An estimate for $S^*_{a,b}(x)$ can now be easily deduced.

THEOREM 4.2. *If* $b > a > 1$, $(a, b) = 1$, *then for* $x \geqslant 2$,

$$S^*_{a,b}(x) = A^* x^{1/a} + B^* x^{1/b} + O(x^{1/c}\log x), \tag{4.12}$$

*where* $A^* = A\zeta(b)U(b)$, $B^* = B\zeta(a)U(a)$, *and*

$$U(s) = \prod_p \left(1 - \frac{2}{p^s}\right), \quad s > 1. \tag{4.13}$$

Proof. By Lemma 4.6, it follows that (cf. (4.6))

$$S^*_{a,b}(x) = \sum_{n \leqslant x} j^*_{a,b}(n) = \sum_{n \leqslant x^{1/k}} \mu^*(n) S_{a,b}\left(\frac{x}{n^k}\right), \tag{4.14}$$

and hence by Theorem 4.1 and the boundedness of $\mu^*(n)$ (cf. Remark 4.1),

$$S^*_{a,b}(x) = Ax^{1/a} \sum_{n \leqslant x^{1/k}} \frac{\mu^*(n)}{n^b} + Bx^{1/b} \sum_{n \leqslant x^{1/k}} \frac{\mu^*(n)}{n^a} + O\left(x^{1/c}\log x \sum_{n \leqslant (x/2)^{1/k}} \frac{1}{n^{k/c}}\right) + O(x^{1/k}).$$

By an argument similar to that of Lemma 4.3, it is seen that

$$\sum_{n=1}^{\infty} \frac{\mu^*(n)}{n^s} = \zeta(s) \prod_p \left(1 - \frac{2}{p^s}\right), \quad s > 1. \tag{4.15}$$

The proof now proceeds like that of Theorem 4.1.

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# Zeta functions of quadratic forms

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*Dedicated to the memory of Dr. R. Vaidyanathaswamy*

**§ 1. Introduction.** The Riemann zeta function has been generalized in two directions; one generalization concerns the zeta functions of algebraic number fields and algebras and the other concerns the zeta functions of Lerch-Epstein associated with definite quadratic forms and of Siegel associated with indefinite quadratic forms. Our object in this paper is to study the zeta functions associated with quadratic forms over involutorial algebras. We deal here with commutative algebras only reserving the non-commutative case for the second part.

Let $K$ be an algebraic number field and $\sigma$ an automorphism of $K$ whose square is the identity. Let $k$ be the fixed field of $\sigma$. Then $(K: k) = 1$ or 2 according as $\sigma$ is or is not the identity automorphism. For any matrix $A$ of $m$ rows and columns with elements in $K$ let $A^\sigma$ denote the matrix, $(a^\sigma_{kl})$ where $A = (a_{kl})$. We say that $A$ is symmetric (hermitian) if $\sigma$ is (or not) the identity and $A' = A^\sigma$. If $\mathfrak{a}$ is a $m$-rowed vector with elements in $K$ we call $\mathfrak{a}' A \mathfrak{a}^\sigma$ the quadratic (hermitian) form associated with $A$. Let first $\sigma = 1$ the identity automorphism. Let $S$ be symmetric, $m$-rowed and non-singular over $K$. Let $K$ have $r_1$ real and $r_2$ complex infinite prime spots and let $S$ be definite at $r_1 - l$ of the real infinite prime spots of $K$, $0 \leqslant l \leqslant r_1$. For every $g \neq 0$ in $K$ which can be represented by $S$ we associate a vector $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_l)$, $\varepsilon_i = \pm 1$ where $\varepsilon_k = g^{(k)}/|g^{(k)}| = \operatorname{sgn} g^{(k)}$. We call $\varepsilon$ the signature of $g$. With each $\varepsilon$ we associate the zeta function

$$\zeta_\varepsilon(S, \mathfrak{a}, s) = N\mathfrak{a}^{2s} \sum_g \frac{M(S, \mathfrak{a}, g)}{(N|g|)^s}$$

where $\mathfrak{a} \neq 0$ is an ideal of $K$, $M(S, \mathfrak{a}, g)$ is the measure of representation of $g$ by $S$ (see § 4) and the summation runs over all $g$ with signature $\varepsilon$ which are representable by $S$ such that for no two $g_1, g_2$ in the summation $g_1 = \varepsilon^2 \bar{\varepsilon} g_2$ holds, $\varepsilon$ being a unit in $K$. There are clearly $2^l$ such Dirichlet series. It is shown (§ 3) that they converge for $\sigma > m/2$ and define in this half plane regular analytic functions of $s$. By generalizing suitably a method due to Siegel, it is shown that these Dirichlet series can be continued analytically into the whole plane where they are meromorphic with atmost two simple poles at $s = m/2$ and $s = 1$. It turns out that the residue of this zeta function at $s = m/2$ is *independent* of $\varepsilon$ and $\mathfrak{a}$. They further-more satisfy functional equations of the type

$$\varphi_\varepsilon(S, \mathfrak{a}, s) = N\|S\|^{-1/2} \sum_{\eta}{}' a_{\varepsilon\eta}(s, a_\eta)\varphi_\eta\left(S^{-1}, \tilde{\mathfrak{a}}, \frac{m}{2} - s\right),$$

$a_{\varepsilon\eta}(s, a_\eta)$ being certain trigonometrical polynomials of $s$. In special cases the functional equation assumes a simpler form. For instance if $|S| > 0$ at all real infinite prime spots of $K$ where $S$ is indefinite, then

$$\varphi_\varepsilon(S, \mathfrak{a}, s) = N\|S\|^{-1/2}(-1)^{a_{\varepsilon_1}+\ldots+a_{\varepsilon_l}/2}\varphi_\varepsilon\left(S^{-1}, \tilde{\mathfrak{a}}, \frac{m}{2} - s\right)$$

for a certain $\varepsilon$.

Suppose now $\sigma \neq$ identity so that $S$ is a non-singular hermitian matrix. Let $\mathfrak{H}$ be the representation space of the units of $S$ and $dv$ the invariant volume element in $\mathfrak{H}$. Let $F$ be a fundamental region for the units of $S$ in $\mathfrak{H}$. We prove first that $\int_F dv$ converges. This is done by suitably parametrizing the $\mathfrak{H}$-space. Let $K$, for simplicity, be the imaginary quadratic field. We can then define zeta functions as above and obtain their functional equations. Instead we follow a method of Hecke and introduce zeta functions with congruence conditions. This has the effect of giving zeta functions some of which are entire functions. The analytic nature and functional equations of these zeta functions are obtained by using the theta series. These zeta functions have at most one simple pole.

The form of the functional equations in the case of hermitian forms and in the case of certain quadratic forms shows that one can associate Hilbert modular forms with these. We shall deal with this topic separately elsewhere.

**§ 2. Notations and terminology.** Capital Roman letters denote matrices. $a$ always stands for a column vector. $N$ denotes norm and $\sigma$ denotes trace. If $S, P$ are matrices we put $S[P]$ for $P'SP$ and $S\{P\}$ for $P'S\overline{P}$, $P'$ and $\overline{P}$ denoting the transpose and complex conjugate respectively of $P$. For a matrix $P$, $|P|$ denotes its determinant and $\|P\|$ the absolute value of $|P|$. Whenever an equation or an inequality is written without superscripts it is understood that these equations and inequalities hold for all conjugates—whenever they have a meaning. For a matrix $A = (a_{kl})$, $dA$ denotes the Euclidean volume element $\prod_{k,l} da_{kl}$, similar meanings where $A$ is real symmetric or complex hermitian. The constants $c_1, c_2 \ldots$ depend only on $m$, $K$ and $S$ in general. The notations and terminology are those in Siegel [7], [8].



**§ 3. Positive systems.** Let $K$ be an algebraic number field of degree $n = r_1 + 2r_2$ over the field of rationals and let $r_1$ and $r_2$ denote the number of real and complex infinite prime spots respectively of $K$. Let $P$ denote the space of $r_1 + r_2$ positive variables $t^{(1)}, \ldots, t^{(r_1+r_2)}$. We denote by $t$ a generic element of $P$. The unit group of $K$ is represented in $P$ as a group of transformations $t \to \varepsilon t\bar{\varepsilon}$, i.e.

$$t^{(i)} \to \varepsilon^{(i)} t^{(i)} \overline{\varepsilon^{(i)}}, \quad i = 1, \ldots, r_1 + r_2,$$

$\varepsilon$ being a unit of $K$. This representation is faithful if we identify $\varepsilon$ and $\omega\varepsilon$, $\omega$ being a root of unity in $K$. There exists in $P$ a fundamental region $G$ for the group of units. Denote by $G_0$ the fundamental region on the norm surface $Nt = t^{(1)} \ldots t^{(r_1)}(t^{(r_1+1)} \ldots t^{(r_1+r_2)})^2 = 1$. Let $[dt]$ denote the volume element

$$(1) \qquad [dt] = \frac{dt^{(1)} \ldots dt^{(r_1+r_2)}}{t^{(1)} \ldots t^{(r_1+r_2)}}$$

and $[dt_0]$ the corresponding volume element on the norm surface. Then

$$(2) \qquad \int_{G_0} [dt_0] = 2^{r_1-1} R$$

where $R$ is the regulator of $K$.

Let $m > 0$ be an integer. A *positive system* $T$ is a set of $n$ matrices

$$(3) \qquad T = \begin{pmatrix} T^{(1)} & & \\ & \ddots & \\ & & T^{(n)} \end{pmatrix}$$

each of $m$ rows such that $T^{(1)}, \ldots, T^{(r_1)}$ are real positive symmetric $T^{(r_1+1)}, \ldots, T^{(r_1+r_2)}$ are positive complex hermitian and $\overline{T^{(r_1+k)}} = T^{(r_1+r_2+k)}$. The positive systems constitute a space $\mathfrak{P}$ of $\frac{1}{2}r_1 m(m+1) + r_2 m^2$ real dimensions. If $m = 1$, $\mathfrak{P}$ coincides with $P$. Denote by $\Gamma(K)$ the group of unimodular matrices over $K$. The mapping $T \to T\{U\}$ of $\Gamma(K)$ in $\mathfrak{P}$ defined by

$$T\{U\} = \begin{pmatrix} T^{(1)}[U^{(1)}] & & \\ & \ddots & \\ & & T^{(n)}\{U^{(n)}\} \end{pmatrix}$$

($U^{(1)}, \ldots, U^{(n)}$ being conjugates of $U$ in $K$) is discontinuous in $\mathfrak{P}$ if we identify $U$ and $\omega U$. Humbert ([3]) constructed for $\Gamma(K)$ in $\mathfrak{P}$ a fundamental region $R$. Note that if $m = 1$, $R$ coincides with $G$. Let $c > 0$. Consider in $\mathfrak{P}$ the point set $\mathfrak{R}_c$ consisting of systems $T = (t_{kl})$ with

$$(4) \qquad \begin{aligned} 0 < t_k^{(a)} &\leqslant c t_l^{(b)}, \quad k \leqslant l, \\ \operatorname{abs} t_{kl}^{(a)} &\leqslant c t_k^{(a)}, \quad k \leqslant l, \\ t_1^{(l)} \ldots t_m^{(l)} &\leqslant c\,|T^{(l)}|, \quad l = 1, \ldots, n. \end{aligned}$$

The fundamental region $R$ of Humbert has the property that there exists a constant $c_1 > 0$ and a finite set $A_1, \ldots, A_\mu$ of $m$ rowed integral matrices all determined uniquely by $K$ and the integer $m$ such that for any $H \in R$ there exists at least one $A_i$ such that $H\{A_i\}$ satisfies (4) with $c_1$ for $c$.

Let $\beta^{(1)}, \ldots, \beta^{(n)}$ be $m$ rowed column vectors of which the first $r_1$ are real and $\beta^{(r_1+k)}$ and $\beta^{(r_1+r_2+k)}$ are complex conjugates, $k = 1, \ldots, r_2$. Let $\mathfrak{a}$ be a non-zero ideal in $K$. One has the formula

$$\vartheta(T, \mathfrak{a}) = \frac{1}{N\|T\|^{1/2}(N\mathfrak{a}\sqrt{|d|})^m} \sum_{\alpha \in \tilde{\mathfrak{a}}} e^{-\pi\sigma(T^{-1}\{\alpha\}) + 2\pi i \sigma(\alpha'\beta)}$$

where

$$\vartheta(T, \mathfrak{a}) = \sum_{a \in \mathfrak{a}} e^{-\pi\sigma(T\{a+b\})};$$

$\tilde{\mathfrak{a}}$ is the complementary ideal to $\mathfrak{a}$ and $T \in \mathfrak{P}$, $d$ is discriminant of $k$.

In particular, if $\beta = 0$ and $t \in P$,

$$(5) \qquad \vartheta(T, \mathfrak{a}, t) = \sum_{a \in \mathfrak{a}} e^{-\pi\sigma(T\{a\}t)} = \frac{1}{N\|T\|^{1/2}(Nt)^{m/2}(N\mathfrak{a}\sqrt{|d|})^m} \sum_{\alpha \in \tilde{\mathfrak{a}}} e^{-\pi\sigma(T^{-1}\{\alpha\}t^{-1})}.$$

Using well-known inequalities for the ordinary theta function in one variable with $m = 1$, we get

$$\vartheta(T, \mathfrak{a}, t) \leqslant \prod_{k=1}^{m} N\big(c_2 + c_3(h_k t)^{-1/2}\big),$$

$c_2, c_3$ being constants depending on $m, k, \mathfrak{a}$ and $N\|T\|$. In particular, if $T \in \mathfrak{R}_{c_1}$, $t \in G$,

$$(6) \qquad \vartheta(T, \mathfrak{a}, t) \leqslant c_4(1 + Nt^{-m/2}) \prod_{k=1}^{m} N(1 + h_k^{-1/2}),$$

$c_4$ depending only on $m$ and $K$ and $N\|T\|$.

**§ 4. Indefinite quadratic forms.** Let $S$ be a symmetric $m$-rowed non-singular matrix with elements in $K$. Let $S^{(1)}, \ldots, S^{(n)}$ be its conjugates. Denote by $S$ the system

$$S = \begin{pmatrix} S^{(1)} & & \\ & \ddots & \\ & & S^{(n)} \end{pmatrix}.$$

Let $\mathfrak{H}$ denote the totality of positive systems $T$ which satisfy $TS^{-1}\bar{T} = \bar{S}$ meaning that the equations

$$(7) \qquad T^{(i)}S^{(i)-1}\bar{T}^{(i)} = \overline{S^{(i)}}, \quad i = 1, \ldots, n,$$



are satisfied. It is known that $\mathfrak{H}$ is a symmetric Riemannian space of $\sum_{k=1}^{r_1} p_k q_k + r_2\frac{m(m-1)}{2}$ dimensions, $p_k, q_k$ being the system of signatures of $S$, $0 \leqslant p_k \leqslant m$. We shall denote by $dv$ the invariant volume element in $\mathfrak{H}$.

Let $S$ have the following form:

$$(8) \qquad S^{(a)} = \begin{pmatrix} 0 & 0 & P^{(a)} \\ 0 & F^{(a)} & Q^{(a)} \\ * & * & G^{(a)} \end{pmatrix} \quad (a = 1, \ldots, n),$$

where $P^{(a)}$ is a square matrix of $g_a$ rows, $0 \leqslant g_a \leqslant \frac{1}{2}m$. We shall give a parametrization of $\mathfrak{H}$. Let $T \in \mathfrak{H}$ and put

$$(9) \qquad T^{(a)} = \begin{pmatrix} H_1^{(a)} & 0 & 0 \\ 0 & H_2^{(a)} & 0 \\ 0 & 0 & H_3^{(a)} \end{pmatrix} \begin{Bmatrix} E & Q_1^{(a)} & Q_2^{(a)} \\ 0 & E & Q_3^{(a)} \\ 0 & 0 & E \end{Bmatrix}$$

where $H_1^{(a)}$ and $H_3^{(a)}$ are square matrices of $g_a$ rows. Using (7) with $Q^{(k)} = 0$, $G^{(k)} = 0$ (see [8]) one obtains

$$(10) \qquad \left.\begin{aligned} F^{(a)-1}[H_2^{(a)}] &= \overline{F^{(a)}} \\ H_1^{(a)}P^{(a)-1}\overline{H_3^{(a)}} &= \overline{P^{(a)\prime}} \\ Q_3^{(a)} &= -F^{(a)-1}Q_1^{(a)\prime}P^{(a)} \\ Q_2^{(a)} &= (A^{(a)} - \tfrac{1}{2}F^{(a)-1}[Q_1^{(a)\prime}])P^{(a)} \end{aligned}\right\} \quad (a = 1, \ldots, n);$$

$A^{(a)} = -A^{(a)\prime}$. For $a = 1, \ldots, r_1 + r_2$ we choose $H_1^{(a)}, Q_1^{(a)}, A^{(a)}$ and the parameters required to parametrize the space of $H_2^{(a)}$ satisfying the first of the above equations (10). They satisfy the conditions $H_1^{(a)} > 0$ (positive symmetric if $a \leqslant r_1$ otherwise positive hermitian), $Q_1^{(a)}$ an arbitrary matrix of $g_a$ rows and $m - 2g_a$ columns (real if $a \leqslant r_1$ and complex otherwise) and $A^{(a)}$ skew symmetric of $g_a$ rows. It is easy to see that in terms of these parameters the volume element is given by

$$(11) \qquad dv = c_5 \prod_{k=1}^{r_1} |H_1^{(k)}|^{(m-2g_k-2)/2} \prod_{k=r_1+1}^{r_1+r_2} |H_1^{(k)}|^{m-2g_k-1} \prod_{k=1}^{n} |H_2^{(k)}|^{-g_k/2} dW$$

where

$$dW = dH_1^{(a)} \ldots dA^{(a)} dv_0;$$

$dv_0$ being the volume element in the $H_2$ space; $c_5$ is a constant depending on $m$ and $g_a$.

Let $S$ be now the matrix of an integral quadratic form $\alpha' S \alpha$ which is non-degenerate. Let $K^{(1)}, \ldots, K^{(n)}$ be the conjugates of $K$ so ordered that $K^{(1)}, \dots, K^{(r_1)}$ are real and the rest pairs of complex conjugate fields. Let $S$ be such that at $l$ of the infinite prime spots, say in the fields $K^{(1)}, \dots, K^{(l)}$, $0 \leqslant l \leqslant r_1$, it is indefinite and at the other $r_1 - l$ infinite real spots it is definite. If $a'Sa$ is a zero form then clearly $r_1 = l$. Let us further-more assume that *$a'Sa$ is neither a binary, ternary nor a quaternionic zero form.*

Let $T$ be in $\mathfrak{H}$ and choose real numbers $a^{(1)}, \dots, a^{(n)}$ such that

$$(12) \qquad \begin{aligned} -1 < a^{(k)} < 1\,, \qquad & k = 1, \dots, l\,, \\ a^{(k)} = 0\,, \qquad & k > l\,. \end{aligned}$$

Put now $H = T - aS$. From the definition of $T$ if follows that $H > 0$ and that

$$(13) \qquad \begin{aligned} |T-aS| = |H| &= N\|S\| \prod_{k=1}^{l} (1-a^{(k)})^{p_k}(1+a^{(k)})^{q_k}\,, \\ (T-aS)^{-1} = H^{-1} &= (1-a^2)^{-1}(T^{-1}+aS^{-1}); \end{aligned}$$

the last equation to be understood in the sense that, for each $k$,

$$(T^{(k)} - a^{(k)}S^{(k)})^{-1} = (1-a^{(k)2})^{-1}(T^{(k)-1} + a^{(k)}S^{(k)-1})\,.$$

Let $c^{(1)}, \dots, c^{(n)}$ be positive real numbers to be chosen presently. We put

$$\vartheta(S, T, \mathfrak{a}, c, t) = \sum_{\alpha \in \mathfrak{a}} e^{-\pi\sigma(cH\{\alpha\}t)}$$

and call it the *theta series associated with $S$*. We now choose $c^{(1)}, \dots, c^{(n)}$ so that

$$(14) \qquad c^{(k)} = (N\|S\|)^{-1/mn}(N\mathfrak{a}\sqrt{|d|})^{-2/n}(1-a^{(k)})^{-p_k/m}(1+a^{(k)})^{-q_k/m}\,.$$

If we put $c^{(k)} = c^{(k)}(S, \mathfrak{a}, a)$ then we have

$$(15) \qquad \tilde{c}^{(k)} = \tilde{c}^{(k)}(S^{-1}, \tilde{\mathfrak{a}}, -a) = c^{(k)-1}(1-a^{(k)2})^{-1}\,.$$

With this choice of $c$, using the transformation formula (5) we get

$$(16) \qquad \vartheta(S, T, \mathfrak{a}, c, t) = \frac{1}{(Nt)^{m/2}} \sum_{\alpha \in \tilde{\mathfrak{a}}} e^{-\pi\sigma(\tilde{c}\tilde{H}\{\alpha\}t^{-1})}$$

where $\tilde{H} = T^{-1} + aS^{-1}$.

Let $\Gamma(S)$ denote the unit group of $S$, that is the group of unimodular matrices $U$ with $U'SU = S$. It is known ([4]) that $\Gamma(S)$ has in $\mathfrak{H}$ a faithful and discontinuous representation if only we identify $U$ and $-U$. Let $F$ denote a fundamental region for $\Gamma(S)$ in $\mathfrak{H}$. We shall prove.

LEMMA 1. *Under the conditions imposed on $S$, for fixed $t$ the integral*

$$\int_F \vartheta(S, T, \mathfrak{a}, c, t)\, dv$$

*converges; the convergence is even uniform on compact sets of $F$.*



Proof. Because of the invariance properties of $dv$ and the properties of $F$ given in § 3, it is enough to prove the lemma in case $F$ is replaced by $J = \mathfrak{H} \cap \mathfrak{R}_{c_0}$ for some $c_0 > c_1$. Since $S$ and $T$ are related by $TS^{-1}\bar{T} = \bar{S}$, we can write

$$S = C'DC\,, \qquad T = C'\bar{C}$$

for a diagonal matrix $D$ with $\pm 1$ in the diagonal. It therefore follows that

$$(17) \qquad |S[\alpha]| = |\alpha' C'D\bar{C}\bar{\alpha}| \leqslant T[\alpha]$$

so that

$$(T-aS)[\alpha] \geqslant T[\alpha] - |a|\,|S[\alpha]| \geqslant (1-|a|)\,T[\alpha]\,.$$

It is therefore enough to consider the integral

$$\int_J \sum_{\alpha \in \mathfrak{a}} e^{-\pi c_2 \sigma(T\{\alpha\}t)} dv$$

for $c_2 > 0$ depending on $a$, $c_1$, $c_0$. Using (6) we see that we are reduced to proving the lemma for the integral

$$(18) \qquad \int_J N \prod_{k=1}^{m} (1 + h_k^{-1/2})\, dv$$

where $T = (h_{kl})$.

We now follow the method in [4]. Using inequalities (38), (39) in [4] it is enough to consider the above integral for each decomposition (8) of $S$. In this case we see that $h_1^{(a)}, \dots, h_{g_a}^{(a)}$ are bounded, $h_{g_a+1}^{(a)}, \dots, h_{m-g_a}^{(a)}$ are bounded both below and above, and $h_{m-g_a+1}^{(a)}, \dots, h_m^{(a)}$ are bounded from below by constants depending only on $m$, $K$ and $S$. Using (4) we see that it is enough to prove the convergence of,

$$(19) \qquad \int_J \prod_{a=1}^{r_1+r_2} \prod_{k=1}^{g_a} (h_k^{(a)})^{\lambda_{ak}} \frac{dh_1^{(1)} \dots dh_{g_{r_1+r_2}}^{(r_1+r_2)}}{h_1^{(1)} \dots h_{g_{r_1+r_2}}^{(r_1+r_2)}}$$

where

$$(20) \qquad \lambda_{ak} = \begin{cases} m - 2k - 1/2\,, & 1 \leqslant a \leqslant r_1\,, \\ m - 2k - 1\,, & a > r_1\,. \end{cases}$$

If we introduce new variables $s_k^{(l)}$ with

$$\left.\begin{aligned} h_1^{(a)} &= s_1^{(a)} \dots s_{g_a}^{(a)}\,, \\ &\dots\dots\dots\dots \\ h_{g_a}^{(a)} &= s_{g_a}^{(a)} \end{aligned}\right\}$$

we have then to prove convergence of integrals of type

$$(21) \qquad \int s_1^{l_1} \dots s_{g_a}^{l_{g_a}} \frac{ds_1 \dots ds_{g_a}}{s_1 \dots s_{g_a}}$$

where $0 < s_k \leqslant c_6$ for a constant $c_6$ and

$$l_k = \begin{cases} \frac{k}{2}(m-k-2)\,, & a \leqslant r_1\,, \\ k(m-k-2)\,, & a > r_1\,. \end{cases}$$

In both cases $l_k \geqslant \frac{1}{2}(m-g_a-2)$. If the $g_k$'s are zero there is nothing to prove since it follows that $J$ itself is compact. If $g_a \neq 0$ then $l_k > 0$ under the conditions on $S$ and this ensures convergence of the integral in (21).

The uniform convergence on compact sets of $F$ follows from the properties of the Humbert domain (4).

We deduce as a consequence of Lemma 1

COROLLARY.

$$\int_F \vartheta(S, T, \mathfrak{a}, c, t)\,dv = \sum_{a \in \mathfrak{a}} \int_F e^{-\pi\sigma(cH\{a\}t)}dv\,. \tag{22}$$

We now prove

LEMMA 2. *If $S$ is not the matrix of a binary zero form $\int_F dv$ is finite.*

Proof. We proceed in exactly the same way as before and obtain the integrals (21) where we have

$$l_k = \begin{cases} \frac{k}{2}(m-k-1)\,, & a \leqslant r_1\,, \\ k(m-k-1)\,, & a > r_1\,. \end{cases}$$

Hence $l_k \geqslant \frac{1}{2}(m-g_a+1)$. Under the conditions on $S$, $l_k > 0$ which gives convergence of (21).

In case $S$ is a binary zero form, the integral is actually divergent ([4]).

It is to be remarked that lemma 2 is actually *not necessary* for our work. In fact it will follow as a consequence of the integral representation for the zeta function but only under the conditions imposed on $S$ in Lemma 1.

Let now

$$\vartheta_0(S, T, \mathfrak{a}, c, t) = \sum_{S[a]\neq 0} e^{-\pi\sigma(cH\{a\}t)}$$

and let $G$ have the meaning of § 3. We have:

LEMMA 3. *If $s$ is a complex variable with $\mathrm{Re}\,s > \frac{1}{2}m$ the integral*

$$\int_G (Nt)^s \int_F \vartheta_0(S, T, \mathfrak{a}, c, t)\,dv[dt]$$

*represents a regular analytic function of $s$.*



Proof. As in the proof of the previous lemma it is enough to consider the inner integral extended over $J$. Let us split $G$ up into $G_1$ and $G_2$ where $G_1$ is that part of $G$ with $Nt \geqslant 1$ and $G_2$ that part with $Nt \leqslant 1$ Then

$$\int_G\int_J = \int_{G_1}\int_J + \int_{G_2}\int_J\,.$$

Consider now $\int_{G_2}$. Let $b = \mathrm{Re}\,s$.

$$\int_{G_2}(Nt)^b\int_J \vartheta_0 dv[dt] \leqslant \int_{G_2}(Nt)^b\int_J \sum_{S[a]\neq 0} e^{-\pi c_7\sigma(T\{a\}t)}dv[dt]$$

and so it is majorised by

$$c_8\int_{G_2}(Nt)^b(1+Nt^{-m/2})\int_J\prod_{k=1}^{m}(1+Nh_k^{-1/2})\,dv[dt]\,.$$

The inner integral is bounded and independent of $t$. The outer integral is easily seen to be convergent for $b-m/2 > 0$ by going over to the norm surface.

We shall now show that the second integral defines actually an entire function. Since in $G_1$, $Nt > 1$, there is at least one $t^{(k)}$, say $t^{(1)}$, which is $> 1$. Now using the arithmetic and geometric inequality we have

$$\sigma(T\{a\}t)-\tfrac{1}{2}T^{(1)}[a^{(1)}] \geqslant c_9\big\{N(T\{a\})(t^{(1)}-\tfrac{1}{2})t^{(2)}\ldots t^{(n)}\big\}^{1/n}\,.$$

From (17) therefore we get, since $|S^{(1)}[a^{(1)}]| > 0$,

$$\sigma(T\{a\}t)-\tfrac{1}{2}T^{(1)}[a^{(1)}] \geqslant c_{10}(Nt\,NT\{a\})^{1/n}\,. \tag{23}$$

Let now

$$a^{(1)} = \begin{pmatrix}\gamma_1\\ \vdots \\ \gamma_m\end{pmatrix} \quad \text{where} \quad \gamma_k = \omega_1^{(1)}x_1^{(k)}+\ldots+\omega_m^{(1)}x_n^{(k)}\,;$$

$\omega_1, \ldots, \omega_m$ being a basis of the ideal $\mathfrak{a}$ and $x_1^{(k)}, \ldots, x_n^{(k)}$ rational integers. Because of the properties of the Humbert domain

$$T^{(1)}[a^{(1)}] \geqslant c_{10}\sum_{k=1}^{m}\sum_{p,q=1}^{n} h_k^{(1)}\omega_p^{(1)}\omega_q^{(1)}x_p^{(k)}x_q^{(k)}\,.$$

If $c_{11}$ is a constant depending on $\omega_1, \ldots, \omega_n$ and $c_{10}$ then, from above,

$$T^{(1)}[a^{(1)}] \geqslant c_{11}\sum_{k=1}^{m}\sum_{p=1}^{n} h_k^{(1)}x_p^{(k)2} \tag{24}$$

using the fact

$$\sigma(T\{a\}t) = \tfrac{1}{2}T^{(1)}[a^{(1)}]+\big(\sigma(T\{a\}t)-\tfrac{1}{2}T^{(1)}[a^{(1)}]\big)$$

and (23), (24) we get

$$\sum_{S[\alpha]\neq 0} e^{-\pi c_2 \sigma(T\{\alpha\}t)} \leqslant e^{-c_{10}(Nt)^{1/n}} \sum_{x_p^{(k)},\ \text{integral}} e^{-c_{11}\pi \sum_{k/p} h_k^{(1)} x_p^{(k)^2} \sigma(T\{\alpha\})}.$$

Using the fact that, by (4), $h_k^{(\alpha)}$ are all of the same order of magnitude for a given $k$ and using well-known inequalities in theta function theory we get

$$\sum_{S[\alpha]\neq 0} e^{-\pi c_2 \sigma(T\{\alpha\}t)} \leqslant e^{-c_{13}(Nt)^{1/n}} \prod_{k=1}^{m} (1+Nh_k^{-1/2}).$$

Going over to the norm surface and using Lemma 1 we see that the integral over $G_1$ converges for all $b$.

The lemma is thus completely proved.

**§ 5. Measure of representation.** Let, as before, $S$ be non-singular in $K$ with system of signatures $(p_1, q_1), \dots, (p_l, q_l)$ and definite at $r_1 - l$ real infinite prime spots. Let $W_1, \dots, W_n$ be $n$ real symmetric matrices so that the $n$ matrices $W^{(1)}, \dots, W^{(n)}$ defined by

$$W^{(k)} = \omega_1^{(k)} W_1 + \dots + \omega_n^{(k)} W_n, \qquad k = 1, \dots, n$$

are non-singular, $W^{(1)}, \dots, W^{(r_1)}$ have the same signature as $S^{(1)}, \dots, S^{(r_1)}$. Here $\omega_1, \dots, \omega_n$ is a basis of integers of $K$. Consider the space $X$ of all real $m$-rowed matrices $X_1, \dots, X_n$ such that

$$X^{(k)} = \omega_1^{(k)} X_1 + \dots + \omega_n^{(k)} X_n, \qquad k = 1, \dots, n$$

satisfy the equations

$$X^{(k)'} S^{(k)} X^{(k)} = W^{(k)}. \tag{25}$$

If $U$ is a unit of $S$ then $U^{(k)}X^{(k)} = \omega_1^{(k)} Y_1 + \dots + \omega_n^{(k)} Y_n$ and $(X_1, \dots, X_n) \to (Y_1, \dots, Y_n)$ gives a transformation of the $X$ space into itself. Let $F_1$ be a fundamental region for this group in the $X$ space. We denote, after Siegel,

$$\mu(S) = N\|S^{(k)}\|^{-1/2} N\|W^{(k)}\|^{1/2} \int_{F_1} \frac{\{dX_1\} \dots \{dX_n\}}{\{dW_1\} \dots \{dW_n\}}$$

and call it the *measure of the unit group* $\Gamma(S)$. Let $F_0$ be the fundamental region in the space $X^{(1)}, \dots, X^{(n)}$ for the group $X^{(k)} \to U^{(k)}X^{(k)}$. We then have

$$\mu(S) = \left(\frac{|d|}{4^{r_2}}\right)^{-m(m-1)/4} N\|S\|^{-1/2} N\|W\|^{1/2} \int_{F_0} \frac{\{dX\}}{\{dW\}}.$$

Using now the fundamental region $F$ in the $\mathfrak{H}$ space of $S$ we get ([7])

$$2\mu(S) = \left(\frac{|d|}{4^{r_2}}\right)^{-m(m-1)/4} 2^{-mr_2} \varrho_m^{r_1+r_2-l} \prod_{k=1}^{l} \varrho_{p_k}\varrho_{q_k} N\|S\|^{-(m+1)/2} \int_F dv. \tag{26}$$



To obtain this we have only to write the product integral and apply the method in [7] for each component.

In case $S$ is totally definite then $r_2 = 0$ and $r_1 = n$. The $\mathfrak{H}$-space is just one point. We then get

$$\mu(S) = |d|^{-m(m-1)/4} \varrho_m^{r_1} N\|S\|^{-(m+1)/2} \frac{1}{E(S)}$$

$E(S)$ being the order of the unit group of $S$. Here $\varrho_k = \prod_{l=1}^{k} \frac{\pi^{l/2}}{\Gamma(l/2)}$.

In a similar way we define the measure of representation. Let $\mathfrak{a}$ be a non-zero ideal and $S[\alpha] = g$ a representation of $g$ by $S$, $\alpha \in \mathfrak{a}$. If $X = (\alpha Y)$ satisfies

$$X'SX = \begin{pmatrix} g & q' \\ q & R \end{pmatrix} = W,$$

we define, as before

$$\mu(S, \mathfrak{a}, \alpha) = N\|S\|^{-1/2} N\|W\|^{1/2} \left(\frac{|d|}{4^{r_2}}\right)^{-(m-1)(m-2)/4} \int_{F_2} \frac{\{dY\}}{\{dq\}\{dR\}},$$

$F_2$ being a fundamental region in the $Y$ space for the group $\Gamma(S, \alpha)$ of units $U$ with $U\alpha = \alpha$. Let us put

$$M(S, \mathfrak{a}, g) = N\|S\|^{m/2} N|g|^{m/2-1} \sum_{\alpha} \mu(S, \mathfrak{a}, \alpha)$$

in case $g \neq 0$ and

$$M(S, \mathfrak{a}, 0) = N\|S\|^{m/2} \sum_{\alpha \neq 0} \mu(S, \mathfrak{a}, 0)$$

when $g = 0$; the summation in both cases running through all non-associated solutions $\alpha$ of the equation $\alpha' S \alpha = g$, $\alpha \in \mathfrak{a}$. We call $M(S, \mathfrak{a}, g)$ the *measure of representation* of $g$ by $S$. We obtain then, by [7],

$$M(S, \mathfrak{a}, g) \prod_{k=1}^{l} \Big( \int_{u>0,\ \operatorname{sgn} g^{(k)}} u^{p_k/2-1} (u - \operatorname{sgn} g^{(k)})^{k/2-1} e^{-\pi t^{(k)}|g^{(k)}|(2u - \operatorname{sgn} g^{(k)})} du \Big) \times$$
$$\times \prod_{k=r_1+1}^{r_1+r_2} \Big( \int_1^\infty (u^2-1)^{m-3/2} e^{-2\pi t^{(k)}|g^{(k)}|u} du \Big) \cdot e^{-\pi \sum_{k=l+1}^{r_1} t^{(k)}|g^{(k)}|}$$
$$= J_l \int_F \sum_{S[\alpha]=g\neq 0} e^{-\pi\sigma(T\{\alpha\}t)} dv$$

when $g \neq 0$ and

$$(2\pi)^{-n(m/2-1)} \left(\Gamma\left(\frac{m}{2}-1\right)\right)^{r_1} \left(\Gamma(m-2)\right)^{r_2} (Nt)^{-(m/2-1)} M(S, \mathfrak{a}, 0)$$
$$= J_{r_1} \int_F \sum_{\substack{S[\alpha]=0 \\ \alpha\neq 0}} e^{-\pi\sigma(T\{\alpha\}t)} dv$$

when $g=0$. Here

$$J_l = \varrho_{m-1}^{r_1+r_2-l}\left(\frac{\Gamma(m-1)}{2^{m-1}\pi^{m/2-1}\Gamma(m/2)}\right)^{r_2}\prod_{k=1}^{l}\varrho_{p_k-1}\varrho_{q_k-1}\left(\frac{|d|}{4^{r_2}}\right)^{-(m-1)(m-2)/4}.$$

In case $S$ is totally definite $M(S,\mathfrak{a},0)=0$ and

$$M(S,\mathfrak{a},g) = \varrho_{m-1}^{r_1}|d|^{-(m-1)(m-2)/4}\frac{A(S,g)}{E(S)};$$

$A(S,g)$ being the number of representations $\mathfrak{a}'S\mathfrak{a}=g$. (This number is finite.) In the proofs of the formulae above we have suppressed some computations. These are easy to carry out by [7].

It is to be noted that $M(S,\mathfrak{a},0)\neq 0$ only in case $S[\mathfrak{a}]$ is a zero form. It is then necessary that $l=r_1$.

**§ 6. Hypergeometric functions.** Consider the hypergeometric function

$$f(a,b,c,x) = \int_0^1 y^a(1-y)^b(1+xy)^c\,dy$$

where $x>0$, $a$ and $b$ are complex numbers having real parts $>-1$. $f(a,b,c,x)$ is a solution of the hypergeometric equation

$$\left[x(x+1)\frac{d^2}{dx^2}+\{(a-c+2)x+(a+b+2)\}\frac{d}{dx}-(a+1)(c+1)\right]W=0.$$

It has two linearly independent solutions

$$f_0(a,b,c,x)=f(a,b,c,x),$$

$$f_{-0}(a,b,c,x)=\frac{1}{x^{a+1}}f(a,c,b,x^{-1}).$$

In the sequel $a=s-\dfrac{m}{2}$, $b=\dfrac{q}{2}-1$, $c=\dfrac{p}{2}-1$. We shall write

$$\begin{aligned} f_1(s,x)&=f_1(s,p,q,x)=f\left(s-\frac{m}{2},\frac{q}{2}-1,\frac{p}{2}-1,x\right),\\ f_{-1}(s,x)&=f_{-1}(s,p,q,x)=\frac{1}{x^{s-m/2+1}}f\left(s-\frac{m}{2},\frac{p}{2}-1,\frac{q}{2}-1,x^{-1}\right). \end{aligned}\tag{27}$$

They satisfy the functional equations (see [6])

$$\begin{aligned} x^{p/2-1}f_1\left(\frac{m}{2}-s,x^{-1}\right)&=a_1(s,q)f_1(s,x)+a_{-1}(s,p)f_{-1}(s,x),\\ x^{p/2-1}f_{-1}\left(\frac{m}{2}-s,x^{-1}\right)&=a_{-1}(s,q)f_1(s,x)+a_1(s,p)f_{-1}(s,x) \end{aligned}\tag{28}$$



where

$$a_1(s,a)=\frac{\sin\pi\left(s-\dfrac{a}{2}\right)}{\sin\pi s},\qquad a_{-1}(s,a)=\frac{\sin\pi a/2}{\sin\pi s}.\tag{29}$$

Here we have used the properties of $f(a,b,c,x)$ as a function of the complex variable $a$ and as a function of the positive variable $x$.

Let $G$ be an abelian group of order $2^l$, $l\geqslant 0$, every element of which except the identity is of order 2. We denote the elements of $G=G_l$ by $\varepsilon$. We may take $\varepsilon=(\varepsilon_1,\dots,\varepsilon_l)$, $\varepsilon_k=\pm 1$ and define multiplication componentwise. Denote by $\varepsilon_0=(1,1,\dots,1)$ the unit element of $G$. We will write the elements of $G$ in a particular order. This is done by induction. Consider the subgroup of $G_l$ with $\varepsilon_l=1$. This is isomorphic to $G_{l-1}$. By induction hypothesis the elements of this subgroup, call it $G'_{l-1}$, are written in a fixed order. Write then the elements of $G_l$ by taking first $(G'_{l-1},1)$ and then $(G'_{l-1},-1)$. In any sum $\sum$ extended over all elements of $G$ we follow this ordering.

Consider now $l$ sets of $2l$ positive integers $(p_1,q_1),\dots,(p_l,q_l)$ with $0<p_k<m$, $p_k+q_k=m$ for every $k$. For each $\varepsilon$ of $G_l$ we take a vector $a_\varepsilon=(a_{\varepsilon_1},\dots,a_{\varepsilon_l})$ such that

$$a_{\varepsilon_k}=\begin{cases} q_k & \text{if}\quad \varepsilon_k=1,\\ p_k & \text{if}\quad \varepsilon_k=-1,\end{cases}\tag{30}$$

$\varepsilon=(\varepsilon_1,\dots,\varepsilon_l)$. In this way we obtain $2^l$ vectors $a_\varepsilon$. If $\varepsilon,\eta$ are two elements of $G_l$ define

$$a_\eta(s,a_\varepsilon)=\prod_{k=1}^{l}a_{\eta_k}(s,a_{\varepsilon_k})\tag{31}$$

with $a_{\eta_k}(s,a_{\varepsilon_k})$ given by (29). We denote by $\alpha_0=\alpha_l$ the matrix

$$\alpha_0=\big(a_\eta(s,a_\varepsilon)\big),\tag{32}$$

$\eta$ being the row index and $\varepsilon$ the column index. We prove.

LEMMA 4. *If $q_k$ is even for some $k$, then there is a submatrix of $\alpha_0$ of $2^{l-1}$ rows and columns with all elements equal to zero. If $q_1,\dots,q_l$ are all even then $\alpha_0$ has the form*

$$\alpha_0=\begin{pmatrix} y_* & * \\ 0 & * \end{pmatrix}$$

*with zero below the diagonal and $y=(-1)^{q_1+\dots+q_l/2}$.*

Proof. Let us, without loss of generality, assume that $q_1$ is even. From (29) $a_{-1}(s,q_1)=0$ and therefore $a_\eta(s,a_\varepsilon)=0$ if $\varepsilon=(1,\varepsilon_2,\dots,\varepsilon_l)$ and $\eta=(-1,\dots)$. This gives a zero matrix of order $2^{l-1}$. In order to prove the second part notice that

$$\alpha_l=\begin{pmatrix}(\alpha_{l-1})\,a_1(s,q_l) & (\alpha_{l-1})\,a_{-1}(s,p_l)\\ (\alpha_{l-1})\,a_{-1}(s,q_l) & (\alpha_{l-1})\,a_1(s,p_l)\end{pmatrix}.$$

By induction, since $q_1, \ldots, q_{l-1}$ are even, $a_{l-1}$ has the property indicated in the lemma. But $q_l$ being even, $a_{-1}(s, q_l) = 0$. Now from (31)

$$a_{\varepsilon_0}(s, a_{\varepsilon_0}) = \prod_{k=1}^{l} a_1(s, q_k) = (-1)^{(q_1+\ldots+q_l)/2}.$$

This proves the lemma completely.

Let now $x_1, \ldots, x_l$ be $l$ positive real parameters. Consider the $2^l$ functions

$$(33) \qquad f_\varepsilon(s, p, q, x) = \prod_{k=1}^{l} f_{\varepsilon_k}(s, p_k, q_k, x_k).$$

These $2^l$ functions are linearly independent and satisfy the functional equations

$$(34) \qquad f_\varepsilon\left(\frac{m}{2}-s, p, q, x^{-1}\right) \prod_{k=1}^{l} x_k^{p_k/2-1} = \sum_{\eta} a_{\varepsilon\eta}(s, a_\eta) f_\eta(s, p, q, x)$$

where $\eta$ runs through all elements of $G_l$. We introduce $2^l$ differential operators

$$(35) \qquad \Delta_\varepsilon = \left(\frac{\partial}{\partial x_1}\right)^{e_1} \cdots \left(\frac{\partial}{\partial x_l}\right)^{e_l}$$

where $e_k = \frac{1}{2}(1-\varepsilon_k)$. The ordering of the $\varepsilon$'s determines the ordering of the $\Delta_\varepsilon$. Let us put $\omega_l = \omega_2(s; x_1, \ldots, x_l)$ the matrix

$$(36) \qquad \omega_l = \left(\Delta_\varepsilon f_\eta(s, p, q, x)\right)$$

where $\varepsilon$ denotes the row and $\eta$ the column. Let $W_l$ denote the determinant of $\omega_l$. We prove,

LEMMA 5.

$$|W_l| = (g_1 \cdot \ldots \cdot g_l)^{2^{l-1}}$$

*where*

$$g_k = f_1(s, p_k, q_k, x_k) \frac{\partial}{\partial x_k} f_{-1}(s, p_k, q_k, x_k) - f_{-1}(s, p_k, q_k, x_k) \frac{\partial}{\partial x_k} f_1(s, x).$$

Proof. As in the second part of Lemma 4 we have, because of the special ordering of $G$,

$$\omega_l(s; x_1, \ldots, x_l) = \begin{pmatrix} (\omega_{l-1}) f_1(s, x_l) & (\omega_{l-1}) f_{-1}(s, x_l) \\ (\omega_{l-1}) \frac{\partial}{\partial x_l} f_1(s, x_l) & (\omega_{l-1}) \frac{\partial}{\partial x_l} f_{-1}(s, x_l) \end{pmatrix}.$$

Assume as induction hypothesis the truth of the lemma for $l-1$ instead of $l$. The result for $l$ follows from the above form of $\omega_l$. We use the truth of Lemma 5 for $l = 1$. This is already in Siegel [6].



Using the value of $g_1, \ldots, g_l$ we get

$$(37) \qquad W_l(s; x_1, \ldots, x_l) = \left\{\gamma(s; x_1, \ldots, x_l)\left(\frac{\Gamma(s-m/2+1)}{\Gamma(s)}\right)^l \prod_{k=1} \Gamma\left(\frac{p_k}{2}\right)\Gamma\left(\frac{q_k}{2}\right)\right\}^{2^{l-1}}$$

where

$$\gamma(s; x_1, \ldots, x_l) = \prod_{k=1}^{l} x_k^{p_k/2-s-1}(1+x_k)^{m/2-2}.$$

(37) shows that $W_l$, as a function of $s$, has a pole at $s = m/2-1$ of order $l \cdot 2^{l-1}$. The hypergeometric function is regular at $s = m/2$ and one has

$$\sum f_\varepsilon\left(\frac{m}{2}, p, q, x\right) = \prod_{k=1}^{l} \left(\frac{\Gamma(p_k/2)\Gamma(q_k/2)}{\Gamma(m/2)}\right) \delta(x_1, \ldots, x_l)$$

where

$$\delta(x) = \delta(x_1, \ldots, x_l) = \prod_{k=1}^{l} x_k^{-q_k/2}(1+x_k)^{m/2-1}.$$

Applying the operators $\Delta_\varepsilon$ to both sides, we get

$$(38) \qquad \prod_{k=1}^{l} \left(\frac{\Gamma(p_k/2)\Gamma(q_k/2)}{\Gamma(m/2)}\right) \omega_l^{-1}\left(\frac{m}{2}; x_1, \ldots, x_l\right) \underline{\Delta_\eta} \delta(x) = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

It is to be noted that on the right of (38) all elements are equal to 1.

**§ 7. Zeta functions.** Let $S$ be the matrix of an indefinite quadratic form with elements in $K$ satisfying conditions in § 4. Let $\mathfrak{a} \neq 0$ be an ideal of $K$. If $a$ is a vector with elements in $\mathfrak{a}$ let $a'Sa = g \neq 0$. Since $S^{(k)}$ for $l < k \leqslant r_1$ is definite, $g^{(k)}$ for these values of $k$ will have the same sign irrespective of $a$. On the other hand if $1 \leqslant k \leqslant l$, $g^{(k)}$ can have any sign. This may be proved by simple arguments of continuity. We can therefore associate with $g$ an element $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_l)$ of $G_l$ so that $\varepsilon_k = \operatorname{sgn} g^{(k)} = g^{(k)}/|g^{(k)}|$, $1 \leqslant k \leqslant l$. We call $\varepsilon$ the *signature* of $g$.

Let $H = T - aS$, where $T$, $a$ are defined as before. Let $c$ be given by (14). Using the duplication formula of the gamma function we obtain by Mellin transformation

$$(39) \qquad \frac{M(S, \mathfrak{a}, g)}{(N|g|)^s} N\|S\|^{s/m}(N\mathfrak{a})^{2s}|d|^s(\Gamma(s))^{r_1+r_2}\left(\Gamma\left(s-\frac{m}{2}+1\right)\right)^{r_2}\pi^{-ns} \times$$
$$\times \prod_{k=1}^{l} x_k^{q_k s/m}(1+x_k)^{1-m/2} f_\varepsilon(s, p, q, x) = h_l \int_P (Nt)^s \int_F \sum_{S[a]=g} e^{-\pi\sigma(cH\{a\}t)} dv[dt]$$

where $\varepsilon$ is the signature of $g$, $P$ is the space of $t^{(1)}, \ldots, t^{(r_1+r_2)}$, $x_k = \frac{1+a_k}{1-a_k} > 0$, $k = 1, \ldots, l$ and

$$h_l = \left(\frac{|d|}{4^{r_2}}\right)^{-(m-1)(m-2)/4} \left(\frac{2}{\pi^{m/2-1}}\right)^{r_2} \varrho_{m-1}^{r_1+r_2-l} \prod_{k=1}^{l} \varrho_{p_k-1}\varrho_{q_k-1} .$$

Let us now define the $2^l$ zeta functions

$$\zeta_\varepsilon(S, \mathfrak{a}, s) = N\mathfrak{a}^{2s} \sum_{g} \frac{M(S, \mathfrak{a}, g)}{(N|g|)^s} \tag{40}$$

where the summation is through a complete system of $g$ of signature $\varepsilon$ which are non-associated in the sense $g_1 \neq g_2$ with $g_2 = g_1 \varepsilon^2 \bar{\varepsilon}$, $\varepsilon$ a unit in $K$. Because of Lemma 3 these series converge absolutely for $\mathrm{Re}\, s > m/2$ and define in this half plane analytic functions.

Put now

$$\varphi_\varepsilon(S, \mathfrak{a}, s) = |d|^s \big(\Gamma(s)\big)^{r_1+r_2} \left(\Gamma\left(s - \frac{m}{2} + 1\right)\right)^{r_2} \pi^{-ns} \zeta_\varepsilon(S, \mathfrak{a}, s) . \tag{41}$$

We then have

$$N\|S\|^{s/m} \prod_{k=1}^{l} x_k^{q_k s/m} (1+x_k)^{1-m/2} \sum_{\varepsilon} f_\varepsilon(s, p, q, x) \varphi_\varepsilon(S, \mathfrak{a}, s)$$
$$= \frac{1}{w} h_l \int_G (Nt)^s \int_F \sum_{S[\mathfrak{a}]\neq 0} e^{-\pi\sigma(cH\{\mathfrak{a}\}t)} dv\, [dt] .$$

From now on we follow the celebrated method of Riemann-Hecke-Siegel and obtain

$$\begin{aligned} & w h_l^{-1} N\|S\|^{s/m} \prod_{k=1}^{l} x_k^{q_k s/m} (1+x_k)^{1-m/2} \sum_{\varepsilon} f_\varepsilon(s, p, q, x) \varphi_\varepsilon(S, \mathfrak{a}, s) \\ & = \int_{G_1} (Nt)^s \Big( \int_{F(S)} \sum_{S[\mathfrak{a}]\neq 0} e^{-\pi\sigma(cH\{\mathfrak{a}\}t)} dv + (Nt)^{m/2-s} \int_{F(S^{-1})} \sum_{S^{-1}[\mathfrak{a}]\neq 0} e^{-\pi(\tilde{c}\tilde{H}\{\mathfrak{a}\}t)} dv \Big) [dt] \\ & \quad - \int_{G_2} (Nt)^s \int_{F(S)} \Big(1 + u \sum_{\substack{\mathfrak{a}\neq 0 \\ S[\mathfrak{a}]=0}} e^{-\pi\sigma(cT\{\mathfrak{a}\}t)} dv\Big) [dt] \\ & \quad + \int_{G_2} (Nt)^{s-m/2} \int_{F(S^{-1})} \Big(1 + u \sum_{\substack{\mathfrak{a}\neq 0 \\ S^{-1}[\mathfrak{a}]=0}} e^{-\pi\sigma(\tilde{c}T^{-1}\{\mathfrak{a}\}t^{-1})} dv\Big) [dt] . \end{aligned} \tag{42}$$

Where $F(S)$ is the fundamental region for the units of $S$ and $F(S^{-1})$ for that of $S^{-1}$, $u = 0$ or $1$ according as $S[\mathfrak{a}]$ is not or is a zero form. $w$ is here the number of roots of unity in $K$.



The last four integrals may be evaluated thus. Let us put

$$\gamma_0 = \int_{G_0} [dt_0] , \quad \gamma_1 = \int_{F(S^{-1})} dv , \quad \gamma_2 = \int_{F(S)} dv \tag{43}$$

and

$$\psi(S, \mathfrak{a}, x) = \big(N\mathfrak{a}\sqrt{|d|}\big)^{m-2} N\|S\|^{\frac{m-2}{2m}} \prod_{k=1}^{r_1} x^{\frac{q_k}{m}\left(\frac{m}{2}-1\right)} (1+x_k)^{-\frac{m}{2}+1} .$$

Then we have, for the last four integrals,

$$\begin{aligned} & \gamma_0\left(\frac{\gamma_1}{s-m/2} - \frac{\gamma_2}{s}\right) + u(2\pi)^{-n(m/2-1)} \left(\Gamma\left(\frac{m}{2}-1\right)\right)^{r_1} \big(\Gamma(m-2)\big)^{r_2} 2^{r_1(m/2-1)} \times \\ & \times \left(\frac{\psi(S^{-1}, \tilde{\mathfrak{a}}, x^{-1}) M(S^{-1}, \tilde{\mathfrak{a}}, 0)}{s-1} - \frac{\psi(S, \mathfrak{a}, x) M(S, \mathfrak{a}, 0)}{s-m/2+1}\right). \end{aligned} \tag{44}$$

The proof of Lemma 3 shows that the first two integrals on the right of (42) are entire functions. (44) therefore shows that $\sum_{\varepsilon} f_\varepsilon(s, p, q, x) \varphi_\varepsilon(S, \mathfrak{a}, s)$ is a meromorphic function with simple poles at $s = 0, 1, m/2-1$ and $m/2$. The points $s = 1, s = m/2-1$ occur only when $S[\mathfrak{a}]$ is a zero form.

Applying the operator $\Delta_\varepsilon$ to both sides of (42) does not change the meromorphic character of either side; what is more the right side has at most simple poles at these four points. Put

$$\sum_{\varepsilon} \Delta_\eta\big(f_\varepsilon(s, p, q, x)\big) \varphi_\varepsilon(S, \mathfrak{a}, s) = A_\eta(s) , \qquad \eta \in G_l ;$$

then $A_\eta(s)$ is a meromorphic function of $s$ with at most simple poles at $s = 0, 1, m/2-1, m/2$. Therefore we get, using the matrix $\omega_l$

$$\underline{\varphi_\varepsilon}(S, \mathfrak{a}, s) = \omega_l^{-1} \underline{A_\eta}(s) . \tag{45}$$

Now $\Delta_\eta f_\varepsilon(s, p, q, x)/\big(\Gamma(s-m/2+1)\big)^l$ is for every $\varepsilon, \eta$ an entire function of $s$. Because of the form of the determinant $W_l$ of $\omega_l$ given in (37), it follows that $\varphi_\varepsilon(S, \mathfrak{a}, s)$ can be continued analytically into the whole plane and have at most simple poles at $s = 1, m/2-1, m/2$ and possibly poles at the poles of $\Gamma(s)$. At all other places they are regular.

Equations (42) and (44) show that the right side of (42) is invariant under the transformations $s \to m/2 - s$, $a_k \to -a_k$ ($x_k \to x_k^{-1}$), $\mathfrak{a} \to \tilde{\mathfrak{a}}$ and $S \to S^{-1}$. This means that

$$N\|S\|^{s/m} \prod_{k=1}^{l} x_k^{q_k s/m} (1+x_k)^{1-m/2} \sum_{\varepsilon} f_\varepsilon(s, p, q, x) \varphi_\varepsilon(S, \mathfrak{a}, s)$$

is invariant under these transformations. Using (34) and the linear independence of $f_\varepsilon(s, p, q, x)$ we get functional equations

$$\underline{\varphi}'_\varepsilon(S, \mathfrak{a}, s) = N\|S\|^{-1/2} \underline{\varphi}'_\varepsilon\left(S^{-1}, \tilde{\mathfrak{a}}, \frac{m}{2} - s\right) \big(a_{\varepsilon\eta}(s, a_\eta)\big) , \tag{46}$$

$\underline{\varphi_\varepsilon}$ denoting the column with $\varphi_\varepsilon$ as element, and denotes transposed matrix.

In particular, if $l=0$ we have only one function and it satisfies

$$\varphi(S,\mathfrak{a},s) = N\|S\|^{-1/2}\varphi\Big(S^{-1},\tilde{\mathfrak{a}},\frac{m}{2}-s\Big);$$

we have thus proved the

THEOREM 1. *The $2^l$ functions $\varphi_\varepsilon(S,\mathfrak{a},s)$ can be continued analytically into functions meromorphic in the whole plane with at most simple poles at $s=1, m/2-1, m/2$ and possibly poles at the poles of $\Gamma(S)$. These functions satisfy functional equations given by* (46).

Suppose that $\eta$ is an element of $G_l$ and *all* $a_{\eta i}$ in $a_\eta$ are *even* integers. By Lemma 4 therefore $\alpha_{\eta\varepsilon}(s,a_\eta)=0$ unless $\eta=\varepsilon$ in which case $\alpha_{\eta\eta}(s,a_\eta) = (-1)^{a_{\eta 1}+\ldots+a_{\eta l}/2}$. We have thus

COROLLARY. *Let $S$ be such that for some $\eta\in G_l$ all $a_{\eta i}$ in $a_\eta$ are even integers. Then $\varphi_\eta(S,\mathfrak{a},s)$ satisfies the functional equation*

$$\varphi_\eta(S,\mathfrak{a},s) = (-1)^{a_{\eta 1}+\ldots+a_{\eta l}/2}N\|S\|^{-1/2}\varphi_\eta\Big(S^{-1},\tilde{\mathfrak{a}},\frac{m}{2}-s\Big). \tag{47}$$

(Note that the hypothesis of this Corollary is empty if $l=0$.)

From the definition of $\varphi_\varepsilon(S,\mathfrak{a},s)$ and the remarks above it follows that $\zeta_\varepsilon(S,\mathfrak{a},s)$ can be continued analytically into the whole plane into a function which is meromorphic with at most simple poles at the points $s=1$, $m/2-1$ and $m/2$ and possibly poles at the poles of $\Gamma(s)$.

We shall study the analytic behaviour of $\zeta_\varepsilon(S,\mathfrak{a},s)$ more closely.

Consider first the point $s=m/2$ at which $\zeta_\varepsilon(S,\mathfrak{a},s)$ has if at all a simple pole. Let $\delta_\varepsilon$ be the residue at this pole for $\varphi_\varepsilon(S,\mathfrak{a},s)$. Put

$$\varrho = N\|S\|^{-1/2}\gamma_0\gamma_1\frac{h_l}{w}.$$

Then from (42) we have

$$\sum_\varepsilon f_\varepsilon\Big(\frac{m}{2},p,q,x\Big)\delta_\varepsilon = \varrho\gamma(x)$$

with $\gamma(x)$ defined in (37). Applying the operator $\Delta_\eta$ to both sides we have

$$\sum_\varepsilon \Delta_\eta\Big(f_\varepsilon\Big(\frac{m}{2},p,q,x\Big)\Big)\delta_\varepsilon = \varrho\Delta_\eta\gamma(x), \qquad \eta\in G_l.$$

This means that

$$\delta_\varepsilon = \varrho\omega_l^{-1}\Delta_\eta\gamma(x);$$

using (38) we deduce

$$\delta_\varepsilon = \frac{\varrho\big(\Gamma(m/2)\big)^l}{\prod_{k=1}^{l}\Gamma(p_k/2)\Gamma(q_k/2)} \tag{48}$$



which is *independent* of $\varepsilon$ and $\mathfrak{a}$. This shows that $\zeta_\varepsilon(S,\mathfrak{a},s)$ has a simple pole at $s=m/2$. Its residue is, restoring the values of $\gamma_0$, $\gamma_1$, and $h_l$, equal to

$$\chi = \frac{2^{r_1+r_2}\cdot\pi^{r_2}\cdot R}{w\sqrt{|d|}}2^{1+r_2}N\|S\|^{-m+2/2}\mu(S^{-1})\prod_{k=1}^{l}\Big(\frac{\Gamma(p_k/2)\Gamma(q_k/2)}{\Gamma(m/2)}\Big). \tag{49}$$

Consider the point $s=0$. The functional equation gives

$$\varphi_\eta(S,\mathfrak{a},s) = N\|S\|^{-1/2}\sum_\varepsilon\varphi_\varepsilon\Big(S^{-1},\tilde{\mathfrak{a}},\frac{m}{2}-s\Big)\alpha_{\eta\varepsilon}(s,a_\eta). \tag{50}$$

Suppose now that $\zeta_\eta(S,\mathfrak{a},s)$ has a singularity at $s=0$. From the form (41) of $\varphi_\eta(S,\mathfrak{a},s)$, it then follows that $\varphi_\eta(S,\mathfrak{a},s)$ has at $s=0$ a pole of order $\geqslant 1+r_1+r_2$. On the right side in (50), $\varphi_\varepsilon(S^{-1},\tilde{\mathfrak{a}},m/2-s)$ has a simple pole at $s=0$ and $\alpha_{\eta\varepsilon}(s,a_\eta)$ has a pole of order $\leqslant l$. Therefore if $r_1+r_2>l$ then $s=0$ has to be a point of regularity for $\zeta_\eta(S,\mathfrak{a},s)$. So let $r_1+r_2=l$. Then $r_2=0$ and $r_1=l$. In equation (50) the left side has a pole at $s=0$ of order $\geqslant l+1$ whereas if $a_{\eta i}$ is *even* for some $i$, $\alpha_{\eta\varepsilon}(s,a_\eta)$ has a pole of order $\leqslant l-1$ so that the right side has a pole of order $\leqslant l$ which is a contradiction. So in this case also $\zeta_\eta(S,\mathfrak{a},s)$ is regular at $s=0$. Finally therefore let $r_2=0, r_1=l$ and all $a_{\eta i}$ in $a_\eta$ odd. Then

$$\alpha_{\eta\varepsilon}(s,a_\eta) = (-1)^{(a_{\eta 1}+\ldots+a_{\eta l}-l)/2}(-1)^e\frac{(\cos\pi s)^e}{(\sin\pi s)^l}$$

where $e$ is the number of times $\varepsilon_i\eta_i$ is positive. But $\varphi_\varepsilon(S^{-1},\tilde{\mathfrak{a}},m/2-s)$ has at $s=0$ a simple pole with residue independent of $\varepsilon$. Since $\frac{(\cos\pi s)^e}{(\sin\pi s)^l}$ has at $s=0$ a pole of order $l$ and $\sum_\varepsilon(-1)^e=0$, it means the right side in (50) has a pole of order $\leqslant l$ while the left side has a pole of order $\geqslant l+1$. This contradiction shows that $\zeta_\eta(S,\mathfrak{a},s)$ is regular at $s=0$.

At the other poles of $\Gamma(s)$ a similar argument applies; in fact, it is even simpler since at these points $\varphi_\eta(S^{-1},\tilde{\mathfrak{a}},m/2-s)$ is regular.

The points $s=m/2-1$ and $s=1$ need be considered only if $S[\alpha]$ is a zero form. In this case $l=r_1$.

Let $S[\alpha]$ be a zero form. Consider the point $s=m/2-1$ at which $\varphi_\varepsilon(S,\mathfrak{a},s)$ has at most a simple pole. Suppose $r_1=0$. Since $r_2>0$, (41) shows that $\zeta_\varepsilon(S,\mathfrak{a},s)$ is regular at this point. Let therefore $r_1>0$. We have from (42)

$$\sum_\varepsilon f_\varepsilon(s,p,q,x)\varphi_\varepsilon(S,\mathfrak{a},s)+\frac{\mu_1}{s-m/2+1}-\frac{\mu_2\prod_{k=1}^{l}x_k^{\frac{m-2q_k}{m}\left(\frac{m}{2}-1\right)}}{s-1} \tag{51}$$

is regular at $s = m/2-1$. Here $\mu_1$ is given by

$$\mu_1 = \frac{h_{r_1}}{w}(2\pi)^{-n(m/2-1)} \times \tag{52}$$

$$\times \left(\Gamma\left(\frac{m}{2}-1\right)\right)^{r_1}\left(\Gamma(m-2)\right)^{r_2} 2^{r_1(m/2-1)} N\|S\|^{(m-2)/2m}\left(N\mathfrak{a}\sqrt{|d|}\right)^{m-2} M(S, \mathfrak{a}, 0).$$

From the definition of the hypergeometric function it is seen that $f_{\varepsilon_k}(s, p_k, q_k, s_k)$ has at $s = m/2-1$ a simple pole with residue 1 whereas its derivative, with regard to $x_k$, is regular at this point. If therefore $m \neq 4$ so that $m/2-1 \neq 1$, we see from (51) that $\sum_{\varepsilon} \varphi_\varepsilon(S, \mathfrak{a}, s)$ has at $s = m/2-1$ a zero of order $l-1$. More specifically

$$\operatorname*{Lt}_{s\to m/2-1} \sum_{\varepsilon} \varphi_\varepsilon(S, \mathfrak{a}, s)\left(s-\frac{m}{2}+1\right)^{1-r_1} = -\mu_1. \tag{53}$$

Consider the matrix $\omega_l$. Put $\omega_l^{-1} = (\beta_{\varepsilon\eta})$, $\varepsilon$ denoting row index and $\eta$ the column index. From the expression for $W_l$, it follows that $W_l^{-1}$ has at $s = m/2-1$ a zero of order $l \cdot 2^{l-1}$. By the above remarks we see that all $\beta_{\varepsilon\eta}$ are regular at $s = m/2-1$. Applying the operators $\Delta_\eta$ to both sides of (51) it is seen that

$$\underline{\varphi}_\varepsilon(S, \mathfrak{a}, s) + \omega_l^{-1}\underline{\Delta}_\eta\left(\frac{\mu_1}{s-m/2+1} - \frac{\mu_2 \prod_{k=1} x_k^{\frac{m-2q_k}{m}\left(\frac{m}{2}-1\right)}}{s-1}\right)$$

is a column vector consisting of functions regular at $s = m/2-1$. Let $\varepsilon' = -\varepsilon_0$ denote the last element of $G_l$ in the prescribed ordering of the elements of $G_l$. From the above remarks it follows that if $\eta \neq -\varepsilon_0$ then $\beta_{\varepsilon\eta}$ has a zero at $s = m/2-1$ whereas if $\eta = \varepsilon' = -\varepsilon_0$.

$$\beta_{\varepsilon\varepsilon'} = \gamma_{\varepsilon\varepsilon'} \prod_{k=1}^{l} x_k^{-p_k/2+m/2}(1+x_k)^{-(m/2-2)},$$

$\gamma_{\varepsilon\varepsilon'}$ being a function of $s$ which does not vanish at $s = m/2-1$ and at this point is independent of $x$. Since $\mu_1$ is a constant $\Delta_\eta \mu_1 = 0$ if $\eta \neq \varepsilon_0$. We therefore see that for *every* $\varepsilon \in G_l$

$$\varphi_\varepsilon(S, \mathfrak{a}, s) + \frac{\mu_2\gamma_{\varepsilon\varepsilon'}}{s-1}\prod_{k=1}^{l}\frac{m-2q_k}{m}\left(\frac{m}{2}-1\right)x_k^{\frac{m-2q_k}{m}\left(\frac{m}{2}-1\right)-\frac{p_k}{2}+\frac{m}{2}-1}(1+x_k)^{m/2-2} \tag{54}$$

is regular at $s = m/2-1$. From this it follows that $\varphi_\varepsilon(S, \mathfrak{a}, s)$ is regular at $s = m/2-1$ if $m \neq 4$. Hence $\zeta_\varepsilon(S, \mathfrak{a}, s)$ is regular at this point.

Suppose $m = 4$ so that $m/2-1 = 1$. It is enough to consider the case $r_2 = 0$ since, if $r_2 > 0$, the form of $\varphi_\varepsilon(S, \mathfrak{a}, s)$ given in (41) shows that $\zeta_\varepsilon(S, \mathfrak{a}, s)$ is regular at $s = 1$. If $r_1 > 0$ and some $q_k$ is an even integer



then (54) shows that $\varphi_\varepsilon(S, \mathfrak{a}, s)$ and so $\zeta_\varepsilon(S, \mathfrak{a}, s)$ are regular at $s = 1$. So let $r_2 = 0$ and all $q_k$ be odd. Then

$$\frac{m-2q_k}{m}\left(\frac{m}{2}-1\right) - \frac{p_k}{2} + \frac{m}{2} - 1 = 0$$

and therefore $\zeta_\varepsilon(S, \mathfrak{a}, s)$ has a simple pole at $s = 1$.

We now finally consider the point $s = 1$ at which $\varphi_\varepsilon(S, \mathfrak{a}, s)$ has at most a simple pole. We may take $m \neq 4$ since it has been studied above. Suppose $r_2 > 0$. The presence of the factor $\left(\Gamma(s-m/2+1)\right)^{r_2}$ in $\varphi_\varepsilon(S, \mathfrak{a}, s)$ shows that if $m$ is even, $\zeta_\varepsilon(S, \mathfrak{a}, s)$ is regular at $s = 1$ since then $\Gamma(s-m/2+1)$ has a pole at $s = 1$. (Note that $m \neq 2$ since then $S[\alpha]$ is binary zero form which is excluded.) So let us assume that if $r_2 > 0$, $m$ is odd. We again use the functional equation (50). If $a_{\eta_k}$ is even for some $k$ then $a_{\eta\varepsilon}(s, a_\eta) = 0$ or has a pole of order $\leqslant l-1$. But $\varphi_\varepsilon(S^{-1}, \tilde{\mathfrak{a}}, m/2-s)$ has at $s = 1$, a zero of order $l-1$. Hence in this case $\zeta_\varepsilon(S, \mathfrak{a}, s)$ is regular at $s = 1$. Suppose now that all $a_{\eta_k}$ are odd and $m$ is odd if $r_2 > 0$. Then $a_{\eta\varepsilon}(s, a_\eta)$ behaves at $s = 1$ like

$$(-1)^{(a_{\eta_1}+\ldots+a_{\eta_l}+l)/2} \cdot \frac{1}{(\sin \pi s)^l}.$$

But $\sum_{\varepsilon} \varphi_\varepsilon(S^{-1}, \tilde{\mathfrak{a}}, m/2-s)$ has by (53) a zero of order $l-1$. Therefore $\varphi_\varepsilon(S, \mathfrak{a}, s)$ and so $\zeta_\varepsilon(S, \mathfrak{a}, s)$ has a simple pole at $s = 1$.

We have thus the

THEOREM 2. *The analytic continuation of $\zeta_\varepsilon(S, \mathfrak{a}, s)$ is a meromorphic function which is regular everywhere except at $s = m/2$ where it has a simple pole and possibly a simple pole at $s = 1$. The point $s = 1$ is a point of regularity if $S[\alpha]$ is not a zero form. In the contrary case $\zeta_\varepsilon(S, \mathfrak{a}, s)$ has a simple pole at $s = 1$ if and only if all $a_{\varepsilon_k}$ in $a_\varepsilon$ are odd when $r_2 = 0$ and in addition $m$ is odd if $r_2 > 0$.*

**§ 8. Remarks.** Let $\lambda \neq 0$ be an element of $K$. Consider the ideals $\mathfrak{a}$ and $\mathfrak{b} = \lambda\mathfrak{a}$. If $\alpha \in \mathfrak{a}$ and $\alpha' S \alpha = g \neq 0$ then $(\lambda\alpha)' S(\lambda\alpha) = \lambda^2 g \neq 0$ and $\lambda\alpha \in \mathfrak{b}$. It follows that $M(S, \mathfrak{a}, g) = M(S, \mathfrak{b}, \lambda^2 g)$. Since $N\mathfrak{b} = (N\lambda)^2 \cdot N\mathfrak{a}$ it shows that $\zeta_\varepsilon(S, \mathfrak{a}, s)$ depends on $\mathfrak{a}$ *only through its ideal class in* $K$. If $\mathfrak{R}$ is this ideal class we can put

$$\zeta_\varepsilon(S, \mathfrak{R}, s) = N\mathfrak{a}^{2s} \sum_{g} \frac{M(S, \mathfrak{a}, g)}{(N|g|)^s};$$

$\mathfrak{a}$ being *any* ideal in the class $\mathfrak{R}$. The functional equation now takes the form

$$\underline{\varphi}'_\varepsilon(S, \mathfrak{R}, s) = N\|S\|^{-1/2}\underline{\varphi}'_\varepsilon\left(S^{-1}, \tilde{\mathfrak{R}}, \frac{m}{2}-s\right)\left(a_\eta(s, a_\varepsilon)\right);$$

$\tilde{\mathfrak{R}}$ being the complementary ideal class.

In case $m=1$ and $S=1$ we have $l=0$ and there is only one zeta function for each class $\mathfrak{R}$ and we have

$$\zeta(s,\mathfrak{R}) = N\mathfrak{a}^{2s}\sum_{g}\frac{1}{(N|g|)^{2s}}, \tag{55}$$

$g$ runs through elements in $\mathfrak{a}$ non associated by units. We then obtain the Hecke functional equation of the Dedekind zeta function. $\zeta(s,\mathfrak{R})$ has $s=\frac{1}{2}$ as the only pole.

Let $S$ be the matrix of a binary form which is not a zero form and which is totally indefinite. We then have

$$a_{\varepsilon\eta}(s, a_\eta) = (-1)^e\frac{(\cos\pi s)^e}{(\sin\pi s)^{r_1}} \tag{56}$$

where $e$ is the number of positive signs in $\varepsilon_\eta$. If therefore

$$\varphi_\varepsilon(S,\mathfrak{a},s) = |d|^s\pi^{-ns}(\Gamma(s))^n\zeta_\varepsilon(S,\mathfrak{a},s)$$

then

$$(\sin\pi s)^{r_1}\varphi_\varepsilon(S,\mathfrak{a},s) = N\|S\|^{-1/2}\sum_{\eta}(-1)^e(\cos\pi s)^e\varphi_\eta(S^{-1},\tilde{\mathfrak{a}},1-s).$$

If we take $r_2=0$ and $r_1=2$ so that $K$ is a real quadratic field we can define the zeta functions

$$(57)\quad \begin{aligned}(N\mathfrak{a})^{-2s}Z_1(S,\mathfrak{a},s) &= \sum_{g}\frac{M(S,\mathfrak{a},g)}{(N|g|)^s}, \quad Ng>0,\\ (N\mathfrak{a})^{-2s}Z_2(S,\mathfrak{a},s) &= \sum_{g}\frac{M(S,\mathfrak{a},g)}{(N|g|)^s}, \quad Ng<0.\end{aligned}$$

They satisfy the functional equations

$$(58)\quad \begin{aligned}\varphi_k(S,\mathfrak{a},s) = N\|S\|^{-1/2}\{&-2\operatorname{cosec}\pi s\cot\pi s\,\varphi_{k+1}(S^{-1},\tilde{\mathfrak{a}},1-s)\\ &+(\cot^2\pi s+\operatorname{cosec}^2\pi s)\varphi_k(S^{-1},\tilde{\mathfrak{a}},1-s)\}, \quad k=1,2,\end{aligned}$$

where $\varphi_3$ is taken as $\varphi_1$ and

$$\varphi_k(S,\mathfrak{a},s) = |d|^s\pi^{-2s}(\Gamma(s))^2Z_k(S,\mathfrak{a},s).$$

These functions are regular everywhere except at $s=1$ where they have simple poles.

**§ 9. Hermitian forms.** Let $K$ be an algebraic number field of degree $n=r_1+2r_2$ over the rational number field and $L=K(\sqrt{d})$, $d\in K$ a quadratic extension of $K$. Let $d<0$ at $l$ of the real infinite prime spots of $K$. We denote by $\bar{a}$ the complex conjugate of $\underline{a}$ and by $a^\sigma$ the image of $\underline{a}$ by the automorphism $L\to L^\sigma$ of $L$ over $K$.



A matix $S$ of $m$ rows with elements in $L$ is said to be hermitian if $S'=S^\sigma$ where $S=(s_{kl})$ and $S'^\sigma=(s^\sigma_{lk})$. Let $S$ be non-singular. We shall associate with $S$ a positive system $T$ in the following way.

Let $k\leqslant l$. Then $L\otimes\overline{K^{(k)}}$ is the complex number field, $\overline{K^{(k)}}$ being the completion of $K$ into the real number field obtained from the valuation of $K$ determined by $K^{(k)}$. Let

$$S^{(k)} = \overline{C^{(k)'}}\begin{pmatrix}E_{p_k} & 0\\ 0 & -E_{q_k}\end{pmatrix}C^{(k)}, \tag{59}$$

$C^{(k)}$ being a complex matrix. Put

$$T^{(k)} = \overline{C^{(k)'}}C^{(k)}.$$

Then

$$T^{(k)}S^{(k)-1}T^{(k)} = S^{(k)}, \tag{60}$$

$T^{(k)}>0$ is hermitian. The totality of $T^{(k)}$ satisfying (60) constitute a space of $2\sum_{k=1}^{l}p_kq_k$ dimensions. The $l$ pairs of integers $p_k, q_k$ form the *system of signatures of* $S$.

Let now $l<k\leqslant r_1$. In this case $L\otimes\overline{K^{(k)}}$ is a direct sum of two real fields. Put now

$$S^{(k)} = (C^{(k)'})^\sigma C^{(k)} = (S^{(k)'})^\sigma$$

where $C^{(k)}$ is an arbitrary real matrix. Write

$$T^{(k)} = C^{(k)'}C^{(k)}, \qquad T^{(k)\sigma} = (C^{(k)'})^\sigma(C^{(k)})^\sigma.$$

Then we have that $T^{(k)}$ and $T^{(k)\sigma}$ are real symmetric positive matrices and satisfy

$$T^{(k)}S^{(k)-1}(T^{(k)})^\sigma = S^{(k)'}. \tag{61}$$

The real positive solutions $T^{(k)}$ of (61) constitute a space of $(r_1-l)\dfrac{m(m+1)}{2}$ dimensions. In a similar way, by considering $L\otimes\overline{K^{(k)}}$ for $k>r_1$ we obtain the space of positive hermitian solutions $T^{(k)}$ satisfying

$$T^{(k)}S^{(k)-1}(T^{(k)})'^\sigma = \overline{S^{(k)'}}. \tag{62}$$

They constitute a space of $r_2m^2$ dimensions. We now associate with $S$ the positive system

$$T = \begin{pmatrix}T^{(1)} & & \\ & \ddots & \\ & & T^{(2n)}\end{pmatrix}$$

where $T^{(1)},\dots,T^{(l)}$ are positive hermitian, $T^{(l+1)}=\overline{T^{(1)}}$ and so forth, $T^{(2l+1)},\dots,T^{(2l+r_1-l)}$ are real positive symmetric, $(T^{(l+r_1+k)}) = (T^{(2l+k)})$ $\sigma$ and similarly the last $2n-2r_1$ are positive hermitian. This system is related to $S$ by

$$TS^{-1}(T^{\sigma})' = \bar{S}'. \tag{63}$$

The $T$'s constitute a space of $2\sum_{k=1}^{l} p_k q_k + (r_1 - l)\frac{m(m+1)}{2} + r_2 m^2$ dimensions. This is the $\mathfrak{H}$ space of $S$.

Let us take $t^{(1)}, \ldots, t^{(2n)}$, $2n$ positive real parameters associated with $L$ in the sense of § 3. Let $S$ be a hermitian $m$-rowed non-singular matrix in $L$ and $T$ an element in the $\mathfrak{H}$ space of $S$. Let $t \in P$ in the sense of § 3. Let $\mathfrak{a}$ be an ideal in $L$. We consider the theta function

$$\vartheta(S, T, \mathfrak{a}, t) = \sum_{a \in \mathfrak{a}} e^{\pi\sigma(T\{a\}t)/2}. \tag{64}$$

We obtain then as in (6) the inequality

$$\vartheta(S, T, \mathfrak{a}, t) \leqslant c_{14}(1+Nt^{-m/2}) \prod_{k=1}^{m} N(1+h_k^{-1}). \tag{65}$$

Where $c_{14}$ is a constant depending on $m$, $S$ and $L$ if $T$ is an element of the Humbert's space given in (4), $t \in G$. $\sigma$ and $N$ in (64) and (65) denote trace and norm from $L$ to the rational number field.

A unimodular matrix $U$ over $L$ which satisfies $U'SU^{\sigma} = S$ is said to be a *unit* of $S$. The units of $S$ form a group $\Gamma(S)$ which has in $\mathfrak{H}$ a representation $T \to U'T\bar{U}$ which is discontinuous and faithful if we identify $U$ and $\omega U$ where $\omega$ is a roof of unity in $L$. Let $F$ be a fundamental region for $\Gamma(S)$ in $\mathfrak{H}$. Let $dv$ denote the invariant volume element in the $\mathfrak{H}$ space. We shall prove

LEMMA 6. *$\int_F \vartheta(S, T, \mathfrak{a}, t)\,dv$ converges and for fixed $t$*

$$\int_F \vartheta(S, T, \mathfrak{a}, t)\,dv = \sum_{a \in \mathfrak{a}} \int_F e^{-\pi\sigma(T\{a\}t)/2},$$

*provided that $S$ is not the matrix of a binary zero form.*

Proof. As in the proof of Lemma 1 we take $S$ in the form

$$S^{(k)} = \begin{pmatrix} 0 & 0 & P^{(k)} \\ 0 & F^{(k)} & Q^{(k)} \\ * & * & G^{(k)} \end{pmatrix};$$

$P^{(k)}$ being a square matrix of $g_k$ rows, $0 \leqslant g_k \leqslant m/2$. Corresponding to this we obtain a parametrization of the $\mathfrak{H}$-space. As in the case of quadratic forms put

$$S^* = C'SC^{\sigma}, \qquad T = \begin{pmatrix} H_1 & 0 & 0 \\ 0 & H_2 & 0 \\ 0 & 0 & H_3 \end{pmatrix} \left\{ \begin{matrix} E & L_1 & L_2 \\ 0 & E & L_3 \\ 0 & 0 & E \end{matrix} \right\},$$



$H_1$ and $H_3$ being square matrices of orders $g_k$. Put $T^* = T\{C\}$ and

$$C_0 C = L = \begin{pmatrix} E & Q_1 & Q_2 \\ 0 & E & Q_3 \\ 0 & 0 & E \end{pmatrix}.$$

(For definitions of $C$ see [8].) Using (63) we obtain the equations

$$\begin{cases} H_2 F^{-1} H_2^{\sigma'} = \bar{F}', \\ H_1^{\sigma} P^{-1} H_3' = \bar{P}, \\ Q_3 = -F^{-1} Q_1^{\sigma'} P, \\ Q_2 = (A - \tfrac{1}{2} Q_1 F^{-1} Q_1^{\sigma'}) P, \end{cases} \tag{66}$$

where $A' = -A^{\sigma}$. We now parametrize the $\mathfrak{H}$ space in the following way. If $k \leqslant l$ choose $H_1, Q_1, A$ and the parameters in $H_2$ as the required parameters. Here $H_1$ is positive $g_k$-rowed hermitian (complex), $Q_1$ a complex matrix of $g_k$ rows and $m-2g_k$ columns, $A$ is a complex matrix of $g_k$ rows. If $2l < k \leqslant l+r_1$ we choose $H_1, H_1^{\sigma}, Q_1, Q_3, A$ and the parameters in $H_2$ as the required parameters. Here $H_1, H_1^{\sigma}, Q_1, A$ are as before except that they are real, $Q_3$ has $m-2g_k$ rows and $g_k$ columns and is arbitrary real. If $2r_1 < k \leqslant 2r_1+r_2$ again $H_1, H_1^{\sigma}, Q_1, Q_3, A$ and $H_2$ are the parameters the only change being all are complex matrices. After a little computation one obtains the volume element $dv$ in terms of these parameters

$$dv = c_{15} \prod_{k=1}^{l} |H_1^{(k)}|^{m-2g_k} \prod_{k=2l+1}^{l+r_1} |H_1^{(k)}|^{m-2g_k-1/2} |H_1^{(k)\sigma}|^{m-2g_k-1/2} \times \tag{67}$$
$$\times \prod_{k=2r_1+1}^{2r_1+r_2} |H_1^{(k)}|^{m-2g_k} |H_1^{(k)\sigma}|^{m-2g_k} f(H_2)\, dv_1 \cdot dv_1^{\sigma} dQ_1 dQ_3 dA\, d\omega$$

where $f(H_2)$ is a product of certain powers of the determinants of $H_2^{(k)}$, $dv_1$, $dv_1^{\sigma}$ are the Euclidean volume element in the product of the spaces $H_1^{(k)}$ and in $H^{(k)\sigma}$ respectively, $dQ_1, dQ_3$ the corresponding volume elements in the products of $Q_1^{(k)}$ and $Q_3^{(k)}$ respectively, $dA$ in the product of $A^{(k)}$ and $d\omega$ the volume element in the $H_2$ space satisfying the first of the equation (66).

It is enough to prove the theorem under the assumption that $S$ is reduced, in the sense of Hermite-Siegel and we have $J = \mathfrak{H} \cap \mathfrak{R}_{c_1}$ instead of $F$. By following the method in [4] one proves that for all $k$

$$h_a^{(k)},\ h_b^{(k)},\ (h_b^{(k)})^{-1},\ (h_c^{(k)})^{-1} \leqslant c_{16} \tag{68}$$

for a constant $c_{16}$ depending only on $m, L$ and $S$. Here $1 \leqslant a \leqslant g_k$, $g_k < b \leqslant m-g_k$, $m-g_k+1 < c \leqslant m$. Furthermore we can use in the integral the expression on the right of (65). As in Lemma 1 we are reduced to proving the convergence of integrals of type

$$\int \prod_{a=1}^{g} s_a^{\mu_a} \frac{ds_1 \ldots ds_g}{s_1 \ldots s_g} \tag{69}$$

where $0 < s_a \leqslant c_{17} \cdot c_{17}$ depends on $c_{16}$ and

$$\mu_a = \begin{cases} a(m-a+1), & k \leqslant l \text{ or } k > 2r_1, \\ \frac{1}{2}a(m-a+1), & \text{otherwise}. \end{cases}$$

For convergence it is necessary that $\mu_a > 0$. But $\mu_a \geqslant \frac{1}{2}(m-a+1) \geqslant \frac{1}{2}(m-g_k+1)$, $a \leqslant g_k$. Since $g_k \leqslant m/2$, it follows that if $m > 2$ there is convergence. If $m = 2$ we have assumed that $S$ is not the matrix of a zero form so that $g_k = 0$ for all $k$. Hence $F$ itself is compact.

In order to prove the second part of the lemma note that on compact subsets of $F$ we can obtain uniform estimates depending only on the subsets because of the properties of $\mathfrak{R}_{c_1}$.

We can as before prove

Lemma 7. *If $m \geqslant 1$, $\int_F dv$ is finite.*

Proof. We proceed exactly as in the above lemma and obtain integrals of type (69) with $\mu_a$ having the values $a(m-a)$ or $\frac{1}{2}a(m-a)$ according as $k$ does not or does satisfy $2l+1 < k \leqslant 2r_1$. As before $\mu_a \geqslant \frac{1}{2}(m-a) \geqslant \frac{1}{2}(m-g_k) \geqslant m/4$. Thus $m > 1$ ensures $\mu_a > 0$ and also the convergence of the integral.

Let us put

$$\vartheta_0(S, T, \mathfrak{a}, t) = \sum_{\substack{\alpha \in \mathfrak{a} \\ \alpha' S \alpha^\sigma \neq 0}} e^{-\pi\sigma(T\{\alpha\}t)/2}.$$

Let $G$ be a fundamental domain in $P$ for the group of units of $L$.

Lemma 8. *If $s$ is a complex number with $\operatorname{Re} s > m$ then*

$$\int_G (Nt)^{s/2} \int_F \vartheta_0(S, T, \mathfrak{a}, t)\, dv[dt]$$

*converges.*

The proof is similar to that of Lemma 3.

We shall prove another lemma which besides being of use in the sequel is of importance for the analytic theory of hermitian forms.

Let $S$ be a complex hermitian matrix of $m$ rows with signature $p, m-p, p \geqslant 0$. Let $F$ be a complex matrix of $m$ rows and $b$ columns so that $S\{F\} = T$ has signature $p, b-p$. Clearly of course $b \geqslant p$. Let $W$ be a non-singular hermitian matrix of $b+c \leqslant m$ rows with signature $p, b+c-p$ and such that $W = \begin{pmatrix} T & Q \\ \bar{Q}' & R \end{pmatrix}$. Let $D$ be the space of complex matrices $X$ with

$$S[FX] = W.$$

On $D$ denote, after Siegel, the volume element by $\frac{\{dX\}}{\{dQ\}\{dR\}}$. We then have

Lemma 9.

$$\int_D \frac{\{dX\}}{\{dQ\}\{dR\}} = |S|^{-c}|T|^{b-m}|W|^{m-b-c}\frac{\mu_{m-b}}{\mu_{m-b-c}}$$

*where if $b = p$ we take $|T| = 1$ and*

$$\mu_a = \prod_{k=1}^{a} \frac{\pi^k}{\Gamma(k)}.$$

We omit the proof as it is exactly the same, with minor changes, as Siegel's proof of Hilfsatz 3 in [7]. In particular, taking $S = -E$ the unit matrix of order $m$, $b = m$ and $c = 0$ we get the volume of the unitary space as $\mu_m$.

A similar formula to one given in Lemma 9 can be proved if we take real quaternion elements. The formula then is

$$\int_D \frac{\{dX\}}{\{dQ\}\{dR\}} = \|S\|^{-c}\|T\|^{b-m-1/2}\|W\|^{m-b-c+1/2}\frac{\nu_{m-b}}{\nu_{m-b-c}}$$

where

$$\nu_a = \prod_{k=1}^{a} \frac{\pi^{2k}}{\Gamma(2k)}.$$

**§ 10. Zeta functions of hermitian forms.** We consider the simple case of hermitian forms over imaginary quadratic fields.

Let $K$ be an imaginary quadratic field with discriminant $d$. Let $|d| = D$. Let $S$ be a non-singular hermitian matrix of $m$ rows and of signature $p, q (= m-p)$ which is not the matrix of a binary zero form. Let $\mathfrak{H}$ be the space of positive hermitian matrices $T$ satisfying

$$(TS^{-1})^2 = E.$$

This is a symmetric space with metric $ds^2 = \sigma(T^{-1}dTT^{-1}dT)$. Let $dv$ be the corresponding volume element.

Let $\Omega(S)$ be the orthogonal group of $S$ namely the group of complex matrices $C$ satisfying $C'S\bar{C} = S$. The unit group $\Gamma(S)$ of $S$ is a discrete subgroup of $\Omega(S)$. Let $1, \omega$ be a basis of integers of $K$. For any complex matrix $X$ put

$$X = X_1 + \omega X_2$$

where $X_1$ and $X_2$ are real. If $W$ is hermitian put $W = W_1 + \omega W_2$ with real $W_1$ and $W_2$. For fixed $W_1, W_2$, consider the space $\Omega(X_1, X_2)$ of solutions of the matrix equation

$$X'S\overline{X} = W;$$

we denote, after Siegel, $\frac{\{dX_1\}\{dX_2\}}{\{dW_1\}\{dW_2\}}$ the volume element in $\Omega(X_1, X_2)$. If $U \epsilon \Gamma(S)$ then $(X_1, X_2) \to (Y_1, Y_2)$ where $UX = Y_1 + \omega Y_2$ defines a mapping of $\Omega(X_1, X_2)$ into itself. Let $F_0$ be a fundamental region in this space for this group. We put

$$\mu(S) = \int_{F_0} \frac{\{dX_1\}\{dX_2\}}{\{dW_1\}\{dW_2\}}$$

and call it the *measure of the unit group* $\Gamma(S)$. Since $X \to UX$ is not the identity mapping we get

$$w\mu(S) = \|S\|^{-m}\left(\frac{D}{4}\right)^{-m(m+1)/4} \mu_p \mu_q \int_F dv; \tag{70}$$

$w$ being the number of roots of unity in $K$ and $F$ is a fundamental region for $\Gamma(S)$ in the $\mathfrak{H}$ space. In case $S$ is definite, this gives

$$\mu(S) = \prod_{k=1}^{m} \frac{(2\pi)^k}{(k-1)!\,D^{k/2}} \frac{\|S\|^{-m}}{E(S)};$$

$E(S)$ being the number of units of $S$.

In a similar way we can define the measure of representation of $g$ by $S$. We put

$$\mu(S, \mathfrak{a}, a) = \left(\frac{D}{4}\right)^{-m(m-1)/4} \int_F \frac{\{dY\}}{\{dq\}\{dR\}}$$

where $S[aY] = \begin{pmatrix} g & q' \\ q & R \end{pmatrix}$ and $F_1$ is a fundamental region in this $Y$ space for the units $U$ of $S$ with $Ua = a$. Put

$$(71)\qquad \begin{aligned} M(S, \mathfrak{a}, g) &= \|S\|^{m-1}|g|^{m-1} \sum_{a} \mu(S, \mathfrak{a}, a), \qquad g \neq 0, \\ M(S, \mathfrak{a}, 0) &= \|S\|^{m-1} \sum_{a} \mu(S, \mathfrak{a}, 0) \end{aligned}$$

we then get

$$\begin{aligned} M(S, \mathfrak{a}, g) \int_{u>0,\,\mathrm{sgn}\,g} u^{p-1}(u-\mathrm{sgn}\,g)^{q-1} e^{-t|g|(2u-\mathrm{sgn}\,g)} du &= \left(\frac{D}{4}\right)^{-m(m-1)/4} \mu_{p-1}\mu_{q-1} \int_F \sum_{S[a]=g\neq 0} e^{-\pi t T\{a\}} dv, \\ M(S, \mathfrak{a}, 0)(2\pi t)^{-(m-1)}\Gamma(m-1) &= \left(\frac{D}{4}\right)^{-m(m-1)/4} \mu_{p-1}\mu_{q-1} \int_F \sum_{\substack{S[a]=0 \\ a\neq 0}} e^{-\pi t T\{a\}} dv. \end{aligned}$$



In case $S$ is definite we obtain from Lemma 9

$$M(S, \mathfrak{a}, g) = \left(\frac{D}{4}\right)^{-m(m-1)/4} \mu_{m-1} \frac{A(S, g)}{E(S)};$$

$A(S, g)$ being the number of integral representations $a'Sa = g$.

It is possible to define zeta functions of hermitian forms in the way we have defined for quadratic forms. We shall, however, use a slightly different procedure due to Hecke.

Let $\mathfrak{a}$ be an integral ideal of norm $N\mathfrak{a} = b$. Let $f > 0$ be a rational integer and $a$ a real number with $-1 < a < 1$. Let $T$ be in $\mathfrak{H}$ and $H = T - aS$. Then $H > 0$. Let $\varrho$ run through a complete system of $m$-rowed integral vectors which are incongruent $\bmod \mathfrak{a}f\sqrt{d}$ and with $\varrho \equiv 0 \pmod{\mathfrak{a}}$. We then consider the theta series

$$\vartheta(S, T, \mathfrak{a}f\sqrt{d}, t, c, \varrho) = \sum_{a\equiv\varrho(\bmod \mathfrak{a}f\sqrt{|d|})} e^{-\pi c H\{a\}t/bfD};$$

$c > 0$ being a constant satisfying

$$c^m(1-a)^p(1+a)^q\|S\|2^{-m}(fD^{1/2})^m = 1. \tag{72}$$

If $c = c(S, \mathfrak{a}, a)$ and $\tilde{c} = c(S^{-1}, \tilde{\mathfrak{a}}, -a)$ then

$$\tilde{c} = 4c^{-1}(1-a^2)^{-1}(fD^{1/2})^{-1}.$$

We now have the theta transformation formula

$$(73)\qquad \vartheta(S, T, \mathfrak{a}f\sqrt{d}, t, c, \varrho) = t^{-m} \sum_{\substack{\omega\,(\bmod \mathfrak{a}'f\sqrt{d}) \\ \omega\equiv 0\,(\bmod \mathfrak{a}')}} e^{2\pi i\sigma(\omega'\varrho/bfD)} \vartheta(S^{-1}, T^{-1}, \mathfrak{a}'f\sqrt{d}, t^{-1}, \tilde{c}, \omega),$$

where $\sigma$ denotes trace from $K$ to the rational number field and $\mathfrak{a}'$ is the conjugate ideal to $\mathfrak{a}$ in $K$.

Let $\Gamma(S, \mathfrak{a}, \varrho)$ denote the subgroup of $\Gamma(S)$ consisting of units $U$ with $U\varrho \equiv \varrho \,(\bmod \mathfrak{a}f\sqrt{d})$. It is a subgroup of finite index $v(S, \mathfrak{a}, \varrho)$ in $\Gamma(S)$. Define $\mu(S, \mathfrak{a}, \varrho)$ and $M(S, \mathfrak{a}f\sqrt{d}, \varrho, g)$ as before but taking this subgroup $\Gamma(S, \mathfrak{a}, \varrho)$ instead of $\Gamma(S)$ and a fundamental region $F_\varrho = F(S, \mathfrak{a}, \varrho)$ of this group. We introduce the zeta functions

$$\zeta(S, \mathfrak{a}f\sqrt{d}, \varrho, s) = N\mathfrak{a}^s \sum_{g>0} \frac{M(S, \mathfrak{a}f\sqrt{d}, \varrho, g)}{g^s}.$$

This series converges for $\mathrm{Re}\, s > m$ and defines there an analytic function of $s$. If we put

$$\xi(S, \mathfrak{a}f\sqrt{d}, \varrho, s) = v(S, \mathfrak{a}, \varrho)^{-1}\left(\frac{2\pi}{fD^{1/2}}\right)^{-s}\Gamma(s)\zeta(S, \mathfrak{a}f\sqrt{d}, \varrho, s). \tag{74}$$

We obtain the integral representation, valid for $\operatorname{Re} s > m$

$$\|S\|^{s/m}x^{qs/m}(1+x)^{1-m}\big(\xi(S, \mathfrak{a}f\sqrt{d}, \varrho, s)f_1(s, 2p, 2q, x) + \xi(-S, \mathfrak{a}f\sqrt{d}, \varrho, s)f_{-1}(s, 2p, 2q, x)\big)$$

$$= \left(\frac{D}{4}\right)^{-m(m-1)/4} \mu_{p-1}\mu_{q-1}\frac{1}{v(S, \mathfrak{a}, \varrho)}\int_0^\infty t^{s-1}\int_{F_\varrho(S)}\sum_{S\{\mathfrak{a}\}\neq 0} e^{-\pi\sigma H\{\mathfrak{a}\}t/bfD}dv\,dt\,.$$

From this by standard methods we obtain

$$\|S\|^{s/m}x^{qs/m}(1+x)^{1-m}\big(\xi(S, \mathfrak{a}f\sqrt{d}, \varrho, s)f_1(s, 2p, 2q, x)$$
$$+\xi(-S, \mathfrak{a}f\sqrt{d}, \varrho, s)f_{-1}(s, 2p, 2q, x)\big)$$
$$= \left(\frac{D}{4}\right)^{-m(m-1)/4}\mu_{p-1}\mu_{q-1}\int_1^\infty\Big\{\frac{t^{s-1}}{v(S, \mathfrak{a}, \varrho)}\int_{F_\varrho(S)}\sum_{S\{\mathfrak{a}\}\neq 0}e^{-\pi\sigma H\{\mathfrak{a}\}t}dv$$
$$+\frac{t^{m-s-1}}{v(S^{-1}, \mathfrak{a}', \omega)}\sum_{\omega}e^{2\pi i\sigma(\omega\varrho)/bfD}\int_{F_\omega(S^{-1})}\sum_{S^{-1}\{\mathfrak{a}\}\neq 0}e^{-\pi\tilde{c}\tilde{H}\{\mathfrak{a}\}t}dv\Big\}dt$$
$$+\left(\frac{D}{4}\right)^{-m(m-1)/4}\mu_{p-1}\mu_{q-1}\int_0^1 t^{s-m-1}\sum_{\omega}\frac{e^{2\pi i\sigma(\omega\varrho/bfD)}}{v(S^{-1}, \mathfrak{a}', \omega)}\int_{F_\omega(S^{-1})}\Big(1+\sum_{\substack{S^{-1}\{\mathfrak{a}\}=0\\ \mathfrak{a}\neq 0}}e^{-\pi\tilde{c}T^{-1}\{\mathfrak{a}\}t^{-1}}\Big)$$
$$\times dv\,dt - \left(\frac{D}{4}\right)^{-m(m-1)/4}\mu_{p-1}\mu_{q-1}\int_0^1 t^{s-1}\int_{F_\varrho(S)}\frac{1}{v(S, \mathfrak{a}, \varrho)}\Big(1+\sum_{\substack{S\{\mathfrak{a}\}=0\\ \mathfrak{a}\neq 0}}e^{-\pi cT\{\mathfrak{a}\}t}\Big)dv\,dt\,.$$

This gives, at once, the analytic continuation of the zeta functions. An analysis similar to that in the case of quadratic forms shows that the functions so continued are meromorphic in the whole plane. Because

$$\frac{1}{v(S^{-1}, \mathfrak{a}', \omega)}\int_{F_\omega(S^{-1})}dv$$

is independent of $\omega$ and

$$\sum_{\substack{\omega\equiv 0\,(\mathrm{mod}\,\mathfrak{a}')\\ \omega(\mathrm{mod}\,\mathfrak{a}'f\sqrt{d})}} e^{2\pi i\sigma(\omega\varrho/bfD)} = 0 \quad \text{if} \quad \varrho\neq 0\ (\mathrm{mod}\,\mathfrak{a}f\sqrt{d})$$

and $= f^2D$ in the contrary case, we obtain the

THEOREM 3. *The functions $\zeta(S, \mathfrak{a}f\sqrt{d}, \varrho, s)$ can be continued analytically into the whole plane into meromorphic functions satisfying the functional equation*

$$\xi(S, \mathfrak{a}f\sqrt{d}, \varrho, s) = \|S\|^{-1}\sum_{\omega}e^{2\pi i\sigma(\omega'\varrho/bfD)}\xi(S^{-1}, \mathfrak{a}'f\sqrt{d}, \omega, m-s)\,.$$

*All except $\zeta(S, \mathfrak{a}f\sqrt{d}, 0, s)$ are entire functions. This function has at $s = m$ the residue*

$$w\cdot f^{2-m}\cdot D\cdot\|S\|^{-(m+1)}\mu(S^{-1}, \mathfrak{a}', 0)\,.$$



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