(cf. the arguments at the end of § 21). Using (32) and the integral formula for $\zeta'(s, \chi)$ (cf. (55)) we deduce that $\zeta'(\sigma_1, \chi) \ll D^{1/8n}$. Combining this with the previous inequality we get (56).

By a more careful account of the number of squares $\mathfrak{c}$ whose norm does not exceed D^a it can be proved that $\delta > c_4 D^{-\varkappa}$ for any $\varkappa > \frac{1}{4}(n+3)$ and $c_4 = c_4(\varkappa)$. But for our prospective arithmetical applications we can do as well with (56).

The theorem of § 1 is an immediate consequence of Lemmas 9-17.

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On sign-changes of the difference $\pi(x) - \mathrm{li}\, x$

by

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1. Let $\nu(T)$ denote the number of sign-changes of the difference $\pi(x) - \mathrm{li}\, x$ for $2 \leqslant x \leqslant T$. Littlewood ([7]) proved in 1914 that $\nu(T)$ tends to infinity together with T. However Littlewood's method, as it stands in [7], does not provide numerical results and in particular does not enable one, even on the Riemann hypothesis, to find an explicit upper bound for the position of the first sign-change of $\pi(x) - \mathrm{li}\, x$. Such numerical estimation has been performed only a few years ago by Skewes [8], the result being

$$\nu\big(\exp\exp\exp\exp(7.705)\big) \geqslant 1 . \tag{1.1}$$

A conditional estimate for the order of growth of $\nu(T)$ has been obtained by Ingham [4]. His theorem reads as follows:

If there exists a ζ-zero $\varrho_0 = \sigma_0 + it_0$ such that $\zeta(s) \neq 0$ in the half-plane $\sigma > \sigma_0$, then

$$\varliminf_{T \to \infty} \frac{\nu(T)}{\log T} > 0 . \tag{1.2}$$

I proved recently [5] the following theorem which leads, when combined with that of Ingham, to an *unconditional* lower estimate for $\nu(T)$:

Let $\varrho_0 = \beta_0 + i\gamma_0$, $\beta_0 \geqslant \frac{1}{2}$, $\gamma_0 > 0$ be an arbitrary ζ-zero. Then, for $T > \max\big(c_1, \exp\exp(\log^2\gamma_0)\big)$, c_1 a numerical constant, we have the inequalities

$$\begin{cases} \max\limits_{2 \leqslant t \leqslant T} \{\Pi(t) - \mathrm{li}\, t\} > T^{\beta_0} \exp\left(-15 \dfrac{\log T}{\sqrt{\log\log T}}\right), \\ \min\limits_{2 \leqslant t \leqslant T} \{\Pi(t) - \mathrm{li}\, t\} < -T^{\beta_0} \exp\left(-15 \dfrac{\log T}{\sqrt{\log\log T}}\right), \end{cases} \tag{1.3}$$

where

$$\Pi(x) = \sum_{m=1}^{\infty} \frac{1}{m} \pi(x^{1/m}) .$$

Having got this we obtain easily

$$\varliminf_{T\to\infty}\frac{\nu(T)}{\log\log T}>0 \tag{1.4}$$

without any unproved conjecture. In fact, in the case of the Riemann hypothesis being true (1.4) follows *a fortiori* from (1.2) while in the opposite case the inequalities (1.3), applied with some $\varrho_0=\beta_0+i\gamma_0$, $\beta_0>\frac{1}{2}$, give

$$\max_{T^{1/2}\leqslant x\leqslant T}(\pi(x)-\mathrm{li}\,x)>0\,,\qquad \min_{T^{1/2}\leqslant x\leqslant T}(\pi(x)-\mathrm{li}\,x)<0\,,$$

valid for all sufficiently large T, which again imply (1.4).

The aim of this paper is to estimate $\nu(T)$ explicitly from below. It is clear that (1.4), however unconditional, is not numerical and having it one cannot determine an explicit finite interval in which $\pi(x)-\mathrm{li}\,x$ changes sign, say, at least 100 times. Such result will be contained in the following

Theorem. *We have*

$$\nu(T)\geqslant\frac{1}{e^{35}}\log\log\log\log T \tag{1.5}$$

for

$$T\geqslant\exp\exp\exp\exp\exp 35\,. \tag{1.6}$$

I did not care much about the numerical constants in (1.5), (1.6); certainly they can be much improved, and probably even so as to cover the quoted result (1.1) of Skewes. Skewes e.g., with a view to making his bound possibly small, has done a certain amount of computations relating to some numerically known ζ-zeros (actually to those with positive imaginary parts <500, 269 in number), of which no use will be made in the present paper. What is more, on choosing the parameters occurring in [5] in a suitable way, one might put the theorem concerned in a form much more useful for numerical purposes which would in turn certainly improve on (1.5), (1.6).

So much for the numerical constants involved. As to the functions $\log\log\log\log T$ resp. $\log\log T$ in (1.4), it would be desirable to replace them by "better" ones, i.e. by functions going more quickly to infinity. This seems to be quite possible and the simplest way for doing so would depend upon finding inequalities similar to (1.3) but with t ranging in a smaller interval, e.g. $\left[T\exp\left(-\frac{\log T}{\log\log T}\log\log\log T\right),\,T\right]$. Such question is, after all, interesting in itself and I hope to return to it in another paper.

I close this introduction by listing a number of numerical results, all relating to the ζ-zeros, of which use will be made in the following.

Let $\varrho=\beta+i\gamma$ denote the zeros of $\zeta(s)$. Let $\frac{1}{2}+i\gamma_1$ be the ζ-zero with minimal positive γ_1 (which is known to be 14.,13...). Let $N(T_1)$ be the number of ϱ's with $0<\beta<1$, $0<\gamma\leqslant T_1$. Suppose $T_1\geqslant\gamma_1$. Then

$$N(T_1)=\frac{T_1}{2\pi}\log\frac{T_1}{2\pi e}+R(T_1)\,,\ \textit{where} \tag{1.7}$$

$$|R(T_1)|<(0.137)\log T_1+(0.443)\log\log T_1+4.350$$

(see [1], also [8], p. 50).

$$\sum_{0<\gamma<T_1}\frac{1}{\gamma}<\frac{1}{4\pi}\log^2T_1 \quad \text{(see [8], p. 50)}. \tag{1.8}$$

$$\sum_{\gamma\geqslant T_1}\frac{1}{\gamma^2}<\frac{1}{2\pi}\,\frac{\log T_1}{T_1} \quad \text{(see [8], p. 50)}. \tag{1.9}$$

$$\sum_{\gamma>0}\frac{1}{\gamma^2}<0.0233 \quad \text{(see [8], p. 50)}. \tag{1.10}$$

(1.11) *Put* $R(x,T_1)=\sum_{|\gamma|>T_1}\frac{x^{\varrho}}{\varrho}$. *Then*

$$|R(x,T_1)|<\frac{x^{1/2}}{10^4}\quad \textit{for}\quad x\geqslant\exp 10^4,\ T_1\geqslant x^2$$

(see [8], p. 51).

(1.12) *If* $|\gamma|\leqslant 10^4$, $0<\beta<1$, *then* $\beta=\frac{1}{2}$ (see [6]).

2. T is supposed to satisfy (1.6) throughout. Write

$$X=\sqrt[3]{\log\log T}\ (\geqslant\exp\exp\exp 34)\,.$$

In this and in the next section we shall deduce (1.5) from the following conjecture:

(C) *Every* ζ*-zero* $\varrho=\beta+i\gamma$ *with* $0<\beta<1$, $|\gamma|\leqslant X^3$, *is such that*

$$|\beta-\tfrac{1}{2}|<\frac{2}{3X^3\log X}\,.$$

Actually we shall prove more, namely, establish the inequality

$$\nu(X)\geqslant\frac{1}{e^{32}}\log\log X-1\,. \tag{2.1}$$

In § 4 we shall deduce (1.5) from (NC)—the negation of (C).

We start with a number of lemmas. First of all let us define, as usual,

$$\psi(x)=\sum_{n\leqslant x}\Lambda(n)\,,\qquad \psi_0(x)=\frac{\psi(x-0)+\psi(x+0)}{2}\,,$$

$$\psi_1(x)=\int_1^x\psi(u)\,du=\sum_{n\leqslant x}(x-n)\Lambda(n)\,.$$

LEMMA 1. *Without any hypothesis*

(2.2) $$|\psi_1(x)-\tfrac{1}{2}x^2| < \tfrac{15}{2}x \quad \text{for} \quad 2 \leqslant x \leqslant e^8 .$$

On the conjecture (C)

(2.3) $$|\psi_1(x)-\tfrac{1}{2}x^2| < \tfrac{1}{10}x^{3/2} \quad \text{for} \quad e^8 < x \leqslant X^{3/2} .$$

Proof. Using the formula

$$\psi_1(x) = \tfrac{1}{2}x^2-\sum_{\varrho}\frac{x^{\varrho+1}}{\varrho(\varrho+1)}-x\frac{\zeta'}{\zeta}(0)+\frac{\zeta'}{\zeta}(-1)-\sum_{r=1}^{\infty}\frac{x^{1-2r}}{2r(2r-1)}$$

and (1.9), (1.10) we have

$$x^{-3/2}\left|\sum_{\varrho}\frac{x^{\varrho+1}}{\varrho(\varrho+1)}\right| \leqslant \sum_{|\gamma|\leqslant X^3}\left|\frac{x^{\varrho-1/2}}{\varrho(\varrho+1)}\right|+\sum_{|\gamma|>X^3}\left|\frac{x^{\varrho-1/2}}{\varrho(\varrho+1)}\right|$$
$$< X^{\frac{3}{2}\cdot\frac{2}{3X^3\log X}}\sum_{\gamma}\frac{1}{\gamma^2}+X^{3/4}\frac{3}{\pi}\cdot\frac{\log X}{X^3} < \frac{1}{20} ,$$

whence (2.3), as

$$\frac{\zeta'}{\zeta}(0) = \log 2\pi , \qquad \left|\frac{\zeta'}{\zeta}(-1)\right| < 1 ,$$

$$\sum_{r=1}^{\infty}\frac{x^{1-2r}}{2r(2r-1)} < \sum_{r=1}^{\infty}\frac{1}{2r(2r-1)} < \frac{3}{4} .$$

Similarly, using (1.9), (1.10), (1.12), we obtain

$$x^{-1}\left|\sum_{\varrho}\frac{x^{\varrho+1}}{\varrho(\varrho+1)}\right| < x^{1/2}\sum_{|\gamma|\leqslant 10^4}\frac{1}{\gamma^2}+x\sum_{|\gamma|>10^4}\frac{1}{\gamma^2} < 0.05\,x^{1/2}+\frac{x}{\pi}\cdot\frac{4\log 10}{10^4}$$
$$< \frac{1}{20}\cdot 60+\frac{3600}{10\,000}\cdot 4 = 4.44 ,$$

and (2.2) follows.

LEMMA 2. *Write*

$$P_0(x) = \{\Pi(x)-\mathrm{li}\,x\}-\frac{\psi_0(x)-x}{\log x} .$$

On the conjecture (C)

(2.4) $$|P_0(x)| \leqslant 2\frac{x^{1/2}}{\log^2 x} \quad \text{for} \quad e^{10^4} \leqslant x \leqslant X^{3/2} .$$

Proof. Putting

$$P(x) = \{\Pi(x)-\mathrm{li}\,x\}-\frac{\psi(x)-x}{\log x}$$

(so that

(2.5) $$P_0(x) = P(x)+\theta\cdot\tfrac{1}{2} , \qquad |\theta| \leqslant 1)$$

and noting the formula ([3], 64)

$$P(x) = \int_2^x \frac{\psi(u)-u}{u\log^2 u}\,du+\frac{2}{\log 2}-\mathrm{li}\,2$$

we get integrating by parts

$$|P(x)| \leqslant \left|\frac{\psi_1(x)-\frac{1}{2}x^2}{x\log^2 x}-\frac{\psi_1(2)-2}{2\log^2 2}+\frac{2}{\log 2}-\mathrm{li}\,2\right|+\int_2^x|\psi_1(u)-\tfrac{1}{2}u^2|\frac{4}{u^2\log^2 u}\,du .$$

Writing the last integral as $\int_2^{e^8}+\int_{e^8}^{x}$ and using Lemma 1 we obtain

$$\int_2^x|\psi_1(u)-\tfrac{1}{2}u^2|\frac{4}{u^2\log^2 u}\,du \leqslant 30\int_2^{e^8}\frac{du}{u\log^2 u}+\frac{2}{5}\int_{e^8}^{x}\frac{u^{1/4}}{\log^2 u}\cdot\frac{du}{u^{3/4}}$$
$$< \frac{30}{\log 2}+\frac{x^{1/4}}{\log^2 x}\cdot\frac{2}{5}\cdot 4x^{1/4} < 45+\frac{8}{5}\cdot\frac{x^{1/2}}{\log^2 x} .$$

Hence and by (2.5)

$$|P_0(x)| \leqslant 50+1.7\frac{x^{1/2}}{\log^2 x} < 2\frac{x^{1/2}}{\log^2 x} .$$

LEMMA 3.

(2.6) $$0 < \Pi(x)-\pi(x) \leqslant 1.43\frac{x^{1/2}}{\log x}$$

for

$$x \geqslant e^{10^4} .$$

Proof. Writing

$$\vartheta(y) = \sum_{p\leqslant y}\log p \qquad (p\text{—prime number})$$

we have for $y \geqslant 2$

$$\vartheta(y) \leqslant 2y\log 2 \qquad ([2],\ \text{p. } 341) .$$

Further, with $\delta = \dfrac{3}{10^3}$,

$$\vartheta(y) \geqslant \sum_{y^{1-\delta}<p\leqslant y}\log p \geqslant (1-\delta)\log y\,\{\pi(y)-\pi(y^{1-\delta})\}$$
$$> (1-\delta)\log y\cdot\pi(y)-(1-\delta)y^{1-\delta}\log y ,$$

so that

$$\pi(y) < \frac{2\log 2}{1-\delta}\cdot\frac{y}{\log y}+y^{1-\delta} .$$

But $\frac{\log y}{y^{\delta}}$ decreases for $y > e^{\frac{1}{3}10^3}$, whence for $y \geqslant e^{\frac{1}{2}10^4}$

$$\pi(y) \leqslant \frac{y}{\log y}\left(\frac{2\log 2}{1-\delta} + \frac{\frac{1}{2}10^4}{e^{\frac{1}{2}10^4\cdot\frac{3}{10^3}}}\right) < 1.42\,\frac{y}{\log y}\,.$$

Consequently

$$0 < \Pi(x) - \pi(x) = \tfrac{1}{2}\pi(x^{1/2}) + \sum_{m=3}^{[\log x/\log 2]} \frac{1}{m}\pi(x^{1/m})$$

$$< \frac{x^{1/2}}{\log x}\cdot 1.42 + \frac{\log x}{3\log 2}\cdot x^{1/3} < 1.43\,\frac{x^{1/2}}{\log x}\,.$$

Lemma 4. *Write*

$$f(x) = \frac{\pi(x) - \operatorname{li} x}{x^{1/2}(\log x)^{-1}}\,, \qquad g(x) = \frac{\psi_0(x) - x}{x^{1/2}}\,;$$

$$F(u) = f(e^u)\,, \qquad G(u) = g(e^u)\,.$$

On the conjecture (C)

(2.7) $$|F(u) - G(u)| \leqslant \tfrac{3}{2}$$

for

$$10^4 \leqslant u \leqslant \tfrac{3}{2}\log X\,.$$

Proof. Using (2.4) and (2.6) we have

$$|f(x) - g(x)| \leqslant \frac{2}{\log x} + 1.43 < 1.5\,.$$

Substituting $x = e^u$ we obtain (2.7).

Lemma 5. *Suppose*

$$\alpha \geqslant e^{10}\,.$$

Writing

$$S_\alpha(u) = 2\sum_{0<\gamma\leqslant\alpha}\frac{\sin(u\gamma)}{\gamma}\left(1 - \frac{\gamma}{\alpha}\right)$$

we have

(2.8) $$S_\alpha\left(\frac{\pi}{2\alpha}\right) \geqslant \frac{1}{8}\log\alpha - \frac{1}{2}\,.$$

Proof. By the definition of $S_\alpha(u)$

$$S_\alpha(u) = 2\int_1^\alpha \frac{\sin(ux)}{x}\left(1 - \frac{x}{\alpha}\right)dN(x)$$

$$= 2\int_{\gamma_1}^\alpha \frac{N(x)}{x}\left\{\frac{1}{x}\sin(ux) - \left(1 - \frac{x}{\alpha}\right)u\cos(ux)\right\}dx\,.$$

Hence

$$S_\alpha\left(\frac{\pi}{2\alpha}\right) = 2\int_{\gamma_1}^{\frac{2}{3}\alpha}\frac{N(x)}{x}\left\{\frac{1}{x}\sin\left(\frac{\pi x}{2\alpha}\right) - \frac{\pi}{2\alpha}\cos\left(\frac{\pi x}{2\alpha}\right) + \frac{x}{\alpha^2}\cdot\frac{\pi}{2}\cos\left(\frac{\pi x}{2\alpha}\right)\right\}dx +$$

$$+2\int_{\frac{2}{3}\alpha}^{\alpha}\frac{N(x)}{x}\left\{\frac{1}{x}\sin\left(\frac{\pi x}{2\alpha}\right) - \left(1 - \frac{x}{\alpha}\right)\frac{\pi}{2\alpha}\cos\left(\frac{\pi x}{2\alpha}\right)\right\}dx \equiv I_1 + I_2\,.$$

In virtue of (1.7), and noting that $\frac{1}{x}\sin\left(\frac{\pi x}{2\alpha}\right) - \frac{\pi}{2\alpha}\cos\left(\frac{\pi x}{2\alpha}\right) \geqslant 0$,

$$I_1 \geqslant 2\int_{\gamma_1}^{\frac{2}{3}\alpha} N(x)\cdot\frac{1}{2}\cdot\frac{\pi}{2}\cdot\frac{1}{\alpha^2}\,dx \geqslant \frac{\pi}{2\alpha^2}\int_{\gamma_1}^{\frac{2}{3}\alpha}\frac{1}{2\pi}x\log\frac{x}{2\pi e}\,dx - \frac{1}{500} > \frac{1}{18}\log\alpha - 0.24\,.$$

Similarly

$$I_2 \geqslant 2\int_{\frac{2}{3}\alpha}^{\alpha}\frac{N(x)}{x}\left\{\frac{\sqrt{3}}{2}\cdot\frac{1}{x} - \frac{\pi}{2\alpha}\cdot\frac{1}{6}\right\}dx \geqslant 2\int_{\frac{2}{3}\alpha}^{\alpha}\frac{N(x)}{x^2}\left(\frac{\sqrt{3}}{2} - \frac{\pi}{12}\right)dx$$

$$> \frac{6}{5}\int_{\frac{2}{3}\alpha}^{\alpha}\frac{N(x)}{x^2}\,dx \geqslant \frac{3}{5\pi}\int_{\frac{2}{3}\alpha}^{\alpha}\frac{\log\frac{x}{2\pi e}}{x}\,dx - \frac{1}{500}$$

$$= \frac{3}{10\pi}\log\frac{2\alpha^2}{3(2\pi e)^2}\log\frac{3}{2} - \frac{1}{500} > (0.07)\log\alpha - 0.26$$

and (2.8) follows.

3. We turn to the proof of

$$(\mathrm{C}) \Rightarrow (2.1)\,.$$

Put

$$\alpha = e^{26}\,, \qquad q = 26^2\,, \qquad N = N(\alpha) = N(e^{26})\ (< e^{28})\,.$$

Further, let, for

$$\nu = 0, 1, 2, \ldots, \left[\frac{\log\log X}{3N\log q}\right] - 1\,,$$

ω_ν be the number satisfying

(3.1) $$(q^N \leqslant)\ q^{(3\nu+1)N} \leqslant \omega_\nu \leqslant q^{(3\nu+2)N}\ \left(\leqslant \frac{\log X}{q^N}\right)$$

such that

(3.2) $$\left|\frac{\gamma\omega_\nu}{2\pi}\right| < \frac{1}{q}\ (\operatorname{mod} 1) \quad \text{for} \quad 0 < \gamma \leqslant \alpha$$

(the existence of ω_ν follows by Dirichlet's theorem).

Putting now

$$\omega_\nu^* = \omega_\nu + \frac{\pi}{2a} \tag{3.3}$$

we get by (3.2) and (1.8)

$$\left|S_a(\omega_\nu^*) - S_a\left(\frac{\pi}{2a}\right)\right| < \frac{4\pi}{q} \sum_{0<\gamma\leqslant a} \frac{1}{\gamma} \leqslant \frac{\log^2 a}{q} = 1 ,$$

whence and by (2.8)

$$S_a(\omega_\nu^*) > \tfrac{1}{8}\log a - \tfrac{3}{2} . \tag{3.4}$$

Substitute $x = e^u$ in the formula

$$\psi_0(x) - x = -\sum_\varrho \frac{x^\varrho}{\varrho} - \frac{\zeta'}{\zeta}(0) - \tfrac{1}{2}\log\left(1 - \frac{1}{x^2}\right),$$

multiply by

$$e^{-\frac{1}{2}u} K_a(u - \omega_\nu^*)$$

where

$$K(y) = \left(\frac{\sin y/2}{y/2}\right)^2, \qquad K_a(y) = aK(ay) ,$$

and integrate with respect to u in $[\frac{1}{2}\omega_\nu^*, \frac{3}{2}\omega_\nu^*] \equiv [h_1, h_2]$. This gives

$$\int_{h_1}^{h_2} K_a(u - \omega_\nu^*) G(u)\, du \tag{3.5}$$

$$= -\sum_\varrho \frac{1}{\varrho} \int_{h_1}^{h_2} K_a(u - \omega_\nu^*) e^{(\varrho-1/2)u} du + \int_{h_1}^{h_2} K_a(u - \omega_\nu^*) r(u)\, du ,$$

where

$$r(u) = e^{-\frac{1}{2}u}\left(\frac{1}{2}\log\frac{1}{1-e^{-2u}} - \frac{\zeta'}{\zeta}(0)\right).$$

Owing to (3.1) and (3.3)

$$|r(u)| < 2e^{-\frac{1}{4}\omega_\nu^*}$$

which together with (3.5) gives, as

$$\int_{-ah_1}^{ah_1} K(y)\, dy < \int_{-\infty}^{+\infty} K(y)\, dy = 2\pi ,$$

$$\int_{-ah_1}^{ah_1} K(y) G\left(\omega_\nu^* + \frac{y}{a}\right) dy \tag{3.6}$$

$$= -\sum_\varrho \frac{e^{(\varrho-1/2)\omega_\nu^*}}{\varrho} \int_{-ah_1}^{ah_1} K(y) e^{\frac{\varrho-1/2}{a}y} dy + \theta_1 \cdot 4\pi e^{-\frac{1}{4}\omega_\nu^*} \qquad (|\theta_1| \leqslant 1) .$$

Now, using the notation of (1.11),

$$|S_1| \overset{\text{def}}{=} \left|\sum_{|\gamma|>X^3} \frac{1}{\varrho} \int_{-ah_1}^{ah_1} K(y) e^{(\varrho-1/2)(\omega_\nu^*+y/a)} dy\right| \leqslant \int_{-ah_1}^{ah_1} K(y) \left|\frac{R(e^{\omega_\nu^*+y/a}, X^3)}{e^{\frac{1}{2}(\omega_\nu^*+y/a)}}\right| dy ,$$

which yields by (1.11)

$$|S_1| < \frac{2\pi}{10^4} . \tag{3.7}$$

Writing further $\varrho - \frac{1}{2} = \beta - \frac{1}{2} + i\gamma = b + i\gamma$, we get by (C) and (1.10) for $1 \leqslant x \leqslant X^{3/2}$

$$\sum_{|\gamma|\leqslant X^3} \left|\frac{x^{b+i\gamma}}{\beta+i\gamma} - \frac{x^{i\gamma}}{i\gamma}\right| \leqslant \left(\sum_{|\gamma|\leqslant X^3} \frac{1}{\gamma^2}\right) \max_{|\gamma|\leqslant X^3} \{|\gamma(x^b - 1)| + \beta\}$$

$$< 0.05\{X^3(e^{1/X^3} - 1) + 1\} < 0.15 .$$

Consequently

$$\left|\int_{-ah_1}^{ah_1} K(y)\left\{\sum_{|\gamma|\leqslant X^3} \frac{e^{(\varrho-1/2)(\omega_\nu^*+y/a)}}{\varrho} - \sum_{|\gamma|\leqslant X^3} \frac{e^{i\gamma(\omega_\nu^*+y/a)}}{i\gamma}\right\} dy\right| \leqslant 0.152\cdot\pi < 0.96$$

(here $x = e^{\omega_\nu^*+y/a} \leqslant e^{\frac{3}{2}\omega_\nu^*} < X^{3/2}$).

Hence and by (3.6), (3.7)

$$\int_{-ah_1}^{ah_1} K(y) G(\omega_\nu^* + y/a)\, dy \tag{3.8}$$

$$= -\sum_{|\gamma|\leqslant X^3} \frac{e^{i\gamma\omega_\nu^*}}{i\gamma} \int_{-ah_1}^{ah_1} K(y) e^{i\gamma y/a} dy + 0.97\cdot\theta_2 \qquad (|\theta_2| \leqslant 1) .$$

Finally

$$|S_2| \overset{\text{def}}{=} \left|\sum_{|\gamma|\leqslant X^3} \frac{e^{i\gamma\omega_\nu^*}}{i\gamma}\left(\int_{-\infty}^{-ah_1} + \int_{ah_1}^{+\infty}\right) K(y) e^{i\gamma y/a} dy\right| \leqslant 4 \sum_{0<\gamma\leqslant X^3} \frac{1}{\gamma}\left|\int_{ah_1}^{\infty} K(y) e^{i\gamma y/a} dy\right|$$

$$= 4\sum_{0<\gamma<a} + 4\sum_{a\leqslant\gamma\leqslant X^3} ,$$

whence and by the inequality (see [8], 57)

$$\left|\int_{ah_1}^{\infty} K(y) e^{i\gamma y/a} dy\right| \leqslant \begin{cases} \dfrac{8}{a\omega_\nu^*} , \\ \dfrac{16}{\omega_\nu^*\gamma} , \end{cases}$$

further by (1.8), (1.9)

$$|S_2| \leqslant \frac{32}{\alpha\omega_\nu^*}\sum_{0<\gamma<\alpha}\frac{1}{\gamma}+\frac{64}{\omega_\nu^*}\sum_{\alpha\leqslant\gamma\leqslant X^3}\frac{1}{\gamma^2} < \frac{8}{\pi\alpha\omega_\nu^*}\log^2\alpha+\frac{32}{\pi\omega_\nu^*}\cdot\frac{\log\alpha}{\alpha} > \frac{1}{100}.$$

Hence and by (3.8)

$$\int_{-\alpha h_1}^{\alpha h_1} K(y)G(\omega_\nu^*+y/\alpha)\,dy = -\sum_{|\gamma|\leqslant X^3}\frac{e^{i\gamma\omega_\nu^*}}{i\gamma}\int_{-\infty}^{+\infty}K(y)e^{i\gamma y/\alpha}dy+\theta_3 \qquad (|\theta_3|\leqslant 1),$$

which yields, in view of

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}K(y)e^{ixy}dy = \begin{cases} 1-|x| & \text{for} \quad -1\leqslant x\leqslant 1,\\ 0 & \text{for} \quad |x|>1,\end{cases}$$

(3.9) $$\frac{1}{2\pi}\int_{-\alpha h_1}^{\alpha h_1}K(y)G\left(\omega_\nu^*+\frac{y}{\alpha}\right)dy = -S_\alpha(\omega_\nu^*)+\frac{\theta_3}{2\pi}.$$

Since owing to (3.1)

$$10^4 < \tfrac{1}{2}\omega_\nu^* \leqslant \omega_\nu^*+\frac{y}{\alpha}\leqslant \tfrac{3}{2}\omega_\nu^* < \tfrac{3}{2}\log X$$

for

$$-\tfrac{1}{2}\alpha\omega_\nu^* \leqslant y \leqslant \tfrac{1}{2}\alpha\omega_\nu^* \ (=\alpha h_1),$$

we get from (3.9), using (2.7) and (3.4),

$$\frac{1}{2\pi}\int_{-\alpha h_1}^{\alpha h_1}K(y)\cdot F(\omega_\nu^*+y/\alpha)\,dy < -\frac{1}{8}\log\alpha+3+\frac{1}{2\pi} < 0.$$

From this it follows that there exists some ω',

$$q^{8\nu N} < \omega' < q^{8(\nu+1)N},$$

such that $F(\omega')<0$.

Similarly, writing

$$\omega_\nu^{**} = \omega_\nu - \frac{\pi}{2\alpha}$$

for ω_ν^* in (3.5), using further the inequality

$$S_\alpha\left(-\frac{\pi}{2\alpha}\right) \leqslant -\frac{1}{8}\log\alpha+\frac{1}{2},$$

we prove that there exists an ω'',

$$q^{8\nu N} < \omega'' < q^{8(\nu+1)N}$$

such that $F(\omega'')>0$. Turning now to the function $\pi(x)-\mathrm{li}\,x$ we find that in each interval

$$\exp(q^{8\nu N}) < x < \exp(q^{8(\nu+1)N}),$$

where

$$\nu = 0, 1, 2, \ldots, \left[\frac{\log\log X}{3N\log q}\right]-1,$$

there is at least one change of sign of $\pi(x)-\mathrm{li}\,x$. Hence

$$\nu(X) \geqslant \frac{\log\log X}{3N\log q}-1$$

and (2.1) follows.

4. Lemma 6. *Suppose that there exists a ζ-zero $\varrho^* = \beta^*+i\gamma^*$ such that*

(4.1) $$\beta^* \geqslant \frac{1}{2}+\frac{16}{\sqrt{\log\log X_1}}, \qquad 0<\gamma^*\leqslant \exp\sqrt{\log\log X_1},$$

where

$$X_1 \geqslant \exp\exp\exp\exp\exp 34.$$

Then

(4.2) $$\max_{2\leqslant x\leqslant X_1}(\pi(x)-\mathrm{li}\,x) > X_1^{\beta^*}\exp\left(-\frac{16\log X_1}{\sqrt{\log\log X_1}}\right),$$
$$\min_{2\leqslant x\leqslant X_1}(\pi(x)-\mathrm{li}\,x) < -X_1^{\beta^*}\exp\left(-\frac{16\log X_1}{\sqrt{\log\log X_1}}\right).$$

This lemma is a corollary of the quoted result of [5], in fact for its proof it would suffice to follow [5] numerically. In the Appendix to this paper I have collected a number of simple numerical results, together with sketchy proofs, which would facilitate one to settle the matter. It may be noted that in most cases the numerical estimations, to be performed in the course of proof of Lemma 6, are quite trivial owing to the enormous size of X_1.

Now we assume the negation of (C):

(NC) *There exists a ζ-zero*

$$\varrho_0 = \beta_0+i\gamma_0,$$

such that

(4.3) $$\beta_0-\frac{1}{2} \geqslant \frac{14}{3(\log\log T)^{3/7}\log\log\log T}, \qquad 0<\gamma_0<(\log\log T)^{3/7}.$$

Passing to the proof of

$$(\mathrm{NC}) \Rightarrow (1.5)$$

we write

$$U = \exp\left(\frac{e^{X^7}}{\log X}\right)$$

and apply Lemma 6 with

$$X_1 = X_1(m) = U^{2^m} \ (\leqslant T),$$

where

$$m = 0, 1, 2, \ldots, [\log\log X]$$

(4.1) is clearly satisfied in virtue of (4.3), whence and from (4.2)

$$\max_{X_1^{1/2}<x\leqslant X_1}\{\pi(x)-\mathrm{li}\,x\}>0\,,\qquad \min_{X_1^{1/2}<x\leqslant X_1}\{\pi(x)-\mathrm{li}\,x\}<0\,,$$

i.e. $\pi(x)-\mathrm{li}\,x$ changes at least once sign in $(X_1^{1/2}, X_1]$ and (1.5) follows.

Appendix

A.1. *Let* $\varrho=\beta+i\gamma$, $\gamma>0$ *be a* ζ*-zero. Then*

$$\beta>\frac{\log^{-9}\gamma}{36^4}\,.$$

To prove this we follow [9] (pp. 42-44). Quite simply

$$\text{(A.1.1)}\qquad \begin{cases}|\zeta(\sigma+it)|\leqslant e\log t+18\,,\\ |\zeta'(\sigma+it)|\leqslant e\log^2 t+20\log t\end{cases}$$

for

$$1-\frac{1}{\log t}\leqslant\sigma\leqslant 2\,,\qquad t\geqslant 14\ \ (>e^2)\,.$$

Further, for

$$1-\frac{1}{\log t}\leqslant\sigma\leqslant 2\,,\qquad t\geqslant e^8\,,$$

we obtain using (A.1.1)

$$\text{(A.1.2)}\qquad \begin{cases}|\zeta(\sigma+it)|\geqslant\dfrac{(\sigma-1)^{3/4}}{3\log^{1/4}t}\,,\\ |\zeta(1+it)-\zeta(\sigma+it)|\leqslant 6(\sigma-1)\log^2 t\,.\end{cases}$$

Hence (writing $\sigma=1+\dfrac{1}{36^4\log^9 t}$ in (A.1.2))

$$|\zeta(1+it)|\geqslant\frac{1}{6.36^3\log^7 t}\,,$$

so that, again by (A.1.2),

$$|\zeta(\sigma+it)|>0\quad\text{for}\quad 1-\frac{\log^{-9}t}{36^4}\leqslant\sigma\leqslant 1\,,\qquad t\geqslant e^8\,.$$

Noting that $1-\beta+i\gamma$ is also a ζ-zero we get therefrom

$$\beta>\frac{\log^{-9}\gamma}{36^4}\,,\qquad \text{Q.E.D.}$$

A.2.

$$\text{(A.2.1)}\qquad \left|\zeta\left(-\frac{3}{2}+it\right)\right|\leqslant\frac{(|t|+2)^2}{12}\,,\qquad -\infty<t<+\infty,$$

$$\text{(A.2.2)}\qquad \left|\frac{\zeta'}{\zeta}\left(-\frac{3}{2}+it\right)\right|\leqslant 32\{8+3\log(|t|+1)\}\,,\qquad -\infty<t<+\infty\,.$$

(A.2.1) follows by the functional equation of $\zeta(s)$, (A.2.2) by the Borel-Carathéodory theorem ([3], p. 50) and the estimate

$$\left|\frac{\zeta(s)}{\zeta(s_0)}\right|<e^{8+3\log(|t_0|+1)}$$

for

$$|s-s_0|\leqslant\tfrac{1}{4}\,,\qquad s_0=-\tfrac{3}{2}+it_0\,,\qquad -\infty<t_0<+\infty\,.$$

A.3.

$$\text{(A.3.1)}\qquad |\zeta(s)|<\frac{6}{5}\quad \textit{for}\quad |s|\leqslant\frac{1}{10}\,,$$

$$\text{(A.3.2)}\qquad \left|\frac{\zeta'}{\zeta}(s)\right|<\frac{5}{2}\quad \textit{for}\quad |s|\leqslant\frac{1}{10}\,.$$

(A.3.1) follows again by the functional equation. $\left|\dfrac{\zeta'}{\zeta}(s)\right|$ might be estimated similarly as in A.2 but it is simpler to use the formula ([3], p. 58).

$$\frac{\zeta'}{\zeta}(s)=b+\frac{1}{s-1}-\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}s+1\right)+\sum_{\varrho}\left(\frac{1}{s-\varrho}+\frac{1}{\varrho}\right),\qquad b=0.549\,\ldots$$

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Added in proof, 9. 10. 1961. Using the inequality $\sin t>\frac{2}{\pi}t$, $0\leqslant t\leqslant\frac{2}{\pi}$, lemma 5 can be proved simpler and even in a stronger form. This would also improve (2.8) and consequently (1.5), (1.6).