ACTA ARITHMETICA
VII (1962)

On the zeros of Hecke's L-functions I

by

E. Fogels (Riga)

Introduction

1. The theory of Dirichlet's functions $L(s, \chi)$ with characters χ modulo $k \to \infty$ has been developed by papers of Titchmarsh, Page, Linnik and other writers (see [6] IV §§ 5,6; X §§ 2,3) as far as they were able to prove the Linnik's estimate $p_0 < k^{O(1)}$ for the least prime $\equiv l \pmod{k}$, $(k, l) = 1$. It is my aim to prove the corresponding result for prime ideals of any class mod $\mathfrak{f}$ in any algebraic field of degree $n \geqslant 1$ (which for $n = 1$ coincides with Linnik's theorem). The necessary auxiliary theorems (which may be intersting in themselves) about the zeros of L-functions of an algebraic field will be given in a series of 3 papers. The arithmetical deductions will then follow in later paper (The results of this series of papers have been announced in a short note to appear in the Doklady Akad. Nauk SSSR).

The L-functions of an algebraic field were introduced in 1917 by Hecke ([2]) and further investigated by Landau ([3]). In this paper, being greatly indebted to Landau's work, we keep his notation $\zeta(s, \chi)$ for that function, although Hecke himself and many recent writers use the symbol L. Actually Hecke's L-functions are in the set of Landau's functions $\zeta(s, \chi)$, but not the contrary; cf. [3], p. 53.

The principal result of this paper I is the following

Theorem. *Let $K, \mathfrak{f}, \zeta(s, \chi)$ denote respectively any algebraic field of degree $n \geqslant 1$, any ideal in K and any Landau function with a character χ modulo $\mathfrak{f}$. Let further*

$$D = |\Delta| \cdot N\mathfrak{f} > D_0 > 1$$

where Δ denotes the discriminant of the field and $N\mathfrak{f}$ the norm of $\mathfrak{f}$. Then there is a positive constant c (which depends only on n) such that in the region

$$(1) \qquad \sigma \geqslant 1 - c/\log D(1+|t|) \geqslant 3/4 \qquad (\sigma = \operatorname{re} s,\ t = \operatorname{im} s)$$

there is no zero of $\zeta(s, \chi)$ in the case of a complex χ. For at most one real χ there may be in (1) a simple zero $= 1 - \delta$ of $\zeta(s, \chi)$; it is real and, if D_0 is large enough, $\delta > D^{-2n}$.

The method used in this paper is on the whole that employed by Titchmarsh ([7]) and Page ([5]), but it is applied to a more complicated function, the properties of which were unknown for $D\to\infty$.

Preliminary theorems

2. Throughout this paper $n, K, \mathfrak{f}, \Delta, D, \zeta(s,\chi)$ keep their meaning as fixed in the theorem. m denotes natural numbers in general, $d(m)$ the number of positive divisors of m. $\varphi(m)$ is the number of natural numbers $l\leqslant m$, prime to m. $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ denote ideals and $\mathfrak{p}$ prime ideals of the field in general, $\mathfrak{o}$ the unity ideal, $\tau(\mathfrak{a})$ the number of divisors of $\mathfrak{a}$. $\mu(\mathfrak{a})$ is the Möbius's function $=(-1)^\nu$, if $\mathfrak{a}$ is a product of $\nu\geqslant 0$ different prime ideals, and $=0$ otherwise. $\mathfrak{H}$ denotes classes of ideals modulo $\mathfrak{f}$ in general and h the number of classes. $a, b, c, c_1, \ldots$ denote positive constants which may depend on n (generally they keep their meaning only throughout the same paragraph). The dependence on other constants is denoted in the usual way.

For positive x by $y\ll x$ or $y=O(x)$ we denote the inequality $|y/x|<c$ for appropriate c. We suppose that the degree of the field is bounded $(n\ll 1)$ whereas $|\Delta|$ and $N\mathfrak{f}$ may increase indefinitely.

The complex variable will be generally denoted by $s=\sigma+it$, but sometimes we use w or z.

We take for granted the elementary properties of Riemann's zeta-function $\zeta(s)$ and the function $\Gamma(s)$ in such extent as is given in Titchmarsh's book [8]. We shall need the following estimates and sum formulae (for the proofs see, for example, [6] I Satz 5.1, 5.2; A Satz 6.2, 1.4, 3.2).

1. *We have*

$$\varphi(m)>c_1 m/\log\log m \qquad (m\geqslant 3)\,. \tag{2}$$

For any $\varepsilon>0$ *and all* $m\geqslant m_0(\varepsilon)$

$$d(m)<\exp\left\{\frac{\log m}{\log\log m}(1+\varepsilon)\log 2\right\}. \tag{3}$$

2. *If* $\sigma\ll 1$ *and* $|t|\to\infty$, *then*

$$|\Gamma(\sigma+it)|=\sqrt{2\pi}\,e^{-|t|\pi/2}|t|^{\sigma-1/2}\{1+O(|t|^{-1})\}\,. \tag{4}$$

3. *Let*

$$f(s)=\sum_m a_m m^{-s}\,,$$

where the series is absolutely convergent for $\sigma>\sigma_0>-\infty$. *Then for any* $y>0$, $w=u+iv$, $b>\max(\sigma_0, u)$

$$\sum_m a_m m^{-w}e^{-my}=\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} y^{w-s}\Gamma(s-w)f(s)\,ds\,. \tag{5}$$

4. *Let* $\lambda_1, \lambda_2, \ldots$ *be a sequence of non-decreasing real numbers with* $\lim\lambda_m=\infty$, *and let* a_m $(m=1,2,\ldots)$ *denote arbitrary real or complex numbers. Then for any real or complex function* $g(\xi)$ *having a continuous derivative in the segment* $\lambda_1\leqslant\xi\leqslant x$ *we have*

$$\sum_{\lambda_1\leqslant\lambda_m\leqslant x} a_m g(\lambda_m)=A(x)g(x)-\int_{\lambda_1}^{x}A(\xi)g'(\xi)\,d\xi\,, \tag{6}$$

where

$$A(\xi)=\sum_{\lambda_1\leqslant\lambda_m\leqslant\xi} a_m\,.$$

3. Let $\chi(\mathfrak{H})$ be any of the h characters of the classes $\mathfrak{H}$ and let for any ideal $\mathfrak{a}$ of the field K

$$\chi(\mathfrak{a})=\begin{cases}\chi(\mathfrak{H}) & \text{if}\quad \mathfrak{a}\in\mathfrak{H}\,,\\ 0 & \text{if}\quad \mathfrak{a}\text{ is not prime to }\mathfrak{f}\,.\end{cases} \tag{7}$$

The principal character will always be denoted by χ_0.

Writing

$$\zeta(s,\chi)=\sum_{\mathfrak{a}}\chi(\mathfrak{a})N\mathfrak{a}^{-s} \qquad (\sigma>1) \tag{8}$$

we have in the half-plane $\sigma>1$

$$\zeta(s,\chi)=\prod_{\mathfrak{p}}\big(1-\chi(\mathfrak{p})N\mathfrak{p}^{-s}\big)^{-1}\,, \tag{9}$$

whence

$$1/\zeta(s,\chi)=\sum_{\mathfrak{a}}\mu(\mathfrak{a})\chi(\mathfrak{a})N\mathfrak{a}^{-s}\,, \tag{10}$$

$$\zeta'/\zeta(s,\chi)=-\sum_{\substack{\mathfrak{p},m\\ m\geqslant 1}}\chi(\mathfrak{p}^m)N\mathfrak{p}^{-ms}\log N\mathfrak{p}\,. \tag{11}$$

The function $\zeta(s,\chi_0)$ is regular in the whole plane, except for a simple pole at $s=1$; the others, $\zeta(s,\chi)$ $(\chi\neq\chi_0)$, are integral functions (see [3] Satz LXIII).

Let $\zeta_K(s)$ be the Dedekind's zeta-function of the field K,

$$\zeta_K(s)=\sum_{\mathfrak{a}}N\mathfrak{a}^{-s}=\prod_{\mathfrak{p}}(1-N\mathfrak{p}^{-s})^{-1} \qquad (\sigma>1)\,.$$

Then we have, by (7), (9),

$$\zeta(s,\chi_0)=\zeta_K(s)\prod_{\mathfrak{p}|\mathfrak{f}}(1-N\mathfrak{p}^{-s})\,. \tag{12}$$

R. Brauer has proved the estimate

$$\operatorname*{Res}_{s=1}\zeta_K(s) = |\Delta|^{o(1)} \qquad (|\Delta|\to\infty)$$

([1], (16)). In consequence of (2)

$$\prod_{\mathfrak{p}|\mathfrak{f}}(1-N\mathfrak{p}^{-1}) \geqslant \left(\frac{\varphi(N\mathfrak{f})}{N\mathfrak{f}}\right)^n = N\mathfrak{f}^{o(1)} \qquad (N\mathfrak{f}\to\infty)\,.$$

Hence, by (12),

$$\operatorname*{Res}_{s=1}\zeta(s,\chi_0) = D^{o(1)} \qquad (D\to\infty)\,. \tag{13}$$

The estimate of $|\zeta(s,\chi)|$ in a strip $\sigma \ll 1$

4. Lemma 1. *For any positive* $\eta \ll 1$

$$\zeta(1+\eta+it,\chi) \ll \eta^{-n}\,, \tag{14}$$

$$\zeta(-\eta+it,\chi) \ll \eta^{-n}D^{1/2+\eta}(1+|t|)^{n/2+n\eta}\,, \tag{15}$$

$$|\zeta(\sigma+it,\chi)| < c(\eta,D)\,e^{n|t|} \qquad (-\eta\leqslant\sigma\leqslant 1+\eta\,,\ |t|\geqslant 1)\,. \tag{16}$$

Proof. Since any prime p in the field K is a product of at most n different prime ideals $\mathfrak{p}$ with $N\mathfrak{p}=p$ (see [4] Satz 815), we have

$$|\zeta(1+\eta+it,\chi)| \leqslant \zeta(1+\eta,\chi_0) \leqslant \zeta_K(1+\eta)$$
$$= \prod_{\mathfrak{p}}(1-N\mathfrak{p}^{-1-\eta})^{-1} \leqslant \prod_{p}(1-p^{-1-\eta})^{-n} = \zeta^n(1+\eta) \ll \eta^{-n}\,.$$

This proves (14).

If $\chi(\mathfrak{a})$ is a primitive character, then $\zeta(s,\chi)$ satisfies the functional equation (see [3], pp. 90, 99-102)

$$\zeta(s,\chi) = (-i)^q W(\chi) A(\mathfrak{f})^{1-2s}\left(\frac{\Gamma\left(\frac{2-s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)}\right)^q\left(\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma(s/2)}\right)^{r_1-q}\left(\frac{\Gamma(1-s)}{\Gamma(s)}\right)^{r_2}\zeta(1-s,\bar{\chi})\,, \tag{17}$$

where r_1 and $2r_2$ denote, respectively, the numbers of the real and not-real conjugate fields of K ($0\leqslant r_1\leqslant n$, $r_1+2r_2=n$), q denotes a non negative integer $\leqslant r_1$, $|W(\chi)|=1$,

$$A(\mathfrak{f}) = 2^{-r_2}\pi^{-n/2}\sqrt{|\Delta| N\mathfrak{f}} \ll D^{1/2}\,.$$

By (4)

$$\frac{\Gamma(1+\eta-it)}{\Gamma(-\eta+it)} \ll (1+|t|)^{1+2\eta}\,, \qquad \left(\frac{\Gamma\left(\frac{1+\eta-it}{2}\right)}{\Gamma\left(\frac{-\eta+it}{2}\right)}\right)^2 \ll (1+|t|)^{1+2\eta}\,,$$

$$\left(\frac{\Gamma\left(\frac{2+\eta-it}{2}\right)}{\Gamma\left(\frac{1-\eta+it}{2}\right)}\right)^2 \ll (1+|t|)^{1+2\eta}\,.$$

From this and (17), (14) we get (15) for a primitive χ.

Now let $\chi(\mathfrak{a})$ be an imprimitive character modulo $\mathfrak{f}$. Then there is an ideal $\mathfrak{f}_0$ which divides $\mathfrak{f}$, and there is a primitive character X modulo $\mathfrak{f}_0$ such that

$$\zeta(s,\chi) = \zeta(s,X)\prod_{\mathfrak{p}|\mathfrak{f},\,\mathfrak{p}\nmid\mathfrak{f}_0}(1-X(\mathfrak{p})N\mathfrak{p}^{-s}) \tag{18}$$

(see [3], p. 102); by (15)

$$\zeta(-\eta+it,X) \ll \eta^{-n}|\Delta N\mathfrak{f}_0|^{1/2+\eta}(1+|t|)^{n/2+n\eta}\,. \tag{19}$$

We deduce, by (3),

$$\tau(\mathfrak{f}) \leqslant d(N\mathfrak{f}) \ll N\mathfrak{f}^{1/8}\,.$$

Hence, writing $\mathfrak{f}=\mathfrak{f}_0\mathfrak{f}_1$,

$$\prod_{\mathfrak{p}|\mathfrak{f},\,\mathfrak{p}\nmid\mathfrak{f}_0}(1-X(\mathfrak{p})N\mathfrak{p}^{\eta-it}) \leqslant \prod_{\mathfrak{p}|\mathfrak{f},\,\mathfrak{p}\nmid\mathfrak{f}_0}(1+N\mathfrak{p}^{\eta}) \ll \prod_{\mathfrak{p}|\mathfrak{f}_1}(1+N\mathfrak{p}^{\eta})$$
$$= \sum_{\mathfrak{d}|\mathfrak{f}_1} N\mathfrak{d}^{\eta} \ll N\mathfrak{f}_1^{\eta}\tau(\mathfrak{f}_1) \ll N\mathfrak{f}_1^{\eta+1/8}\,.$$

From this and (18), (19) we get (15) for imprimitive χ.

Again let χ be a primitive character modulo $\mathfrak{f}$ and let

$$\Phi(s,\chi) = A(\mathfrak{f})^s\,\Gamma^q\left(\frac{s+1}{2}\right)\Gamma^{r_1-q}\left(\frac{s}{2}\right)\Gamma^{r_2}(s)\,\zeta(s,\chi)\,.$$

It is proved by Landau (see [3], formulae (52), (55), (41)) that, in the region G ($-\eta\leqslant\sigma\leqslant 1+\eta$, $|t|\geqslant 1$), $|\Phi(s,\chi)| < c_1(\eta,D)$. Hence, by (4), we have in G

$$|\zeta(s,\chi)| < c_2(\eta,D)\,e^{n|t|}\,,$$

which implies (16) for a primitive χ. From this and (18) we get (16) for any χ.

5. Let in the region $(-\pi/2a \leqslant \sigma \leqslant \pi/2a, t \geqslant 0)$ $F(s)$ be a regular function satisfying the inequality

$$F(\sigma+it) \ll \exp e^{\gamma t} \tag{20}$$

with $\gamma < a$ and let on the boundary $|F(s)| \leqslant M$. Then by a theorem of Phragmén-Lindelöf (see [8], § 5.65) we have in the region $|F(s)| \leqslant M$. Replacing s by $sca/\pi+\beta+it_0$, where c, β, t_0 are appropriate real constants $(c>0, t_0 \geqslant 0)$ we get the theorem for any region $(\sigma_0 \leqslant \sigma \leqslant \sigma_0+c, t \geqslant t_0)$ in which $F(s)$ satisfies (20) with $\gamma < \pi/c$.

Lemma 2. *Let in the region $G(\alpha \leqslant \sigma \leqslant \beta, t \geqslant t_0 > 1)$ $f(s)$ be a regular function satisfying the inequality*

$$f(\sigma+it) \ll \exp e^{\gamma t} \quad \textit{with} \quad \gamma < \pi(\beta-\alpha)^{-1} \tag{21}$$

and let

$$f(\alpha+it) \ll t^a, \quad f(\beta+it) \ll t^b \quad (t > t_0).$$

Then we have in G

$$f(\sigma+it) \ll t^{a(\beta-\sigma)/(\beta-\alpha)+b(\sigma-\alpha)/(\beta-\alpha)}. \tag{22}$$

For functions $f(s)$ satisfying in G the inequality $f(s) \ll t^c$ the proof is given in [4] (Satz 405) where it is based on a weaker form of Phragmén-Lindelöf theorem ([4], Satz 404). If replaced by the aforesaid stronger form, then the proof holds for functions $f(s)$ satisfying (21).

6. Let in the strip $\sigma_1 \leqslant \sigma \leqslant \sigma_3$, $f(s)$ be a bounded and regular function (with exception at most the point $s=\infty$), not identically $=0$. Let further $\sigma_1 < \sigma_2 < \sigma_3$ and let M_r $(r=1,2,3)$ denote the upper bound of $|f(s)|$ on the line $\sigma=\sigma_r$. Then, by a theorem of Doetsch (see, for example, [6], A, Satz 9.1)

$$M_2^{\sigma_3-\sigma_1} \leqslant M_1^{\sigma_3-\sigma_2} M_3^{\sigma_2-\sigma_1}. \tag{23}$$

Lemma 3. *Let in the strip $S(\alpha \leqslant \sigma \leqslant \beta)$, $F(s)$ be a regular function satisfying the inequalities*

$$F(\sigma+it) \ll \exp e^{\gamma|t|} \quad \textit{with} \quad \gamma < \pi(\beta-\alpha)^{-1}$$

and

$$F(\alpha+it) \ll U(1+|t|)^{\nu}, \quad F(\beta+it) \ll V \tag{24}$$

where $\nu > 0$, $U > 1$, $V > 1$ are independent of t. Then in S

$$|F(\sigma+it)| < c_1(\alpha,\beta)\, U^{(\beta-\sigma)/(\beta-\alpha)} V^{(\sigma-\alpha)/(\beta-\alpha)} (1+|t|)^{\nu(\beta-\sigma)/(\beta-\alpha)}. \tag{25}$$

Proof. By (22) (and the corresponding result with s replaced by $-s$) we have in S

$$|F(\sigma+it)| < c(\alpha,\beta,U,V)(1+|t|)^{\nu(\beta-\sigma)/(\beta-\alpha)}. \tag{26}$$

The function

$$g(s) = \left(\frac{s-\alpha+2(\beta-\alpha)}{s-\alpha} \cdot \frac{\Gamma\left(1-\dfrac{s-\alpha}{2(\beta-\alpha)}\right)}{\Gamma\left(\dfrac{s-\alpha}{2(\beta-\alpha)}\right)}\right)^{\beta-\alpha} \tag{27}$$

is regular in S and $\neq 0$ at any finite $s \in S$, whence at no such s is a branch-point (concerning the case of non-integer $\beta-\alpha$). Any branch of this functions is a single-valued function in S; further on we use the principal branch of $g(s)$ which is positive for $t=0$, $\alpha \leqslant \sigma \leqslant \beta$.

Writing

$$|g(s)| = G(\sigma,t)\cdot(1+|t|)^{\beta-\sigma} \tag{28}$$

we have, by (27), (4),

$$c_1 < G(\sigma,t) < c_2 \quad (\alpha \leqslant \sigma \leqslant \beta) \tag{29}$$

for appropriate c_1, c_2 (which may depend on α, β). Any fixed branch of the function

$$f(s) = F(s)/g(s)^{\nu/(\beta-\alpha)}\, U^{(\beta-s)/(\beta-\alpha)} V^{(s-\alpha)/(\beta-\alpha)} \tag{30}$$

is regular in S. Taking the principal values for the powers of U and V it satisfies the inequalities

$$|f(\alpha+it)| \leqslant c_3, \quad |f(\beta+it)| \leqslant c_4, \tag{31}$$

by (24), (28), (29), and is bounded in S, since, by (26), (28), (29),

$$|f(\sigma+it)| \leqslant c_5(\alpha,\beta,U,V).$$

Hence, if M_σ denotes the upper bound of $f(s)$ on the line $\sigma+it$ $(-\infty < t < \infty$, σ fixed) we have, by (23) and (31), $M_\sigma \leqslant c_6$. From this and (30), (28), (29) we get (25).

7. Lemma 4. *For any positive $\delta \leqslant 1/\log D < \frac{1}{2}$ we have uniformly in $-\delta \leqslant \sigma \leqslant 1+\delta$*

$$\zeta(s,\chi) \ll \delta^{-n} D^{(1-\sigma)/2}(1+|t|)^{(1+\delta-\sigma)n/2} \tag{32}$$

provided that $|s-1| > \frac{1}{8}$ when $\chi=\chi_0$.

Proof. Suppose first $\chi \neq \chi_0$. Then $\zeta(s,\chi)$ is regular in the strip $-\delta \leqslant \sigma \leqslant 1+\delta$ and (32) follows from Lemma 3 in which we can use $\alpha=-\delta$, $\beta=1+\delta$, $\nu=n/2+n\eta$, $n\eta=\delta$, $V=n^n\delta^{-n} \ll \delta^{-n}$, $U \ll \delta^{-n}D^{1/2}$, by (14), (15).

In the case of $\chi=\chi_0$ we use the function

$$F(s) = \frac{s-1}{s-2}\zeta(s,\chi_0)$$

which is regular in $-\delta \leqslant \sigma \leqslant 1+\delta$ and satisfies the inequalities

$$F(1+\delta+it) \ll \delta^{-n}, \quad F(-\delta+it) \ll \delta^{-n} D^{1/2}(1+|t|)^{\delta+n/2},$$

by (14), (15). By the argument used before

$$(33) \quad \frac{s-1}{s-2}\zeta(s,\chi_0) \ll \delta^{-n} D^{(1-\sigma)/2}(1+|t|)^{(1+\delta-\sigma)n/2} \quad (-\delta \leqslant \sigma \leqslant 1+\delta).$$

This proves (32) for $\chi = \chi_0$, $|s-1| > \frac{1}{8}$.

Using Lemmas 1 and 3 we deduce that

$$(34) \quad \zeta(s,\chi) \ll D^{c+1/2}(1+|t|)^{nc+n/2} \quad (\chi \neq \chi_0),$$

$$(35) \quad (s-1)\zeta(s,\chi_0) \ll D^{c+1/2}(1+|t|)^{1+nc+n/2}$$

uniformly in $-c \leqslant \sigma \leqslant 1+c$ $(\frac{1}{8} \leqslant c \ll 1)$.

On the zeros of $\zeta(s,\chi)$ in some regions

8. In the half-plane $\sigma < 0$ the functions $\zeta(s,\chi)$ have no other zeros than the trivial ones $= -2m$ or $= -2m+1$ (or both; see [3], Satz LXIII). All other zeros (the "critical" ones) lie in the strip $0 \leqslant \sigma < 1$.

LEMMA 5. *If $N_\chi(T)$ denotes the number of zeros of $\zeta(s,\chi)$ in the rectangle* $(0 \leqslant \sigma \leqslant 1, |t-T| \leqslant 1)$, *then*

$$(36) \quad N_\chi(t) \ll \log D(1+|t|).$$

Multiple zeros are (as always) counted according to their order of multiplicity.

Proof. Let first $\chi \neq \chi_0$. By (10), (8), (14),

$$\left|\frac{1}{\zeta(2+it,\chi)}\right| = \left|\sum_{\mathfrak{a}} \frac{\mu(\mathfrak{a})\chi(\mathfrak{a})}{N\mathfrak{a}^{2+it}}\right| \leqslant \zeta(2,\chi_0) \ll 1,$$

whence

$$(37) \quad |\zeta(2+it,\chi)| > c_1.$$

Write $s_0 = 2+it$. By (34) we deduce the existence of a constant c_2 such that for all s in the circle $|s-s_0| \leqslant 12$

$$(38) \quad |\zeta(s,\chi)| < \exp\{c_2 \log D(1+|t|)\}.$$

Let $\nu(x) = \nu(x, s_0, \chi)$ denote the number of zeros of $\zeta(s,\chi)$ in $|s-s_0| \leqslant x$. Then, by (37), (38) and Jensen's theorem (see [8], § 3.61),

$$(39) \quad \int_0^{12} \frac{\nu(x)}{x}dx = \frac{1}{2\pi}\int_0^{2\pi} \log|\zeta(s_0+12e^{i\theta},\chi)|d\theta - \log|\zeta(s_0,\chi)| < c_3 \log D(1+|t|).$$

Since

$$\int_0^{12} \frac{\nu(x)}{x}dx \geqslant \int_3^{12} \frac{\nu(x)}{x}dx \geqslant \nu(3)\log 4,$$

and $N_\chi(t) \leqslant \nu(3)$, using (39), (38) we deduce (36).

If $\chi = \chi_0$, then we use the function $(s-1)\zeta(s,\chi_0)$ and similar arguments.

9. Our further deductions are based on the following Landau's lemma (see [9], III § 9).

If $f(s)$ is regular and

$$|f(s)/f(s_0)| < e^M \quad (M > 1)$$

in the circle $|s-s_0| \leqslant r$, then

$$(40) \quad \frac{f'}{f}(s) - \sum \frac{1}{s-\varrho} \ll M/r \quad (|s-s_0| \leqslant r/4)$$

where ϱ runs through the zeros of $f(s)$ in $|s-s_0| \leqslant r/2$.

LEMMA 6. *If $e_0 = 1$ for $\chi = \chi_0$ and $= 0$ for $\chi \neq \chi_0$, then in the strip* $S(-1 \leqslant \sigma \leqslant 5)$

$$(41) \quad \frac{\zeta'}{\zeta}(s,\chi) - \sum_{|s-\varrho|<1} \frac{1}{s-\varrho} + \frac{e_0}{s-1} \ll \log D(1+|t|).$$

Proof. If $\chi \neq \chi_0$, then we have, by (37), (38) and (40) (with $s_0 = 2+it$, $r = 12$), in S

$$(42) \quad \frac{\zeta'}{\zeta}(s,\chi) - \sum_{\varrho \in C} \frac{1}{s-\varrho} \ll \log D(1+|t|)$$

where C denotes the circle $|s-s_0| \leqslant 6$. By (36) the sum in (42) differs from that in (41) by $\ll \log D(1+|t|)$ and we get the required result.

If $\chi = \chi_0$, then we use the function $(s-1)\zeta(s,\chi_0)$ and similar arguments.

10. LEMMA 7. *If $\nu = \nu(r; T, \chi)$ denotes the number of zeros of $\zeta(s,\chi)$ in the circle $|s-1-iT| \leqslant r$ with $r \in [1/\log D(1+|t|), 2]$, then*

$$(43) \quad \nu(r; t, \chi) \ll r \log D(1+|t|).$$

Proof. Since for $r > \frac{1}{2}$ (43) is a consequence of (36), we take $r \leqslant \frac{1}{2}$. Considering that any prime p in the field K is a product of at most n prime ideals $\mathfrak{p}$ with $N\mathfrak{p} = p$, we have, by (11),

$$\left|\frac{\zeta'}{\zeta}(1+r,\chi_0)\right| \leqslant n \sum_{\substack{p,m \\ m \geqslant 1}} \frac{\log p}{p^{m(1+r)}} = -n\frac{\zeta'}{\zeta}(1+r) < c_1/r.$$

From this and (41) (with $s = 1+r+it$, $e_0(s-1)^{-1} \ll r^{-1} \ll \log D(1+|t|)$) we deduce the inequalities

$$\frac{c_1}{r} \geqslant \left|\frac{\zeta'}{\zeta}(1+r,\chi_0)\right| \geqslant \left|\frac{\zeta'}{\zeta}(s,\chi)\right| \geqslant \mathrm{re}\sum_{|s-\varrho|<1}\frac{1}{s-\varrho} - c_2\log D(1+|t|)$$

$$\geqslant \nu\frac{2}{5r} - c_2\log D(1+|t|)$$

implying (43).

On the zeros of $\zeta(s,\chi)$ near the line $\sigma=1$

11. LEMMA 8. *If a is a sufficiently small absolute constant, $0<a<1$, then*

(44) $$\left|\frac{\zeta'}{\zeta}(\sigma_0,\chi_0)\right| < \frac{\frac{5}{4}}{\sigma_0-1} \quad \textit{for} \quad \sigma_0 = 1+a/\log D.$$

Proof. On the stretch $\sigma>1$ of the real axis $\zeta'/\zeta(\sigma,\chi_0)<0$, by (11). By (41)

$$\frac{\zeta'}{\zeta}(\sigma,\chi_0) = \frac{-1}{\sigma-1} + \mathrm{re}\Big\{\sum_{|\varrho-\sigma|<1}\frac{1}{\sigma-\varrho} + \theta c_1\log D\Big\}$$

($|\theta|<1$), whence (since $\mathrm{re}\sum(\sigma-\varrho)^{-1}\geqslant 0$ for $\sigma>1$)

$$\mathrm{re}\Big\{\sum_{|\varrho-\sigma|<1}\frac{1}{\sigma-\varrho} + \theta c_1\log D\Big\} \in \Big[-c_1\log D,\ \frac{1}{\sigma-1}\Big].$$

If a is small enough, then

$$\frac{1}{4}\cdot\frac{\log D}{a} > c_1\log D,$$

which implies (44).

12. LEMMA 9. *There is an absolute constant $c_1>0$ such that no function $\zeta(s,\chi)$ has a zero in the region ($\sigma>1-c_1/\log D|t|$, $|t|\geqslant 3$).*

Proof. Let $\zeta(s,\chi)$ have a zero $\beta+i\gamma$ ($|\gamma|\geqslant 3$) and let $\sigma_0 = 1+a/\log|D\gamma|$ with a satisfying (44). If s_0 denotes any of the numbers $\sigma_0+i\gamma$, $\sigma_0+2i\gamma$, then we have, by (10), (8), (14),

$$|1/\zeta(s_0,\chi)| \leqslant \sum_{\substack{\mathfrak{a}\\(\mathfrak{a},\mathfrak{f})=\mathfrak{o}}} N\mathfrak{a}^{-\sigma_0} = \zeta(\sigma_0,\chi_0) \ll a^{-n}\log^n D|\gamma|,$$

whence in $|s-s_0|\leqslant\frac{1}{2}$

$$|\zeta(s,\chi)/\zeta(s_0,\chi)| < a^{-n}|D\gamma|^{c_2}$$

by (34), (35). And we may replace χ by χ^2. From this and [4], Theorem 374 (with $r=\frac{1}{2}$, $f(s)=\zeta(s,\chi)$ and $=\zeta(s,\chi^2)$, $M = c_2\log|D\gamma|+n\log 1/a$) we deduce

$$-\mathrm{re}\,\zeta'/\zeta(\sigma_0+2i\gamma,\chi^2) < 8(c_2\log|D\gamma|+n\log 1/a),$$

(45) $$-\mathrm{re}\,\zeta'/\zeta(\sigma_0+i\gamma,\chi) < 8(c_2\log|D\gamma|+n\log 1/a)-(\sigma_0-\beta)^{-1}.$$

Writing $\chi(\mathfrak{a}) = e^{i\varphi(\mathfrak{a})}$, when $(\mathfrak{a},\mathfrak{f})=\mathfrak{o}$, we have, by (11),

$$-3\zeta'/\zeta(\sigma_0,\chi_0)-4\,\mathrm{re}\,\zeta'/\zeta(\sigma_0+i\gamma,\chi)-\mathrm{re}\,\zeta'/\zeta(\sigma_0+2i\gamma,\chi^2)$$

$$= \sum_{\substack{\mathfrak{p},m\\ \mathfrak{p}\nmid\mathfrak{f},m\geqslant 1}}\frac{3+4\cos\{\varphi(\mathfrak{p}^m)-m\gamma\log N\mathfrak{p}\}+\cos\{2\varphi(\mathfrak{p}^m)-2m\gamma\log N\mathfrak{p}\}}{N\mathfrak{p}^{m\sigma_0}}\log N\mathfrak{p}$$

$$= \sum_{\substack{\mathfrak{p},m\\ \mathfrak{p}\nmid\mathfrak{f},m\geqslant 1}}\frac{2\big(1+\cos\{\varphi(\mathfrak{p}^m)-m\gamma\log N\mathfrak{p}\}\big)^2}{N\mathfrak{p}^{m\sigma_0}}\log N\mathfrak{p} \geqslant 0.$$

Hence, by (44), (45),

$$\frac{15}{4(\sigma_0-1)} + 40(c_2\log|D\gamma|+n\log 1/a) - \frac{4}{\sigma_0-\beta} \geqslant 0,$$

whence

$$1-\beta > \frac{a}{\log|D\gamma|}\Big\{\frac{16}{15+160c_2a+160na\log(1/a)/\log|D\gamma|}-1\Big\}.$$

Since $a\log 1/a\to 0$ as $a\to 0$, the expression in brackets is $>a_1>0$ when a is small enough, whence $1-\beta > aa_1/\log|D\gamma|$, is the desired result.

13. LEMMA 10. *There is an absolute constant $c_1>0$ such that no function $\zeta(s,\chi)$ with a complex character vanishes in the region ($|t|\leqslant 5$, $\sigma>1-c_1/\log D$).*

Proof. We can use the arguments of the previous lemma. For $\sigma_0 = 1+a/\log D$, $s_0=\sigma_0+i\gamma$, $|\gamma|\leqslant 5$ we have

$$|\zeta(s,\chi)/\zeta(s_0,\chi)| < a^{-n}D^{c_3} \qquad (|s-s_0|\leqslant\tfrac{1}{2}),$$

$$-\mathrm{re}\,\zeta'/\zeta(\sigma_0+2i\gamma,\chi^2) < 8(c_3\log D+n\log 1/a),$$

$$-\mathrm{re}\,\zeta'/\zeta(\sigma_0+i\gamma,\chi) < 8(c_3\log D+n\log 1/a)-(\sigma_0-\beta)^{-1},$$

$$-\zeta'/\zeta(\sigma_0,\chi_0) < 5/4(\sigma_0-1),$$

whence

$$\frac{15}{4(\sigma_0-1)} + 40(c_3\log D+n\log 1/a) - 4(\sigma_0-\beta)^{-1} \geqslant 0,$$

$$1-\beta \geqslant \frac{a}{\log D}\Big\{\frac{16}{15+160c_3a+160na\log(1/a)/\log D}-1\Big\},$$

and $1-\beta > c_4/\log D$, if $a>0$ is small enough.

14. LEMMA 11. *There is an absolute constant $c_0 > 0$ such that no function $\zeta(s, \chi)$ with a real character $\chi \neq \chi_0$ vanishes in the region* $(0 < |t| \leqslant 5,\ \sigma > 1 - c_0/\log D)$.

Proof. First let $\beta + i\gamma$ be a zero of the function $\zeta(s, \chi)$ such that $c/\log D \leqslant |\gamma| \leqslant 5$. Since $\chi^2 = \chi_0$, by the arguments of § 12,

$$(46)\qquad -3\zeta'/\zeta(\sigma_0, \chi_0) - 4\operatorname{re}\zeta'/\zeta(\sigma_0 + i\gamma, \chi) - \operatorname{re}\zeta'/\zeta(\sigma_0 + 2i\gamma, \chi_0) \geqslant 0 .$$

Let $\sigma_0 = 1 + a/\log D$ be defined by (44). We substitute for the first and second term into (46) from (44), (45), respectively, but we increase n by unity and write $c_1 \log D$ instead of $c_2 \log|D\gamma|$. To provide a suitable substitute for the last term, write

$$G(s) = \zeta(s, \chi_0)/\zeta(s) , \qquad s_0 = \sigma_0 + 2i\gamma .$$

By (10)

$$|1/G(s_0)| = |\zeta(s_0)| \cdot |1/\zeta(s_0, \chi_0)| \ll a^{-n-1} \log^{n+1} D .$$

Hence in $|s - s_0| \leqslant \frac{1}{2}$

$$|G(s)/G(s_0)| < a^{-n-1} D^{c_1} ,$$

by (35) and the fact that $\zeta(s)$ does not vanish in the rectangle $(0 \leqslant \sigma \leqslant 1$, $|t| \leqslant 14)$ (see [9], II § 12, XV § 1 and the references given there). From this and [4] Theorem 374 (with $r = \frac{1}{2}, f(s) = G(s)$) we deduce

$$-\operatorname{re} G'/G(s_0) < 8(c_1 \log D + (n+1)\log 1/a) ,$$

whence

$$-\operatorname{re}\zeta'/\zeta(s_0, \chi_0) < 8(c_1 \log D + (n+1)\log 1/a) + c^{-1}\log D .$$

Now we have (cf. § 12)

$$15/4(\sigma_0 - 1) + \big(40 c_1 \log D + (n+1)\log 1/a\big) - 4/(\sigma_0 - \beta) + c^{-1}\log D \geqslant 0$$

or

$$15/4(\sigma_0 - 1) + 40\,\big(c_2 \log D + (n+1)\log 1/a\big) - 4/(\sigma_0 - \beta) \geqslant 0 ,$$

whence the required result follows for $|\gamma| > c/\log D$.

15. Now suppose that $\varrho_1 = \beta + i\gamma$ is a zero of $\zeta(s, \chi)$ such that $\beta \geqslant 1 - 1/c\log D$, $0 < \gamma < 1/c\log D$. Writing

$$\sigma_0 = 1 + 1/b\log D , \qquad s_0 = \sigma_0 + i\gamma ,$$

we have, by (10), (14), $1/\zeta(s_0, \chi) \ll b^n \log^n D$, whence in the circle $|s - s_0| \leqslant \frac{1}{2}$

$$|\zeta(s, \chi)/\zeta(s_0, \chi)| < e^M , \qquad M = c_3 \log D + n\log b ,$$

by (34). $\varrho_2 = \beta - i\gamma$ is another zero of $\zeta(s, \chi)$ Both zeros lie in the circle $|s - s_0| \leqslant \frac{1}{4}$.

Now we use Theorem 374 of [4] in the form

$$(47)\qquad -\operatorname{re} f'/f(s_0) < 4M/r - \operatorname{re} \sum_{|\varrho - s_0| < r/2} \frac{1}{s_0 - \varrho}$$

with $r = \frac{1}{2}$, $f(s) = \zeta(s, \chi)$. Since

$$\sum_{\varrho} \operatorname{re} \frac{1}{s_0 - \varrho} \geqslant \frac{1}{\sigma_0 - \beta} + \frac{\sigma_0 - \beta}{(\sigma_0 - \beta)^2 + 4\gamma^2} ,$$

we have

$$-\operatorname{re}\zeta'/\zeta(s_0, \chi) < 8(c_3 \log D + n\log b) - \left[\frac{1}{\sigma_0 - \beta} + \frac{\sigma_0 - \beta}{(\sigma_0 - \beta)^2 + 4\gamma^2}\right] .$$

Taking b large enough we have, by (44),

$$\operatorname{re}\zeta'/\zeta(s_0, \chi) < |\zeta'/\zeta(\sigma_0, \chi)| < 5/4(\sigma_0 - 1) ,$$

whence

$$(48)\qquad \frac{1}{\sigma_0 - \beta} + \frac{\sigma_0 - \beta}{(\sigma_0 - \beta)^2 + 4\gamma^2} < c_4 \log D + 8n\log b + 5/4(\sigma_0 - 1)$$

or

$$\frac{\sigma_0 - \beta}{(\sigma_0 - \beta)^2 + 4\gamma^2} < c_4 \log D + 8n \cdot \log b + \frac{1}{4(\sigma_0 - 1)} + \frac{1 - \beta}{(\sigma_0 - 1)(\sigma_0 - \beta)}$$
$$< c_4 \log D + 8n \cdot \log b + \frac{b}{4}\log D + \frac{b^2}{c}\log D$$
$$< (b/2 + b^2/c)\log D .$$

Since

$$\sigma_0 - \beta = 1 + 1/b\log D - \beta \leqslant (1/b + 1/c)/\log D ,$$

we hawe

$$1 - \beta + 1/b\log D < (b/2 + b^2/c)\log D \cdot [(1/b + 1/c)^2 + 4/c^2]\log^{-2} D .$$

This is impossible, if

$$1/b > (b/2 + b^2/c)[(1/b + 1/c)^2 + 4/c^2] .$$

There are positive numbers satisfying the latter inequality (since for a fixed b the right-hand side tends to $1/2b$ as $c \to \infty$), whence the lemma.

16. LEMMA 12. *Let χ be a real character $\neq \chi_0$. For a sufficiently small absolute constant $c > 0$ there is at most one real zero $> 1 - c/\log D$ of the function $\zeta(s, \chi)$.*

Proof. Let β, β' be two real zeros of $\zeta(s, \chi)$ and $\beta' \geqslant \beta$. By the same argument as used in the proof of (48) we get the inequality

$$\frac{1}{\sigma_0 - \beta} + \frac{1}{\sigma_0 - \beta'} < c_5 \log D + 8n\log b + 5/4(\sigma_0 - 1) ,$$

whence

$$\frac{2}{\sigma_0-\beta} < c_5\log D + 8n\log b + \tfrac{1}{4} b\log D < \tfrac{3}{4} b\log D\,,$$

if b is large enough. Hence

$$\sigma_0-\beta > 4/3\, b\log D\,,$$

$$\beta < \sigma_0 - 4/3\, b\log D = 1+1/b\log D - 4/3\, b\log D = 1-1/3\, b\log D = 1-c/\log D\,,$$

say. This proves the required result.

17. LEMMA 13. *For a sufficiently small absolute constant $c_1 > 0$ the function $\zeta(s, \chi_0)$ does not vanish in the region $0 < |t| \leqslant 3$, $\sigma > 1-c_1/\log D$.*

Proof. Let $\beta+i\gamma$ $(0<\gamma\leqslant 3)$ be a zero of the function $\zeta(s,\chi_0)$, $\sigma_0 = 1+a/\log D$ be defined by (44) and let s_0 denote any of the numbers $\sigma_0+i\gamma$, $\sigma_0+2i\gamma$. Writing $G(s) = \zeta(s,\chi_0)/\zeta(s)$ we have in the circle $|s-s_0| \leqslant \frac{1}{2}$

(49) $$|G(s)/G(s_0)| < a^{-n-1}D^{c_2}$$

(cf. § 14). Hence, by Theorem 374 of [4],

$$-\operatorname{re} G'/G(\sigma_0+2i\gamma) < 8\big(c_2\log D + (n+1)\log 1/a\big)\,,$$

$$-\operatorname{re} G'/G(\sigma_0+i\gamma) < 8\big(c_2\log D + (n+1)\log 1/a\big) - (\sigma_0-\beta)^{-1}\,.$$

Suppose first $\gamma > c/\log D$. Then $\zeta'/\zeta(\sigma_0+i\gamma)$ and $\zeta'/\zeta(\sigma_0+2i\gamma)$ are in modulus less than $(2/c)\log D$, whence adding $\operatorname{re}\zeta'/\zeta(\sigma_0+2i\gamma)$, $\operatorname{re}\zeta'/\zeta(\sigma_0+i\gamma)$ to the last inequalities we deduce

$$-\operatorname{re}\zeta'/\zeta(\sigma_0+2i\gamma,\chi_0) > 8\big(c_3\log D + (n+1)\log 1/a\big)\,,$$

$$-\operatorname{re}\zeta'/\zeta(\sigma_0+i\gamma,\chi_0) < 8\big(c_3\log D + (n+1)\log 1/a\big) - (\sigma_0-\beta)^{-1}\,.$$

Repeating the arguments used in § 12 we prove the lemma for $|t| \geqslant c/\log D$.

18. Now let the function $G(s) = \zeta(s,\chi_0)/\zeta(s)$ have a zero $\varrho_1 = \beta+i\gamma$ such that $0<\gamma<1/c\log D$, $\beta \geqslant 1-1/c\log D$ and let $\sigma_0 = 1+1/b\log D$, $s_0 = \sigma_0+i\gamma$. Suppose (if possible)

(50) $$\operatorname{re} G'/G(s_0) \geqslant 5/4(\sigma_0-1)\,.$$

By (41)

$$\operatorname{re}\zeta'/\zeta(s_0,\chi_0) = \operatorname{re}\frac{-1}{s_0-1} + \operatorname{re}\sum_{|s_0-\varrho|<1}\frac{1}{s_0-\varrho} + \theta c_4\log D\,,\qquad |\theta|<1\,,$$

whence (since $G'/G(s_0) = \zeta'/\zeta(s_0,\chi_0) - \zeta'/\zeta(s_0)$)

$$\operatorname{re} G'/G(s_0) = -\operatorname{re}\zeta'/\zeta(s_0) + \operatorname{re}\frac{-1}{s_0-1} + \operatorname{re}\sum_{|s_0-\varrho|<1}\frac{1}{s_0-\varrho} + \theta c_4\log D\,,$$

or

$$\operatorname{re} G'/G(s_0) = \operatorname{re}\sum_{|s_0-\varrho|<1}\frac{1}{s_0-\varrho} + \theta c_5\log D\,.$$

Hence, by (50),

(51) $$\operatorname{re}\sum_{|s_0-\varrho|<1}\frac{1}{s_0-\varrho} > \frac{1.2}{\sigma_0-1}\,,$$

if b is large enough. By (41)

$$\frac{\zeta'}{\zeta}(\sigma_0,\chi_0) = \frac{-1}{\sigma_0-1} + \operatorname{re}\sum_{|s_0-\varrho|<1}\frac{1}{\sigma_0-\varrho} + \theta c_6\log D\,.$$

Since $\zeta'/\zeta(\sigma_0,\chi_0)$ is negative and is less in modulus than $5/4(\sigma_0-1)$, by (44), whereas $\operatorname{re}\sum(\sigma_0-\varrho)^{-1} > 0$, we have

$$\operatorname{re}\sum_{|s_0-\varrho|<1}\frac{1}{\sigma_0-\varrho} < \frac{1.1}{\sigma_0-1}\,.$$

Hence, by (51),

$$\operatorname{re}\sum_{|s_0-\varrho|<1}\Big(\frac{1}{s_0-\varrho} - \frac{1}{\sigma_0-\varrho}\Big) > \frac{0.1}{\sigma_0-1}$$

and thus

$$|s_0-\sigma_0|\sum_{|s_0-\varrho|<1}\frac{1}{|s_0-\varrho|\,|\sigma_0-\varrho|} > \frac{0.1}{\sigma_0-1}\,,$$

whence

$$\frac{1}{c\log D}\sum_{|s_0-\varrho|<1}\frac{1}{|s_0-\varrho|\,|\sigma_0-\varrho|} > \frac{b}{10}\log D\,,$$

or

$$\sum_{|s_0-\varrho|<1}\frac{1}{|s_0-\varrho|\,|\sigma_0-\varrho|} > \frac{bc}{10}\log^2 D\,.$$

Taking $c > b$ we have

$$\left|\frac{\sigma_0-\varrho}{s_0-\varrho}\right| \geqslant \frac{1}{\sqrt{2}} \qquad\text{or}\qquad \frac{1}{|\sigma_0-\varrho|} \leqslant \frac{\sqrt{2}}{|s_0-\varrho|}\,.$$

This combined with the previous inequality gives

$$\sum_{|s_0-\varrho|<1}\frac{\sqrt{2}}{|s_0-\varrho|^2} > \frac{bc}{10}\log^2 D$$

or

(52) $$\sum_{|s_0-\varrho|<1}|s_0-\varrho|^{-2} > \frac{bc}{10\sqrt{2}}\log^2 D\,.$$

Let $\nu(r)$ denote the number of zeros of $\zeta(s, \chi_0)$ in a circle having its centre at $s_1 = 1+i\gamma$ and radius $r \geqslant 1/\log D$. By (43) $\nu(r) \ll r\log D$. Hence for $b > 1$

$$\sum_{|s_0-\varrho|<1} |s_0-\varrho|^{-2} \leqslant \sum_{|s_1-\varrho|\leqslant 1/\log D} |s_0-\varrho|^{-2} + \sum_{1/\log D<|s_1-\varrho|<1} |s_0-\varrho|^{-2}$$

$$< \sum_{|s_1-\varrho|\leqslant 1/\log D} |s_0-s_1|^{-2} + \sum_{1/\log D<|s_1-\varrho|<1} |s_1-\varrho|^{-2}$$

$$\ll b^2\log^2 D + \int_{1/\log D}^{1} \frac{\nu(r)}{r^3} dr \ll b^2\log^2 D\,,$$

by (6). Being a contradiction to (52) (if c is large enough) this disproves (50). Hence, for appropriate b, c,

$$\text{(53)} \qquad \operatorname{re} G'/G(s_0) < 5/4(\sigma_0-1)\,.$$

19. We are now in a position to finish the proof of Lemma 13.

If $\varrho_1 = \beta+i\gamma$ is a zero of $G(s)$, then $\varrho_2 = \beta - i\gamma$ is another one. In the circle $|s-s_0| \leqslant \frac{1}{2}$

$$|G(s)/G(s_0)| < e^M, \qquad M = c_4\log D + (n+1)\log b$$

(cf. (49)). Hence, by (47) (with $f = G$, $r = \frac{1}{2}$)

$$-\operatorname{re} G'/G(s_0) < 8\big(c_4\log D + (n+1)\log b\big) - \left[\frac{1}{\sigma_0-\beta} + \frac{\sigma_0-\beta}{(\sigma_0-\beta)^2+4\gamma^2}\right]$$

$$\left(\text{since} \quad \operatorname{re} \sum_{|s_0-\varrho|<r/2} \frac{1}{s_0-\varrho} \geqslant \frac{1}{\sigma_0-\beta} + \frac{\sigma_0-\beta}{(\sigma_0-\beta)^2+4\gamma^2}\right)$$

whence, by (53),

$$\text{(54)} \qquad \frac{1}{\sigma_0-\beta} + \frac{\sigma_0-\beta}{(\sigma_0-\beta)^2+4\gamma^2} < c_5\log D + 8(n+1)\log b + \frac{5}{4(\sigma_0-1)}\,,$$

or

$$\frac{\sigma_0-\beta}{(\sigma_0-\beta)^2+4\gamma^2} < c_5\log D + 8(n+1)\log b + \frac{1}{4(\sigma_0-1)} + \frac{1-\beta}{(\sigma_0-1)(\sigma_0-\beta)}$$

$$< c_5\log D + 8(n+1)\log b + \tfrac{1}{4}b\log D + (b^2/c)\log D$$

$$< (b/2 + b^2/c)\log D\,,$$

if b is large enough. We conclude the proof by arguments used at the end of § 15.

20. Lemma 14. *For a sufficiently small absolute constant $c > 0$ there is at most one real zero $> 1 - c/\log D$ of the function $\zeta(s, \chi_0)$.*

Proof. Let β, β' be two real zeros of $\zeta(s, \chi_0)$ and let $\beta' \geqslant \beta$. By the same argument as used in the proof of (54) we get the inequality

$$\frac{1}{\sigma_0-\beta} + \frac{1}{\sigma_0-\beta'} < c_6\log D + 8(n+1)\log b + \frac{5}{4(\sigma_0-1)}\,,$$

whence

$$\frac{2}{\sigma_0-\beta} < c_6\log D + 8(n+1)\log b + \tfrac{5}{4}\,b\log D < \tfrac{3}{2}\,b\log D\,,$$

if b is large enough. This proves the lemma (cf. § 16).

Lemma 15. *For appropriate $c_1 > 0$ there is at most one function $\zeta(s, \chi)$ of character χ mod $\mathfrak{f}$ having a real zero in $\sigma \geqslant 1 - c_1/\log D$.*

Proof. Let there be two real and different characters χ_1, χ_2 such that $\zeta(s, \chi_1)$ and $\zeta(s, \chi_2)$ have real zeros $\beta_1 > \frac{1}{2}$ and $\beta_2 > \frac{1}{2}$, respectively. We suppose first that $\chi_1 \neq \chi_0$ and $\chi_2 \neq \chi_0$.

For $\sigma_0 = 1 + 1/b\log D$ we have (cf. § 13)

$$-\zeta'/\zeta(\sigma_0, \chi_1) < 8(c_3\log D + n\log b) - (\sigma_0-\beta_1)^{-1}\,,$$
$$-\zeta'/\zeta(\sigma_0, \chi_2) < 8(c_3\log D + n\log b) - (\sigma_0-\beta_2)^{-1}\,,$$
$$-\zeta'/\zeta(\sigma_0, \chi_1\chi_2) < 8(c_3\log D + n\log b)\,,$$
$$-\zeta'/\zeta(\sigma_0, \chi_0) < 5/4(\sigma_0-1)\,.$$

Since the sum of the left-hand sides in this set of inequalities is $\geqslant 0$, by (11), we deduce

$$(\sigma_0-\beta_1)^{-1} + (\sigma_0-\beta_2)^{-1} < 24(c_3\log D + n\log b) + \tfrac{5}{4}\,b\log D < \tfrac{3}{2}\,b\log D$$

(if b is large enough) and conclude the proof as in § 16.

Now let $\chi_2 = \chi_0$, the other premises remaining unchanged. Then we use the inequalities (cf. § 17)

$$-G'/G(\sigma_0) < 8(c_2\log D + (n+1)\log b) - (\sigma_0-\beta_2)^{-1}\,,$$
$$-\zeta'/\zeta(\sigma_0) < 5/4(\sigma_0-1)\,,$$
$$-\zeta'/\zeta(\sigma_0, \chi_1) < 8(c_3\log D + n\log b) - (\sigma_0-\beta_1)^{-1}\,,$$

where $G(s) = \zeta(s, \chi_0)/\zeta(s)$. Since, by (11),

$$-\zeta'/\zeta(\sigma_0, \chi_1) - \zeta'/\zeta(\sigma_0, \chi_0) \geqslant 0\,,$$

we have

$$(\sigma_0-\beta_1)^{-1} + (\sigma_0-\beta_2)^{-1} < c_4\log D + (16n+8)\log b + \tfrac{5}{4}\,b\log D$$

and may go on as before.

21. Lemma 16. *Let β_0 be the real zero of the function $\zeta(s, \chi_0)$ such that $\beta_0 > 1 - c/\log D$ for arbitrarily small $c > 0$. Writing $\delta_0 = 1 - \beta_0$ we have for any positive $\varepsilon < \frac{1}{2}$*

$$\delta_0 > c_1(\varepsilon) D^{-\varepsilon}\,.$$

Proof. We have, by (33), in the circle $C(|s-1|\leqslant\eta=\varepsilon/2)$

$$(s-1)\zeta(s,\chi_0)\ll D^{\eta}$$

(the constant of $\ll$ depending on η), whence in C

$$G(s)=\zeta(s,\chi_0)/\zeta(s)=(s-1)\zeta(s,\chi_0)/(s-1)\zeta(s)\ll D^{\eta}.$$

Hence, in $|s-1|\leqslant\eta/2$

$$G'(s)=\frac{1}{2\pi i}\int\limits_{|w-s|=\eta/2}\frac{G(w)}{(w-s)^2}dw\ll D^{\eta}. \tag{55}$$

By (13) $G(1)=D^{o(1)}$ $(D\to\infty)$, whence $G(1)>c_2(\eta)D^{-\eta}$. Since $G(\beta_0)=0$, we have

$$c_2(\eta)D^{-\eta}<G(1)-G(\beta_0)=\delta_0 G'(\sigma_1),\qquad \beta_0<\sigma_1<1.$$

Hence, by (55),

$$\delta_0>c_2(\eta)D^{-\eta}/G'(\sigma_1)>c_1(\varepsilon)D^{-\varepsilon}.$$

22. LEMMA 17. *Let β be the real zero of the function $\zeta(s,\chi)$ with a real character $\chi\neq\chi_0$ such that $\beta>1-c/\log D$ for arbitrarily small $c>0$ and let $\delta=1-\beta$. Then for all large D*

$$\delta>D^{-2n}. \tag{56}$$

Proof. Writing

$$g(\mathfrak{c})=\sum_{\mathfrak{b}/\mathfrak{c}}\chi(\mathfrak{b}) \tag{57}$$

we have for any $\mathfrak{a},\mathfrak{b}$, prime to each other,

$$g(\mathfrak{a})g(\mathfrak{b})=\sum_{\mathfrak{d}|\mathfrak{a}}\chi(\mathfrak{d})\sum_{\mathfrak{d}_1|\mathfrak{b}}\chi(\mathfrak{d}_1)=\sum_{\mathfrak{d}\mathfrak{d}_1|\mathfrak{a}\mathfrak{b}}\chi(\mathfrak{d}\mathfrak{d}_1)=\sum_{\mathfrak{d}_2|\mathfrak{a}\mathfrak{b}}\chi(\mathfrak{d}_2)=g(\mathfrak{a}\mathfrak{b}).$$

Since, by (57),

$$g(\mathfrak{p}^k)=\begin{cases}1+1+\ldots+1>1 & \text{if}\quad \chi(\mathfrak{p})=1,\\ 1-1+-\ldots=1 & \text{if}\quad \chi(\mathfrak{p})=-1,\ k\ \text{even},\\ 1-1+-\ldots=0 & \text{if}\quad \chi(\mathfrak{p})=-1,\ k\ \text{odd},\end{cases}$$

using the multiplicative property of $g(\mathfrak{a})$, we deduce that any $g(\mathfrak{a})$ is a non-negative integer and $g(\mathfrak{a})\geqslant 1$, if $\mathfrak{a}$ is a square. By (8)

$$\zeta(s,\chi)\zeta(s,\chi_0)=\sum_{\mathfrak{c}}g(\mathfrak{c})N\mathfrak{c}^{-s}\qquad(\sigma>1).$$

Hence, by (5), for any $\nu>0$

$$\sum_{\mathfrak{c}}\frac{g(\mathfrak{c})}{N\mathfrak{c}}e^{-\nu N\mathfrak{c}}=\frac{1}{2\pi i}\int\limits_{2-i\infty}^{2+i\infty}\nu^{1-s}\Gamma(s-1)\zeta(s,\chi)\zeta(s,\chi_0)\,ds.$$

In the region $\sigma>0$ the integrand has no other singularities than a double pole at $s=1$ with the residue

$$E+(c_0-\log\nu)\mu$$

where

$$\mu=\zeta(1,\chi)\operatorname*{Res}_{s=1}\zeta(s,\chi_0),\qquad E=\lim_{s\to1}\{\zeta(s,\chi)\zeta(s,\chi_0)-\mu(s-1)^{-1}\},$$

$$c_0=\lim_{z\to0}\{\Gamma(z)-1/z\}\ll 1.$$

Taking $\nu=D^{-a}$ and moving the contour of integration to the line $\sigma=1/\log D$ we get, by (32), (4),

$$\sum_{\mathfrak{c}}\frac{g(\mathfrak{c})}{N\mathfrak{c}}e^{-\nu N\mathfrak{c}}=E+(c_0+a\log D)\mu+O(D^{1-a}\log^{2n+1}D).$$

Replace a by $a+1$. By subtraction

$$\sum_{\mathfrak{c}}\frac{g(\mathfrak{c})}{N\mathfrak{c}}(e^{-D^{-a-1}N\mathfrak{c}}-e^{-D^{-a}N\mathfrak{c}})=\mu\log D+O(D^{1-a}\log^{2n+1}D).$$

We have

$$\sum_{N\mathfrak{c}\leqslant D^a}\frac{g(\mathfrak{c})}{N\mathfrak{c}}e^{-D^{-a-1}N\mathfrak{c}}(1-e^{-(D^{-a}-D^{-a-1})N\mathfrak{c}})>\tfrac{1}{2}\sum_{N\mathfrak{c}\leqslant D^a}\frac{g(\mathfrak{c})}{N\mathfrak{c}}(1-e^{-\frac{1}{2}D^{-a}N\mathfrak{c}})$$
$$>\tfrac{1}{8}D^{-a}\sum_{N\mathfrak{c}\leqslant D^a}g(\mathfrak{c}),$$

whence

$$\tfrac{1}{8}D^{-a}\sum_{N\mathfrak{c}\leqslant D^a}g(\mathfrak{c})<\mu\log D+c_2D^{1-a}\log^{2n+1}D.$$

Since $g(\mathfrak{c})\geqslant1$ for every square $\mathfrak{c}$, and, by (2), for $a=2n+1$, $D>D_0$ there are at least $D^{a/2n}/\log D$ squares $\mathfrak{c}=1^2,2^2,3^2,\ldots$ with $N\mathfrak{c}\leqslant D^a$, $(N\mathfrak{c},D)=1$, we have

$$\tfrac{1}{8}D^{-a}\sum_{N\mathfrak{c}<D^a}g(\mathfrak{c})\geqslant\tfrac{1}{8}D^{-a+1+1/2n}/\log D>2c_2D^{1-a}\log^{2n+1}D$$

(provided that D_0 is large enough), whence

$$\mu\log D>c_3D^{-2n+1/2n}/\log D.$$

Since $\mu=\zeta(1,\chi)\operatorname*{Res}_{s=1}\zeta(s,\chi_0)$, we have, by (13),

$$\zeta(1,\chi)>D^{-2n+1/4n}$$

or

$$\delta\cdot\zeta'(\sigma_1,\chi)>D^{-2n+1/4n},\qquad\beta<\sigma_1<1$$

(cf. the arguments at the end of § 21). Using (32) and the integral formula for $\zeta'(s,\chi)$ (cf. (55)) we deduce that $\zeta'(\sigma_1,\chi)\ll D^{1/8n}$. Combining this with the previous inequality we get (56).

By a more careful account of the number of squares $\mathfrak{c}$ whose norm does not exceed D^a it can be proved that $\delta > c_4 D^{-\varkappa}$ for any $\varkappa > \frac{1}{4}(n+3)$ and $c_4 = c_4(\varkappa)$. But for our prospective arithmetical applications we can do as well with (56).

The theorem of § 1 is an immediate consequence of Lemmas 9-17.

References

[1] R. Brauer, *On the zeta-functions of algebraic number fields*, Amer. Journ. Math. LXIX (1947), pp. 243-250.

[2] E. Hecke, *Über die L-Funktionen und den Dirichletschen Primzahlsatz für einen beliebigen Zahlkörper*, Göttinger Nachrichten, Math. ph. Klasse (1917), pp. 299-318.

[3] E. Landau, *Über Ideale und Primideale in Idealklassen*, Math. Zeitschr. 2 (1918), pp. 52-154.

[4] — *Vorlesungen über Zahlentheorie II, III*, Leipzig. 1927.

[5] A. Page, *On the number of primes in an arithmetic progression*, Proc. London Math. Soc. 39 (1935), pp. 116-141.

[6] K. Prachar, *Primzahlverteilung*, Berlin 1957.

[7] E. C. Titchmarsh, *A divisor problem*, Rendiconti di Palermo LIV (1930), pp. 414-429.

[8] — *The theory of functions*, Oxford 1939.

[9] — *The theory of Riemann zeta-function*, Oxford 1951.

Reçu par la Rédaction le 23. 1. 1961

ACTA ARITHMETICA
VII (1962)

On sign-changes of the difference $\pi(x)-\mathrm{li}\,x$

by

S. Knapowski (Poznań)

1. Let $\nu(T)$ denote the number of sign-changes of the difference $\pi(x)-\mathrm{li}\,x$ for $2\leqslant x\leqslant T$. Littlewood ([7]) proved in 1914 that $\nu(T)$ tends to infinity together with T. However Littlewood's method, as it stands in [7], does not provide numerical results and in particular does not enable one, even on the Riemann hypothesis, to find an explicit upper bound for the position of the first sign-change of $\pi(x)-\mathrm{li}\,x$. Such numerical estimation has been performed only a few years ago by Skewes [8], the result being

$$\nu\big(\exp\exp\exp\exp(7.705)\big)\geqslant 1\,. \tag{1.1}$$

A conditional estimate for the order of growth of $\nu(T)$ has been obtained by Ingham [4]. His theorem reads as follows:

If there exists a ζ-zero $\varrho_0=\sigma_0+it_0$ such that $\zeta(s)\neq 0$ in the half-plane $\sigma>\sigma_0$, then

$$\varlimsup_{T\to\infty}\frac{\nu(T)}{\log T}>0\,. \tag{1.2}$$

I proved recently [5] the following theorem which leads, when combined with that of Ingham, to an *unconditional* lower estimate for $\nu(T)$:

Let $\varrho_0=\beta_0+i\gamma_0$, $\beta_0\geqslant\frac{1}{2}$, $\gamma_0>0$ be an arbitrary ζ-zero. Then, for $T>\max\big(c_1,\exp\exp(\log^2\gamma_0)\big)$, c_1 a numerical constant, we have the inequalities

$$\begin{cases}\max\limits_{2\leqslant t\leqslant T}\{\Pi(t)-\mathrm{li}\,t\}>T^{\beta_0}\exp\left(-15\dfrac{\log T}{\sqrt{\log\log T}}\right),\\[2ex] \min\limits_{2\leqslant t\leqslant T}\{\Pi(t)-\mathrm{li}\,t\}<-T^{\beta_0}\exp\left(-15\dfrac{\log T}{\sqrt{\log\log T}}\right),\end{cases} \tag{1.3}$$

where

$$\Pi(x)=\sum_{m=1}^{\infty}\frac{1}{m}\pi(x^{1/m})\,.$$