whence

$$\int_{T\varphi^2}^{T} \frac{|M(x)|}{x^{3/2}} dx \geqslant c_{25}$$

and finally

$$\int_{T\varphi^2}^{T} \frac{|M(x)|}{x} dx \geqslant c_{26}\varphi T^{1/2}.$$

This clearly implies (1.8).

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On the zeros of Hecke's L-functions II

by

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Introduction

1. In the first paper (see [1]) it has been proved in particular that the Hecke-Landau function $\zeta(s,\chi)$ of the field K of degree $n \geqslant 1$ with a complex character χ modulo $\mathfrak{f}$ has no zero in a rectangle

$$1-A_0/\log D \leqslant \sigma \leqslant 1, \quad |t| \leqslant D^2$$

(where $D = |\Delta| N\mathfrak{f} \geqslant D_0 > 1$, Δ denotes the discriminant of the field and $A_0 > 0$ depends only on n). For at most one real χ in that rectangle may be a simple zero $\beta' = 1-\delta$ of $\zeta(s,\chi)$; it is real and, if D_0 is large enough, then

$$(1) \qquad \delta > D^{-2n}.$$

β', if it exists, is called the "exceptional" zero. The corresponding character $\chi = \chi'$ and function $\zeta(s,\chi')$ also are called the "exceptional" ones. Consider, that χ' is a real character, not necessarily different from the principal one.

In this paper we shall prove the following

Theorem. *There is an absolute constant $A > 0$ (which depends only on n) such that for*

$$\delta_0 = \begin{cases} \delta & \text{if} \quad \delta \leqslant A/\log D, \\ A/\log D & \text{otherwise}, \end{cases}$$

$$\lambda_0 = A \log \frac{eA}{\delta_0 \log D} \in [A, \tfrac{1}{2}\log D]$$

in the rectangle $(1-\lambda_0/\log D \leqslant \sigma \leqslant 1,\ |t| \leqslant D)$ *there is no zero of the function* $Z(s) = \prod_{\chi} \zeta(s,\chi)$ *with at most one exception* β'.

We may suppose that the exceptional zero exists. If it does not, then this theorem (with $\delta_0 = A_0/\log D$) is a simple consequence of that proved in [1]. And so it is (with $\delta_0 = \frac{1}{6}A_0/\log D$) if $\delta_0 \in [\frac{1}{6}A_0/\log D, A_0/\log D]$. Hence, in what follows we suppose that

$$\delta < \tfrac{1}{6}A_0/\log D.$$

The method used in this paper is on the whole that employed by Rodosskiĭ for L-functions of Dirichlet (cf. [3], X § 3). He has based his proof on a transformation of the function $L'/L(s,\chi)$, although it is more convenient to deal directly with the function itself (cf. Lemma 2 of this paper and [3], X, Lemmas 3.4-3.9 or X, Lemmas 2.4-2.9).

2. The notation (and the convention $n \ll 1$) remains generally the same as that used in [1]. The constant A_0 keeps its meaning throughout this paper. We shall need the following estimates

(2) $$\sum_{p\leqslant x} \log p \ll x \quad (x > 0),$$

(3) $$\sum_{p\leqslant x} \log p/p = \log x + O(1) \quad (x \geqslant 1),$$

(4) $$\sum_{p\leqslant x} 1/p = \log\log x + c_1 + O(1/\log x) \quad (x\to\infty),$$

(5) $$\prod_{p\leqslant x} (1-1/p)^{-1} = c_2 \log x + O(1) \quad (x\to\infty),$$

where p runs through all primes (see, for example, [3], I, Satz 3.1, 4.1). Other quotations from [3] and [1] will follow during the proofs.

Two general lemmas

3. LEMMA 1. *Let $A, \lambda, E, a, \gamma_0, \gamma$ denote parameters such that $A > 1$ may increase indefinitely, $0 < c \leqslant \lambda \leqslant \frac{2}{5}A$, $1 < E \ll 1$,*

(6) $$a = \lambda/A, \quad e^{\lambda}/A \leqslant \gamma_0 \leqslant e^{A}, \quad \gamma = \min(1, e^{\lambda}/A).$$

Let further S be some set of points $\varrho = \beta + i\tau$ in the strip $\frac{1}{2} \leqslant \sigma \leqslant 1$. If there is a point $\varrho_0 = \beta_0 + i\tau_0 \in S$ in the region $(1-a \leqslant \sigma \leqslant 1, |t| \leqslant \gamma_0)$, then there is also a "convenient" point $\varrho_1 = \beta_1 + i\tau_1 \in S$ such that $\beta_1 \geqslant \beta_0$, $|\tau_1 - \tau_0| < 5E\lambda\gamma$ and in the region

(7) $$\beta_1 + 1/EA \leqslant \sigma \leqslant 1, \quad |t-\tau_1| \leqslant 5\gamma$$

there is no point $\varrho \in S$.

Proof. If there is no point $\varrho \in S$ in the region $R_0(\beta_0 + 1/EA \leqslant \sigma \leqslant 1$, $|t-\tau_0| \leqslant 5\gamma)$, then ϱ_0 is a "convenient" point and we have nothing to prove. If there are points ϱ in R_0, then there is at least one point $\varrho_2 = \beta_2 + i\tau_2$ (say) in R_0 having the maximal β_2. And if in the corresponding region $R_2(\beta_2 + 1/EA \leqslant \sigma \leqslant 1, |t-\tau_2| \leqslant 5\gamma)$ there is no $\varrho \in S$, then ϱ_2 is a "convenient" point, etc. Repeating this argument we get the "convenient" point ϱ_1 in less than $E\lambda$ steps (since there are no points ϱ on the right of the line $\sigma = 1$).

4. LEMMA 2. *Let the function $F(s)$ be meromorphic in $\sigma > -\frac{1}{2}$ and satisfy the following conditions.*

(i) *The only singularities of $F(s)$ in $\sigma > -\frac{1}{2}$ are simple poles ϱ on the left of the line $\sigma = 1$ with residues $m_\varrho \geqslant 1$, such that*

(8) $$\sum_{|\varrho-1-it|<r} m_\varrho \ll rA$$

for $1/A \leqslant r \leqslant 2$, $|t| \leqslant e^{2A}$.

(ii) *In the notation of Lemma 1, there is a pole ϱ_0 of $F(s)$ in the rectangle $(1-a \leqslant \sigma \leqslant 1, |t| < \gamma_0)$ and we have in the region $1-a \leqslant \sigma \leqslant 4$, $|t-\tau_1| < 1+5\gamma$*

(9) $$F(s) - \sum_{|s-\varrho|<1} \frac{m_\varrho}{s-\varrho} \ll A$$

(iii) *In $\sigma > 1$*

(10) $$F(s) = \sum_{m=1}^{\infty} a_m m^{-s}, \quad a_p \ll p^{c_1/A} \log p,$$

and the terms of the expansion (10) with $m \neq p$ constitute a series which is absolutely and uniformly convergent in $\sigma \geqslant 1-a$ having the sum $\ll 1$.

Write

(11) $$Z = \exp\left\{A \frac{\log(1+\lambda)}{2+2\lambda}\right\},$$

(12) $$F_1(s) = \sum_{Z<p<M} a_p p^{-s}, \quad F_2(s) = \sum_{p\geqslant M} a_p p^{-s} \quad (\sigma > 1)$$

where $M \geqslant e^{2A}$ is a fixed number $< \infty$.

Under these circumstances either (I) *there is a point $s_1 = \sigma_1 + it_1$ in*

(13) $$R_{\mathrm{I}}(1-a \leqslant \sigma \leqslant 1+3a, \ |t-\tau_1| \leqslant 4a)$$

satisfying

$$|F_1(s_1)| \geqslant A$$

or (II) *there is a point $s_2 = \sigma_2 + it_2$ in*

$$R_{\mathrm{II}}(1+\tfrac{1}{2}a \leqslant \sigma \leqslant 1+\tfrac{3}{2}a, \ |t-\tau_1| \leqslant a)$$

such that

(14) $$|F_2(s_2)| > A\exp(-c_0\lambda).$$

Proof. Let us introduce the point $s_3 = \varrho_1 + 3/EA$ at which, by (9),

(15) $$\mathrm{re}\,F(s_3) \geqslant \tfrac{1}{3}EA - cA.$$

Writing

(16) $$f(s) = \sum_{\substack{m \\ m\neq p}} a_m m^{-s} + \sum_{p\leqslant Z} a_p p^{-s}, \quad F_3(s) = F(s) - f(s)$$

we have in $\sigma \geqslant 1-\alpha$

(17) $$f(s) \ll \Lambda\,,$$

by (10), (11), (6), (3). Hence, by (15),

$$|F_3(s_3)| > 2\Lambda$$

(provided that E is large enough). This implies the inequality

(18) $$|F_1(s_3)|+|F_2(s_3)| > 2\Lambda\,,$$

since

(19) $$F_3(s) = F_1(s)+F_2(s)$$

by (10), (12), (16) (in the first instance in $\sigma > 1$ and further by analytic continuation).

Now suppose that (I) does not hold. Then we have for all $s \in R_{\mathrm{I}}$

(20) $$|F_1(s)| < \Lambda$$

and hence, by (18), $|F_2(s_3)| > \Lambda$ (since $s_3 \in R_{\mathrm{I}}$).

Let C_1 be the circle passing through the point s_3 and having its centre at $1+\alpha+i\tau_1$; then its radius is

(21) $$r_1 = 1+\alpha-(\beta_1+3/E\Lambda)\,.$$

Write

$$M_1 = \max_{s \in C_1} |F_2(s)|\,.$$

If z_1 denotes that point of C_1 (or one of such points) where $|F_2(s)|$ takes its maximal value, then

(22) $$|F_2(z_1)| = M_1 > \Lambda\,.$$

Denoting $\sigma_0 = \beta_1+1/E\Lambda$,

$$\mathrm{re}\sum_{\substack{|s-\varrho|<1\\ \beta\leqslant\sigma_0}} \frac{m_\varrho}{s-\varrho} = P(s)\,, \qquad \mathrm{re}\sum_{\substack{|s-\varrho|<1\\ \beta>\sigma_0}} \frac{m_\varrho}{s-\varrho} = Q(s)$$

we have in R_{I}

(23) $$\mathrm{re}F(s) = P(s)+Q(s)+O(\Lambda)\,,$$

by (9).

Let $\nu(u) = \nu(u,t)$ (for $u \geqslant \gamma$, $|t| \leqslant e^{2\Lambda}$) denote the sum $\sum m_\varrho$ over the points $\varrho = \beta+i\tau$ in the rectangle R_t defined by $\sigma_0 \leqslant \beta \leqslant 1$, $|\tau-t| \leqslant u$. Since R_t can be covered by $\ll u\colon(\lambda/\Lambda)$ circles of radius $2\lambda/\Lambda$ having their centres on the line $\sigma = 1$, by (8) $\nu(u) \ll u\Lambda$. Hence, by (13), (7) and [1] (6), if $s \in R_{\mathrm{I}}$ and $\gamma < 1$,

(24) $$Q(s) \ll \alpha\Big(\int_\gamma^1 \frac{\nu(u)}{u^3}\,du+\nu(1)\Big) \ll \Lambda\,.$$

(If $\gamma = 1$, then $Q = 0$ as an empty sum.)

Let now s, s^* be any two points in the region

$$\beta_1+2/E\Lambda \leqslant \sigma \leqslant 1+3\alpha\,, \qquad |t-\tau_1| \leqslant 4\alpha$$

satisfying $|s-s^*| \leqslant 1/E\Lambda$. If s replaced by s^*, then the term

$$P_\varrho = \mathrm{re}\frac{m_\varrho}{s-\varrho} = m_\varrho\frac{\mathrm{re}(s-\varrho)}{|s-\varrho|^2}$$

of $P(s)$ changes into $k_\varrho P_\varrho$, where

$$k_\varrho = \frac{\mathrm{re}(s^*-\varrho)}{\mathrm{re}(s-\varrho)}\cdot\frac{|s-\varrho|^2}{|s^*-\varrho|^2} \in [\tfrac14, 4]\,.$$

Hence the ratio $P(s):P(s^*)$ remains in the segment $[\frac14, 4]$. From this and (23), (24), (16), (17), (19), (20), (22) we deduce that for all s in the circle $C_0(|s-z_1| < r,\ r = 1/E\Lambda)$

$$\begin{aligned} P(s) &\leqslant 4P(z_1) \leqslant 4\,|\mathrm{re}F(z_1)|+c_2\Lambda < 4\,|F(z_1)|+c_2\Lambda < 4\,|F_3(z_1)|+c_3\Lambda \\ &\leqslant 4\,|F_1(z_1)|+4\,|F_2(z_1)|+c_3\Lambda < 4\,|F_2(z_1)|+c_4\Lambda < c_5M_1\,. \end{aligned}$$

Hence, by the same formulae, in C_0

$$|\mathrm{re}F_2(s)| < c_6M_1\,.$$

Now by the theorem of Borel-Carathéodory (see, for example, [3], A, Satz 4.2) in all points of the circle

(25) $$C_0'(|s-z_1| \leqslant r'\,,\ r' = 1/E^2\Lambda)$$

we have

$$|F_2(s)-F_2(z_1)| \leqslant \frac{2r'}{r-r'}\{\max_{s \in C_0'}|\mathrm{re}F_2(s)|-\mathrm{re}F_2(z_1)\} < \tfrac12 M_1\,,$$

provided that E is large enough. Hence, by (22), at that point $s = s'$ (say) of the circle $C_2(|1+\alpha+i\tau_1-s| \leqslant r_2)$ with

(26) $$r_2 = r_1-r'\,,$$

where C_2 touches C_0', we have

$$|F_2(s')| > \tfrac12 M_1$$

and thus

(27) $$\max_{s \in C_2}|F_2(s)| = M_2 > \tfrac12 M_1\,.$$

Writing

$$M_3 = \max_{s \in C_3}|F_2(s)|\,, \qquad C_3(|1+\alpha+i\tau_1-s| \leqslant r_3)\,, \qquad r_3 = \tfrac12\alpha$$

we have, by (27), (21), (26), (25), (22) and Hadamard's three circle theorem (see, for example, [3], A, Satz 9.2)

$$\begin{aligned} M_3 &\geqslant M_1^{(\log r_3/r_2)/\log r_1/r_2}\,M_2^{(\log r_1/r_3)/\log r_1/r_2} \\ &\geqslant M_1 2^{-(\log r_1-\log r_3)/\log 1:(r_2/r_1)} \geqslant \Lambda\exp(-c_7E^2\lambda)\,, \end{aligned}$$

whence (14) follows.

Proof of the inequality (28) for $\chi_0 \neq \chi'$

5. Lemma 3. *There are positive absolute constants A_1, A_2 such that if*

$$\delta < A_1/\log D$$

and $\varrho = \beta + i\tau \neq \beta'$ is a zero in $|t| \leqslant D$ of any function $\zeta(s, \chi)$ with a character mod $\mathfrak{f}$, *then*

$$\beta < 1 - \frac{A_2}{\log D} \log \frac{eA_1}{\delta \log D}. \tag{28}$$

The proof of this lemma will be the object of the following paragraphs 6-16.

Let $\varrho_0 = \beta_0 + i\tau_0$ be a zero of $\zeta(s, \chi)$ such that $|\tau_0| \leqslant D$, $\beta_0 \geqslant 0$; write

$$\beta_0 = 1 - \lambda/\log D. \tag{29}$$

Since, by (1),

$$\log \frac{eA_1}{\delta \log D} < c_1 \log D,$$

if $\beta_0 < \frac{3}{5}$ or $\lambda \geqslant \frac{2}{5}\log D$, then (28) holds (with β_0 instead of β) for any $A_2 \leqslant \frac{2}{5}c_1^{-1}$. Hence it remains to prove the lemma for $\lambda < \frac{2}{5}\log D$.

6. We shall begin the proof by verifying that the function

$$F(s) = \begin{cases} \zeta'/\zeta(s, \chi) + \zeta'/\zeta(s-\delta, \chi\chi') & \text{if} \quad \chi = \chi_0 \neq \chi', \\ \zeta'/\zeta(s, \chi) + \zeta'/\zeta(s+\delta, \chi\chi') & \text{if} \quad \chi \neq \chi_0 \neq \chi', \\ \zeta'/\zeta(s, \chi) & \text{if} \quad \chi \neq \chi_0 = \chi' \end{cases} \tag{30}$$

satisfies the conditions of Lemma 2 with

$$A = \log D, \quad \lambda \in [A_0, \tfrac{2}{5}\log D], \quad \gamma_0 = D, \quad M = D^{4n}. \tag{31}$$

Consider that for $\chi = \chi_0 \neq \chi'$ the pole of $\zeta'/\zeta(s, \chi_0)$ at $s = 1$ (with residue -1) is cancelled out by the pole at $s = 1$ (with residue 1) of $\zeta'/\zeta(s-\delta, \chi')$. And for $\chi \neq \chi_0 \neq \chi'$ the same remark applies to the pole at $s = \beta'$ (with residue -1) of $\zeta'/\zeta(s+\delta, \chi_0)$ and the corresponding pole of $\zeta'/\zeta(s, \chi')$ with residue 1. In the other cases there are no poles of ζ'/ζ with negative residues.

We have (8) in a consequence of (31) and [1] (43), [1] (36).

(9) holds for $\gamma_0 = D$, $A = \log D$, by [1] (41), since $|\tau_1| < D^{3/2}$ (cf. Lemma 1).

The condition (iii) of Lemma 2 results from (30) and [1] (11).

7. Since any prime p in the field K is a product of at most n prime ideals $\mathfrak{p}$ with $N\mathfrak{p} = p$, in the exposition (10) for any of the functions (30) we have, by [1] (11),

$$|a_p| < n(1 + p^{\delta})\log p. \tag{32}$$

By (2), [1] (6),

$$\sum_{p \geqslant x} \frac{\log p}{p^{1+a/3}} < c_1/ax^{a/3} \qquad (a > 0, x > 1).$$

From this and (32) (since $\delta < A_0/6\log D$, $a \geqslant A_0/\log D$, by (31) and § 1) we deduce that there is an absolute constant $A \geqslant 8n$ such that

$$\sum_{p > D^A} \frac{|a_p|}{p^{1+a/2}} < 2n \sum_{p > D^A} \frac{\log p}{p^{1+a/3}} < \tfrac{1}{4}\exp(-c_0\lambda)\log D \tag{33}$$

for c_0 defined by (14).

Since, by (11), (31),

$$Z = D^{\{\log(1+\lambda)\}/(2+2\lambda)}$$

we have, by (4),

$$\sum_{Z \leqslant p < D^{4n}} p^a \frac{1}{p} < D^{4na}(\log\log D^{4n} - \log\log Z + 1) < \exp(c_2\lambda),$$

$$\sum_{D^{4n} < p < D^A} \frac{|a_p|}{p^{1+a/2}} < 2n \sum_{D^{4n} < p < D^A} \frac{\log p}{p^{1+a/3}} < c_3 e^{-n\lambda}\log D.$$

Hence, if $B \lll 1$ is large enough,

$$e^{-B\lambda} \sum_{Z \leqslant p < D^{4n}} \frac{1}{p^{1-a}} < 1/12n^2, \tag{34}$$

$$e^{-B\lambda} \sum_{D^{4n} < p < D^A} \frac{|a_p|}{p^{1+a/2}} < \tfrac{1}{4}\exp(-c_0\lambda)\log D. \tag{35}$$

If there is a positive number $q < D^A$ such that either $1 - q^{-\delta} \geqslant e^{-B\lambda}$ or $q^{\delta} - 1 \geqslant e^{-B\lambda}$ then we have respectively

$$e^{B\lambda} \geqslant \frac{1}{1 - \exp(-\delta \log q)} \geqslant \frac{1}{1 - \exp(-A\delta \log D)} \geqslant \frac{1}{A\delta \log D}$$

(since $1/(1 - e^{-x}) \geqslant 1/x$ for $x > 0$) or

$$e^{B\lambda} \geqslant \frac{1}{\exp(\delta \log q) - 1} \geqslant \frac{1}{\exp(A\delta \log D) - 1} > \frac{1}{c_4(A)\delta \log D},$$

whence

$$\lambda > B^{-1}\log \frac{1}{c_5(A)\delta \log D}.$$

From this and (29) we deduce (28) (with β_0 instead of β). Hence, we may suppose that

$$\left.\begin{matrix} 1 - N\mathfrak{p}^{-\delta} \\ N\mathfrak{p}^{\delta} - 1 \end{matrix}\right\} < e^{-B\lambda} \quad \text{if} \quad N\mathfrak{p} < D^A. \tag{36}$$

8. In the following paragraphs 8-11 we shall consider the first and second case of (30), namely that of $\chi_0 \neq \chi'$. Then the coefficients a_p with indices $p = N\mathfrak{p}$ in (10) satisfy

$$-a_p = \begin{cases} \sum\limits_{\substack{\mathfrak{p} \\ N\mathfrak{p}=p}} \{1+\chi'(\mathfrak{p})N\mathfrak{p}^{\delta}\}\log N\mathfrak{p} & \text{if} \quad \chi = \chi_0 , \\ \sum\limits_{\substack{\mathfrak{p} \\ N\mathfrak{p}=p}} \chi(\mathfrak{p})\{1+\chi'(\mathfrak{p})N\mathfrak{p}^{-\delta}\}\log N\mathfrak{p} & \text{if} \quad \chi \neq \chi_0 . \end{cases}$$

Let $\mathfrak{p}$ in this paragraph denote prime ideals such that $N\mathfrak{p} = p$, $\mathfrak{p}\nmid\mathfrak{f}$. If in Lemma 2 the alternative (I) holds, then we have, by (13), (12), (31),

$$\sum_{\substack{\mathfrak{p} \\ Z<N\mathfrak{p}\leqslant D^{4n}}} \frac{1+\chi'(\mathfrak{p})N\mathfrak{p}^{-\delta}}{N\mathfrak{p}^{1-\alpha}}\log N\mathfrak{p} \geqslant \log D ,$$

resp.,

$$\sum_{\substack{\mathfrak{p} \\ Z<N\mathfrak{p}\leqslant D^{4n}}} \frac{N\mathfrak{p}^{\delta}+\chi'(\mathfrak{p})}{N\mathfrak{p}^{1-\alpha}}\log N\mathfrak{p} \geqslant \log D ,$$

whence

$$\sum_{\substack{\mathfrak{p} \\ Z<N\mathfrak{p}\leqslant D^{4n}}} \frac{1+\chi'(\mathfrak{p})N\mathfrak{p}^{-\delta}}{N\mathfrak{p}^{1-\alpha}} > 1/4n ,$$

resp.,

$$\sum_{\substack{\mathfrak{p} \\ Z<N\mathfrak{p}\leqslant D^{4n}}} \frac{N\mathfrak{p}^{\delta}+\chi'(\mathfrak{p})}{N\mathfrak{p}^{1-\alpha}} > 1/4n$$

and thus, by (36), (34),

(37) $$\sum_{\substack{\mathfrak{p} \\ Z<N\mathfrak{p}\leqslant D^{4n} \\ \chi'(\mathfrak{p})=1}} \frac{1}{N\mathfrak{p}^{1-\alpha}} > 1/24n .$$

Let us distribute the ideals $\mathfrak{p}$ of the last sum into sets $S_1, \ldots, S_\nu$, the set S_j containing all the $\mathfrak{p}$ satisfying

$$2^{-j}\log D^{4n} < \log N\mathfrak{p} \leqslant 2^{1-j}\log D^{4n} , \qquad j = 1, \ldots, \nu ;\ \nu \ll \log(2+2\lambda) .$$

For at least one set $S_j = S$ (say) we have, by (37),

$$\sum_{\mathfrak{p}\in S} \frac{1}{N\mathfrak{p}^{1-\alpha}} > c_1/\log(2+2\lambda) .$$

Raising to the power 2^j and dividing through by $2^j!\,D^{8n\alpha} < \exp(c_2\lambda)$ we get the inequality

(38) $$\sum_{\substack{\mathfrak{u} \\ D^{4n}<N\mathfrak{u}\leqslant D^{8n}}} 1/N\mathfrak{u} > \exp(-c_3\lambda) ,$$

where $\mathfrak{u}$ denotes ideals of which all the prime divisors are the $\mathfrak{p}$ having $\chi'(\mathfrak{p}) = 1$, $N\mathfrak{p} > Z$.

If in Lemma 2 the alternative (II) holds, then, by (14), (36), (35), (33)

$$4\sum_{\substack{\mathfrak{p} \\ D^{4n}<N\mathfrak{p}\leqslant D^{A} \\ \chi'(\mathfrak{p})=1}} \frac{\log N\mathfrak{p}}{N\mathfrak{p}^{1+\alpha/2}} > \exp(-c_0\lambda)\log D - e^{-B\lambda}\sum_{\substack{\mathfrak{p} \\ D^{4n}<N\mathfrak{p}\leqslant D^{A} \\ \chi'(\mathfrak{p})=-1}} \frac{\log N\mathfrak{p}}{N\mathfrak{p}^{1+\alpha/2}} - \sum_{\substack{\mathfrak{p} \\ N\mathfrak{p}>D^{A}}} \frac{(1+N\mathfrak{p}^{\delta})\log N\mathfrak{p}}{N\mathfrak{p}^{1+\alpha/2}} > \tfrac{1}{2}\exp(-c_0\lambda)\log D ,$$

whence

(39) $$\sum_{\substack{\mathfrak{u} \\ D^{4n}<N\mathfrak{u}\leqslant D^{A}}} 1/N\mathfrak{u} > \exp(-c_4\lambda) ,$$

$\mathfrak{u}$ being defined as in (38). Consider that (39), being a weaker inequality than (38), (since $A > 8n$) may be used in the former case as well.

9. In this and the following paragraph $\mathfrak{a}$ and $\mathfrak{b}$ denote ideals, prime to $\mathfrak{f}$, such that every prime divisor of $\mathfrak{b}$ is in norm less than Z, whereas those of $\mathfrak{a}$ have the norms $\epsilon\,(Z, D^{4n}]$. Hence $(\mathfrak{a}, \mathfrak{b}) = \mathfrak{o}$ and if

$$g(\mathfrak{c}) = \sum_{\mathfrak{b}|\mathfrak{c}} \chi'(\mathfrak{b})$$

then, by [1] § 22,

$$g(\mathfrak{a}\mathfrak{b}) = g(\mathfrak{a})g(\mathfrak{b}) \leqslant \tau(\mathfrak{a})g(\mathfrak{b}) ,$$

whence

$$\sum_{\substack{\mathfrak{b} \\ N\mathfrak{b}\leqslant D^{4n}}} \frac{g(\mathfrak{b})}{N\mathfrak{b}} \prod_{\substack{\mathfrak{p} \\ Z<N\mathfrak{p}\leqslant D^{4n}}} (1-1/N\mathfrak{p})^{-2} = \sum_{\substack{\mathfrak{b} \\ N\mathfrak{b}\leqslant D^{4n}}} \frac{g(\mathfrak{b})}{N\mathfrak{b}} \sum_{\mathfrak{a}} \frac{\tau(\mathfrak{a})}{N\mathfrak{a}} \geqslant \sum_{\substack{\mathfrak{c} \\ N\mathfrak{c}\leqslant D^{4n}}} \frac{g(\mathfrak{c})}{N\mathfrak{c}}$$

or

(40) $$\sum_{\substack{\mathfrak{b} \\ N\mathfrak{b}\leqslant D^{4n}}} \frac{g(\mathfrak{b})}{N\mathfrak{b}} > c_5(1+\lambda)^{-2n} \sum_{\substack{\mathfrak{c} \\ N\mathfrak{c}\leqslant D^{4n}}} \frac{g(\mathfrak{c})}{N\mathfrak{c}} ,$$

by (5), (11), (31). Defining $\mathfrak{u}$ as in (39) we have (since $g(\mathfrak{u}) \geqslant 1$)

(41) $$\sum_{\substack{\mathfrak{b} \\ N\mathfrak{b}\leqslant D^{4n}}} \frac{g(\mathfrak{b})}{N\mathfrak{b}} \sum_{\substack{\mathfrak{u} \\ D^{4n}<N\mathfrak{u}\leqslant D^{A}}} \frac{1}{N\mathfrak{u}} < \sum_{\substack{\mathfrak{b} \\ N\mathfrak{b}\leqslant D^{4n}}} \frac{g(\mathfrak{b})}{N\mathfrak{b}} \sum_{\substack{\mathfrak{u} \\ D^{4n}<N\mathfrak{u}\leqslant D^{A}}} \frac{g(\mathfrak{u})}{N\mathfrak{u}} \leqslant \sum_{\substack{\mathfrak{c} \\ D^{4n}<N\mathfrak{c}\leqslant D^{A+4n}}} \frac{g(\mathfrak{c})}{N\mathfrak{c}} .$$

10. Writing

$$U_1 = \sum_{\substack{\mathfrak{c} \\ N\mathfrak{c}>D^{a}}} \frac{g(\mathfrak{c})}{N\mathfrak{c}} \{\exp(-D^{-b}N\mathfrak{c}) - \exp(-D^{-a}N\mathfrak{c})\} ,$$

$$U_2 = \sum_{\substack{\mathfrak{c} \\ N\mathfrak{c}\leqslant D^{a}}} \frac{g(\mathfrak{c})}{N\mathfrak{c}} \{\exp(-D^{-b}N\mathfrak{c}) - \exp(-D^{-a}N\mathfrak{c})\} ,$$

where $a \geqslant 2n+1$, $b \geqslant a+1$, we have

$$U_1 = \sum_{\substack{\mathfrak{c} \\ N\mathfrak{c} > D^a}} \frac{g(\mathfrak{c})}{N\mathfrak{c}} \exp(-D^{-b}N\mathfrak{c})[1-\exp\{-(D^{-a}-D^{-b})N\mathfrak{c}\}]$$

$$> \sum_{\substack{\mathfrak{c} \\ D^a < N\mathfrak{c} \leqslant D^b}} \frac{g(\mathfrak{c})}{N\mathfrak{c}} e^{-1}(1-e^{-1/2}) > \tfrac{1}{8} \sum_{\substack{\mathfrak{c} \\ D^a < N\mathfrak{c} \leqslant D^b}} \frac{g(\mathfrak{c})}{N\mathfrak{c}}.$$

By the arguments of [1], § 22

$$U_1 + U_2 = (b-a)\mu \log D + O(D^{1-a}\log^{2n+1} D), \qquad \mu = \zeta(1, \chi') \operatorname*{Res}_{s=1} \zeta(s, \chi_0),$$

and $U_2 > 0$ exceeds the absolute value of the remaining term $O(D^{1-a}\log^{2n+1}D)$. Hence

$$(b-a)\mu \log D > \tfrac{1}{8} \sum_{\substack{\mathfrak{c} \\ D^a < N\mathfrak{c} \leqslant D^b}} \frac{g(\mathfrak{c})}{N\mathfrak{c}}.$$

Taking $a = 4n$, $b = 4n+A$ we have, by (41), (40), (39),

(42) $$\mu > \frac{c_6}{\log D} \sum_{\substack{\mathfrak{c} \\ D^{4n} < N\mathfrak{c} \leqslant D^{4n+A}}} \frac{g(\mathfrak{c})}{N\mathfrak{c}} > \frac{c_6}{\log D} \sum_{\substack{\mathfrak{b} \\ N\mathfrak{b} \leqslant D^{4n}}} \frac{g(\mathfrak{b})}{N\mathfrak{b}} \sum_{\substack{\mathfrak{u} \\ D^{4n} < N\mathfrak{u} \leqslant D^{A}}} \frac{1}{N\mathfrak{u}}$$

$$> \frac{c_7}{\log D}(1+\lambda)^{-2n}\exp(-c_4\lambda) \sum_{\substack{\mathfrak{c} \\ N\mathfrak{c} \leqslant D^{4n}}} \frac{g(\mathfrak{c})}{N\mathfrak{c}} > \frac{\exp(-c_8\lambda)}{\log D} \sum_{\substack{\mathfrak{c} \\ N\mathfrak{c} \leqslant D^{4n}}} \frac{g(\mathfrak{c})}{N\mathfrak{c}}.$$

11. Now we take in [1] (5) $y = D^{-3n}$, $w = \beta'$, $f(s) = \zeta(s, \chi')\zeta(s, \chi_0)$ and move the contour of integration to the line $\sigma = \frac{1}{2}$. By [1] (4), [1] (32) and the theorem of residues

(43) $$\sum_{\mathfrak{c}} g(\mathfrak{c}) N\mathfrak{c}^{-\beta'} e^{-yN\mathfrak{c}} = D^{3n\delta}\Gamma(\delta)\mu + R, \qquad R \ll D^{-3n/2+1/2}\log^{2n} D.$$

Since $g(\mathfrak{c}) \geqslant 0$ and $g(\mathfrak{o}) = 1$, the principal term on the right-hand side exceeds twice the modulus of the remaining term. For $N\mathfrak{c} \leqslant D^{4n}$ we have $N\mathfrak{c}^{\delta} \ll 1$, whence

$$\sum_{\substack{\mathfrak{c} \\ N\mathfrak{c} \leqslant D^{4n}}} g(\mathfrak{c}) N\mathfrak{c}^{-\beta'} e^{-yN\mathfrak{c}} = \sum_{\substack{\mathfrak{c} \\ N\mathfrak{c} \leqslant D^{4n}}} \frac{g(\mathfrak{c})}{N\mathfrak{c}} N\mathfrak{c}^{\delta} e^{-yN\mathfrak{c}} < c_9 \sum_{\substack{\mathfrak{c} \\ N\mathfrak{c} \leqslant D^{4n}}} \frac{g(\mathfrak{c})}{N\mathfrak{c}}.$$

Since each of these sums exceeds 1 and

$$\Big| \sum_{\substack{\mathfrak{c} \\ N\mathfrak{c} > D^{4n}}} \frac{g(\mathfrak{c})}{N\mathfrak{c}} e^{-yN\mathfrak{c}} - R \Big| < \tfrac{1}{2}$$

for $D > D_0$, we have

$$\sum_{\mathfrak{c}} g(\mathfrak{c}) N\mathfrak{c}^{-\beta'} e^{-yN\mathfrak{c}} - R < c_{10} \sum_{\substack{\mathfrak{c} \\ N\mathfrak{c} \leqslant D^{4n}}} \frac{g(\mathfrak{c})}{N\mathfrak{c}}.$$

Hence, by (43), (42)

$$\sum_{\substack{\mathfrak{c} \\ N\mathfrak{c} \leqslant D^{4n}}} \frac{g(\mathfrak{c})}{N\mathfrak{c}} > c_{11}\mu/\delta > c_{11} \frac{\exp(-c_8\lambda)}{\delta \log D} \sum_{\substack{\mathfrak{c} \\ N\mathfrak{c} \leqslant D^{4n}}} \frac{g(\mathfrak{c})}{N\mathfrak{c}},$$

whence

$$1 > c_{11} \frac{\exp(-c_8\lambda)}{\delta \log D}, \qquad \lambda > \frac{1}{c_8} \log \frac{c_{11}}{\delta \log D}.$$

From this and (29) we get (28) for $\chi' \neq \chi_0$.

Proof of (28) for $\chi_0 = \chi' \neq \chi$

12. In the following paragraphs 12-15 we shall consider the case of $\chi \neq \chi_0 = \chi'$ to which in (30) corresponds the function

$$F(s) = -\sum_{\substack{\mathfrak{p}, m \\ m \geqslant 1}} \frac{\chi(\mathfrak{p}^m)\log N\mathfrak{p}}{N\mathfrak{p}^{ms}} \qquad (\sigma > 1).$$

Suppose first that in Lemma 2 the alternative (I) holds. Then

$$\sum_{\substack{Z < p < D^{4n} \\ (p = N\mathfrak{p}, \mathfrak{p} \nmid \mathfrak{f})}} \frac{\log p}{p^{1-\alpha}} > \frac{1}{n} \log D,$$

whence

(44) $$\sum_{\substack{p \\ Z < p < D^{4n} \\ (p = N\mathfrak{p}, \mathfrak{p} \nmid \mathfrak{f})}} \frac{1}{p^{1-\alpha}} > 1/4n^2.$$

Distribute the primes p of the last sum into sets $S_1, \ldots, S_\nu$, the set S containing all the p satisfying

$$2^{-j}\log D^{4n} < \log p \leqslant 2^{1-j}\log D^{4n} \qquad (j = 1, \ldots, \nu;\ \nu \ll \log(2+2\lambda)).$$

For at least one set $S_j = S$ (say) we have, by (44),

$$\sum_{p \in S} \frac{1}{p^{1-\alpha}} > c_1/\log(2+2\lambda).$$

Raising to the power 2^j and dividing through by $2^j! D^{8n\alpha} < \exp(c_2\lambda)$ we get the inequality

(45) $$\sum_{D^{4n} < u \leqslant D^{8n}} \frac{1}{u} > \exp(-c_3\lambda)$$

where u denotes integers all the prime divisors of which are $p = N\mathfrak{p} > Z(\mathfrak{p} \nmid \mathfrak{f})$.

If in Lemma 2 the alternative (II) holds, then

$$\sum_{\substack{\mathfrak{p}\\ N\mathfrak{p}=p>D^{4n}}} \frac{\log p}{p^{1+a/2}} > \exp(-c_0\lambda)\log D\,,$$

whence, by (33), for a sufficiently large $A \geqslant 8n$

$$\sum_{\substack{\mathfrak{p}\\ D^{4n}<p=N\mathfrak{p}<D^{A}}} \frac{\log p}{p^{1+a/2}} > \tfrac{1}{2}\exp(-c_0\lambda)\log D$$

or

$$\sum_{\substack{\mathfrak{p}\\ D^{4n}<p=N\mathfrak{p}<D^{A}}} \frac{1}{p} > \frac{1}{2An}\exp(-c_0\lambda)\,.$$

Hence for a suitable c_4 in both cases (I), (II) of Lemma 2

(46) $$\sum_{D^{4n}<u<D^{A}} \frac{1}{u} > \exp(-c_4\lambda)$$

the numbers u having the same meaning as in (45).

13. Writing

$$\zeta_1(s) = \zeta(s)\prod_{p|N\mathfrak{f}}(1-p^{-s}) = \sum_{\substack{m=1\\ (m,N\mathfrak{f})=1}}^{\infty} m^{-s}(\sigma>1)\,,\qquad G(s) = \zeta(s,\chi_0)/\zeta_1(s)\,,$$

we have

$$\zeta(s,\chi_0) = \zeta_1(s)G(s) = \sum_{\substack{m=1\\ (m,N\mathfrak{f})=1}}^{\infty}\frac{1}{m^s}\cdot\sum_{\substack{m=1\\ (m,N\mathfrak{f})=1}}^{\infty}\frac{\gamma_m}{m^s} = \sum_{m=1}^{\infty}\frac{g(m)}{m^s}\,,$$

where

$$g(m) = \sum_{d/m}\gamma_d$$

for $(m, N\mathfrak{f}) = 1$ is the number of ideals of K having the norm $= m$, and $= 0$ otherwise. This is a multiplicative function $\geqslant 0$. Let during this and the following paragraph a and b denote natural numbers prime to $N\mathfrak{f}$ such that any prime dividing b is $< Z$, whereas the primes dividing a satisfy the inequalities $Z < p \leqslant D^{4n}$. Since $(a, b) = 1$, we have

$$g(ab) = g(a)g(b) \leqslant d_n(a)g(b)$$

where the numbers $d_n(a)$ are defined by the expansion

$$\zeta_1^n(s) = \sum_{\substack{m=1\\ (m,N\mathfrak{f})=1}}^{\infty}\frac{d_n(m)}{m^s}\qquad(\sigma>1)$$

(cf. [2], Hilfssatz 1). Hence

$$\sum_{b\leqslant D^{4n}}\frac{g(b)}{b}\prod_{\substack{Z<p<D^{4n}\\ p\nmid N\mathfrak{f}}}\Big(1-\frac{1}{p}\Big)^{-n} = \sum_{b<D^{4n}}\frac{g(b)}{b}\sum_{a}\frac{d_n(a)}{a} \geqslant \sum_{m\leqslant D^{4n}}\frac{g(m)}{m}\,.$$

From this and (5) we deduce that

(47) $$\sum_{b\leqslant D^{4n}}\frac{g(b)}{b} > c_5(1+\lambda)^{-n}\sum_{m\leqslant D^{4n}}\frac{g(m)}{m}\,.$$

Since $g(u) \geqslant 1$,

(48) $$\sum_{b\leqslant D^{4n}}\frac{g(b)}{b}\sum_{D^{4n}<u<D^{A}}\frac{1}{u} < \sum_{b\leqslant D^{4n}}\frac{g(b)}{b}\sum_{D^{4n}<u<D^{A}}\frac{g(u)}{u} \leqslant \sum_{D^{4n}<m\leqslant D^{4n+A}}\frac{g(m)}{m}\,.$$

14. Now we use the identity

$$\sum_{m=1}^{\infty}\frac{g(m)}{m}e^{-ym} = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} y^{1-s}\Gamma(s-1)\zeta(s,\chi_0)\,ds\qquad(y>0)$$

(cf. [1] (5)) with $y = D^{-4n}, D^{-4n-A}$. Moving the contour of integration to the line $\sigma = \frac{1}{2}$ and subtracting we get, by the arguments of § 10,

$$\tfrac{1}{8}\sum_{D^{4n}<m\leqslant D^{4n+A}}\frac{g(m)}{m} < A\mu_0\log D + O(D^{-1})\,,\qquad \mu_0 = \operatorname*{Res}_{s=1}\zeta(s,\chi_0) = D^{o(1)}\,(D\to\infty)$$

(cf. [1] (13)), whence for D large enough

$$A\mu_0\log D > \tfrac{1}{10}\sum_{D^{4n}<m\leqslant D^{4n+A}}\frac{g(m)}{m}\,.$$

Comparing this with (48), (46), (47) we deduce

(49) $$\mu_0 > \frac{c_6}{\log D}\sum_{D^{4n}<m\leqslant D^{4n+A}}\frac{g(m)}{m} > \frac{c_7}{\log D}\sum_{b\leqslant D^{4n}}\frac{g(b)}{b}\sum_{D^{4n}<u<D^{A}}\frac{1}{u}$$
$$> \frac{c_8}{\log D}(1+\lambda)^{-n}\exp(-c_4\lambda)\cdot\sum_{m\leqslant D^{4n}}\frac{g(m)}{m} > \frac{\exp(-c_9\lambda)}{\log D}\sum_{m\leqslant D^{4n}}\frac{g(m)}{m}\,.$$

15. Using in [1] (5) $w = \beta'$, $y = D^{-3n}$, $f(s) = \zeta(s,\chi_0) = \sum_m g(m)m^{-s}$ $(\sigma>1)$ and moving the contour of integration to the line $\sigma = \frac{1}{2}$, we obtain by [1] (4), [1] (32) and the theorem of residues

(50) $$\sum_{m} g(m)m^{-\beta'}e^{-ym} = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} y^{\beta'-s}\Gamma(s-\beta')\zeta(s,\chi_0)\,ds$$
$$= y^{-\delta}\Gamma(\delta)\mu_0 + R\,,\qquad R \ll D^{-1}\,.$$

Since $y^{-\delta} > 1$, $\mu_0 > D^{-1/8}$, $\Gamma(\delta) > 1/2\delta > \log D$ for $D > D_0$, the principal term on the right-hand side exceeds twice the modulus of the remaining term. For $m \leqslant D^{4n}$ we have $m^\delta \ll 1$, whence

$$\sum_{m \leqslant D^{4n}}' \frac{g(m)}{m^{\beta'}} e^{-ym} = \sum_{m \leqslant D^{4n}}' \frac{g(m)}{m} m^{\delta} e^{-ym} < c_{10} \sum_{m \leqslant D^{4n}}' \frac{g(m)}{m}.$$

Since $g(1) = 1$, each of these sums exceed 1 and for $D > D_0$

$$\Big|\sum_{m > D^{4n}} g(m) m^{-\beta'} e^{-ym} - R\Big| < \tfrac{1}{2}.$$

Hence

$$\sum_{m}' g(m) m^{-\beta'} e^{-ym} - R < c_{11} \sum_{m \leqslant D^{4n}}' \frac{g(m)}{m}$$

and, by (50), (49)

$$\sum_{m \leqslant D^{4n}}' \frac{g(m)}{m} > c_{12}\mu_0/\delta > c_{12} \frac{\exp(-c_9\lambda)}{\delta \log D} \sum_{m \leqslant D^{4n}}' \frac{g(m)}{m},$$

whence

$$1 > c_{12} \frac{\exp(-c_9\lambda)}{\delta \log D}, \qquad \lambda > \frac{1}{c_9} \log \frac{c_{12}}{\delta \log D}.$$

From this and (29) we get the lemma for $\chi \neq \chi_0 = \chi'$.

Proof of (28) for $\chi = \chi_0 = \chi'$

16. In this case we suppose the existence of a zero $\varrho_0 = \beta_0 + i\tau_0 \neq \beta'$ of $\zeta(s, \chi_0)$ in $\frac{3}{5} \leqslant \sigma \leqslant 1$, $|t| \leqslant D$ (otherwise there is nothing to prove). By the arguments of § 3 we get the "convenient" zero $\varrho_1 = \beta_1 + i\tau_1$ having the property that there are no zeros of $\zeta(s, \chi_0)$ in $\sigma > \beta_1 + 1/E \log D$, $|t - \tau_1| \leqslant 5\gamma \leqslant 5$. If $|\tau_1| \geqslant 7$, then we use Lemma 2 with $F(s) = \zeta'/\zeta(s, \chi_0)$, $A = \log D$, $\gamma_0 = D$ and get the required result arguing as in §§ 12-15. (1)

Let us suppose $|\tau_1| < 7$. Since there are no zeros of $\zeta(s)$ in the region $R(\sigma > -\frac{1}{2}, |t| \leqslant 14)$ (see [4], II § 12, XV § 1), the function

$$G(s) = \zeta(s, \chi_0)/\zeta_1(s), \qquad \text{where} \qquad \zeta_1(s) = \zeta(s) \prod_{p | N\mathfrak{f}} (1 - p^{-s})$$

is regular in R (cf. [1] (12)). Write

(51) $$f(s) = \zeta(s, \chi_0) \zeta_1(s + \delta) G(s - \delta).$$

This function is regular in R, since the poles $s = 1$ of $\zeta(s, \chi_0)$ and $s = \beta'$ of $\zeta_1(s + \delta)$ are cancelled out by the zeros at the same points of the function

(1) Since in the proof of Lemma 2 the condition $m_\varrho > 0$ is used only in the strip $|t - \tau_1| < 1 + 5\gamma$, the pole at $s = 1$ (with residue -1) of $\zeta'/\zeta(s, \chi_0)$ in the present case does not play any rôle.

$G(s - \delta)$ and $\zeta(s, \chi_0)$, respectively. Let S be the set of zeros $\neq \beta'$ of $\zeta(s, \chi_0)$ in R. The set of zeros of the function $f(s)$ in the same region consists of S and its displacement by δ in the direction of the positive real axis. Hence by the same displacement we get the "convenient" zero of $f(s)$ from that of $\zeta(s, \chi_0)$. The arguments used in § 6 prove that for $F(s) = f'/f(s)$, when A, λ, γ_0, M defined by (31) and $|\tau_1| < 7$, the conditions of Lemma 2 are satisfied.

During the rest of this paragraph let p and $\mathfrak{p}$ denote, respectively, primes and prime ideals not dividing $N\mathfrak{f}$, and $k(p)$ be the number of different prime ideals in K having the norm $p\,[0 \leqslant k(p) \leqslant n]$. Further let $\Phi(s), \Phi_1(s), \ldots$ denote Dirichlet's series $\sum_q c_q q^{-s}$ with $q = p^2, p^3, \ldots$, absolutelly convergent in $\sigma > \frac{1}{2}$.

By (51), [1] (9), we have in $\sigma > 1$

$$G(s) = \prod_{\mathfrak{p}} (1 - N\mathfrak{p}^{-s})^{-1} : \prod_{p} (1 - p^{-s})^{-1},$$

$$\frac{G'}{G}(s) = \sum_p' \frac{p^{-s}\log p}{1 - p^{-s}} - \sum_{\mathfrak{p}}' \frac{N\mathfrak{p}^{-s} \log N}{1 - N\mathfrak{p}^{-s}} = \sum_p' \frac{[1 - k(p)]\log p}{p^s} + \Phi(s),$$

$$\frac{\zeta'}{\zeta}(s, \chi_0) = \sum_p' \frac{-k(p)\log p}{p^s} + \Phi_1(s), \qquad \frac{\zeta_1'}{\zeta_1}(s) = -\sum_p' \frac{\log p}{p^s} + \Phi_2(s),$$

$$\frac{f'}{f}(s) = \sum_p' \frac{-k(p)\log p}{p^s} - \sum_p' \frac{p^{-\delta}\log p}{p^s} + \sum_p' \frac{[1 - k(p)]p^{\delta}\log p}{p^s} + \Phi_3(s)$$

$$= \sum_p' \frac{\log p}{p^s} \{-k(p)(1 + p^{\delta}) + p^{\delta} - p^{-\delta}\} + \Phi_3(s).$$

Hence for the coefficients in (10) we have

(52) $$-a_p = \{k(p)(1 + p^{\delta}) + p^{-\delta} - p^{\delta}\}\log p.$$

λ, A, B being defined as in §§ 5,7 we may suppose

(53) $$\left.\begin{matrix} 1 - p^{-\delta} \\ p^{\delta} - 1 \end{matrix}\right\} < e^{-B\lambda} \quad \text{if} \quad p < D^A.$$

Let us suppose first that in Lemma 2 the alternative (I) holds. Then

$$\sum_{\substack{p \\ Z < N\mathfrak{p} = p < D^{4n}}} \frac{|a_p|}{p^{1-a}} > \log D,$$

whence, by (52), (53),

$$\sum_{\substack{p \\ Z<p=Np<D^{4n}}}' \frac{\log p}{p^{1-a}} > \frac{1}{3n}\log D\,, \qquad \sum_{\substack{p \\ Z<p=Np<D^{4n}}}' \frac{1}{p^{1-a}} > 1/12n^2\,.$$

By the arguments used in § 12 we get the inequality

$$(54) \qquad \sum_{D^{4n}<u<D^{8n}} 1/u > \exp(-c_1\lambda)\,,$$

where u denotes integers all the prime divisors of which are $p = Np > Z$, prime to $N\mathfrak{f}$.

If the alternative (II) of Lemma 2 holds, then

$$\sum_{\substack{p \\ p=Np>D^{4n}}}' \frac{|a_p|}{p^{1+a/2}} > \exp(-c_0\lambda)\log D\,.$$

Hence, by (52), (53) and the analogous of (33),

$$\sum_{\substack{p \\ D^{4n}<p=Np<D^{A}}}' \frac{\log p}{p^{1+a/2}} > \frac{1}{6n}\exp(-c_0\lambda)\log D\,,$$

if $A \geqslant 8n$ is large enough (cf. § 8), whence

$$\sum_{\substack{p \\ D^{4n}<p=Np<D^{A}}}' \frac{1}{p} > \frac{1}{6An}\exp(-c_0\lambda)\,.$$

Comparing with (54) we deduce that in both cases (I), (II) of Lemma 2

$$\sum_{D^{4n}<u<D^{A}} 1/u > \exp(-c_2\lambda)\,,$$

the integers u being defined as in (54). We complete the proof by arguments used in §§ 13-15.

Proof of the theorem

17. By § 1 there is a $c_1 > 0$ such that in the region

$$(55) \qquad R(1-c_1/\log D \leqslant \sigma \leqslant 1\,,\ |t| \leqslant D)$$

there is at most one zero $\varrho = \beta'$ of $Z(s)$. If β' is not in R, then the theorem holds for $A = c_1$. If $\beta' \in R$, then $\delta \leqslant c_1/\log D$ and taking $A = c_1$ we have $\delta_0 = \delta$. By Lemma 3 there is no zero $\varrho \neq \beta'$ of $Z(s)$ in the region $\sigma \geqslant \sigma_0$, $|t| \leqslant D$, if

$$1-\sigma_0 = \frac{A_2}{\log D}\log\frac{eA_1}{\delta\log D}\,.$$

This is

$$> \frac{A}{\log D}\log\frac{eA}{\delta_0\log D}$$

if $A = c_1 < \min(A_1, A_2)$, which can be taken for granted (otherwise replace c_1 in (55) by a smaller constant).

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