А. О. Гельфонд, *Исчисление конечных разностей*, Москва 1952.

[3] G. H. Hardy and J. E. Littlewood, *Contributions to theory of the Riemann zeta-function and the theory of the distribution of primes*, Acta Math. 41 (1918), pp. 119-196.

[4] S. Knapowski, *Contributions to the theory of the distribution of prime numbers in arithmetical progressions* I, Acta Arithm. 6 (1961), pp. 415-434.

[5] Mellin, Acta Soc. Fennicae 20 (1895), pp. 1-39.

[6] N. Nielsen, *Handbuch der Theorie der Gammafunktion*, Leipzig 1906.

[7] E. C. Titchmarsh, *The theory of the Riemann zeta-function*, Oxford 1951.

[8] P. Turán, *Eine neue Methode in der Analysis und deren Anwendungen*, Budapest 1953.

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A theorem on "ordered" polynomials in a finite field

by

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Let F denote the finite field $GF(q)$ of order q, where $q = p^n$ is odd. Put $\psi(a) = +1, -1$ or 0 according as a is a non-zero square, a non-square or zero in F. Then we have

$$\psi(a) = a^m, \tag{1}$$

where $q = 2m+1$. The writer has proved the following theorem.

Theorem A. *Let $f(x)$ be a permutation polynomial such that*

$$f(0) = 0, \quad f(1) = 1 \tag{2}$$

and

$$\psi\big(f(x) - f(y)\big) = \psi(x-y) \tag{3}$$

for all $x, y \in F$. Then we have

$$f(x) = x^{p^j} \tag{4}$$

for some j in the range $0 \leqslant j < n$.

We recall that a polynomial $f(x)$ with coefficients in F is a permutation polynomial if the numbers $f(a)$, $a \in F$, are distinct. Also two polynomials $f(x)$, $g(x)$ are defined as equal if $f(a) = g(a)$ for all $a \in F$; this is equivalent to the statement

$$f(x) \equiv g(x) \pmod{x^q - x}.$$

Now it is evident that the hypothesis (3) implies that $f(x)$ is a permutation polynomial. Also we may drop the hypothesis (2) and replace Theorem A by the following slightly more general theorem.

Theorem B. *Let $f(x)$ be a polynomial with coefficients $\in F$ such that*

$$\psi\big(f(x) - f(y)\big) = \lambda\psi(x-y)$$

for all $x, y \in F$, where $\lambda = \pm 1$ is fixed. Then we have

$$f(x) = ax^{p^j} + b \tag{5}$$

for some j in the range $0 \leqslant j < n$ and where $a, b \in F$, $\psi(a) = \lambda$.