L_p-deviations from zero of polynomials
with integral coefficients

by

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Dedicated to the memory of my father

1. Introduction. Let p(x) and u(x) be two non-negative summable functions defined on the interval [a, b], which assume the value zero only on a set of measure zero. Let \( \phi_1(x), \phi_2(x), \ldots \) be a finite or denumerably infinite system of linearly independent functions defined on \([a, b]\) which belong to \(L^2_p([a, b]) \cap L^n_u([a, b]), p \geq 1\) \((L^q_v([a, b]))\) is the class of those functions \( f(x) \) for which the product \( v(x)|f(x)|^q \) is summable).

Let \( \{\omega_k(x)\} \) be the orthonormal system with weight \( p(x) \) that is obtained by the orthogonalization of the original system \( \{\phi_k(x)\} \) according to the Schmidt procedure. Then

\[
\omega_k(x) = \beta_{1k}\phi_1(x) + \ldots + \beta_{kk}\phi_k(x), \quad \beta_{kk} = \left( \frac{\Delta_{k-1}}{\Delta_k} \right)^{1/2},
\]

and

\[
\phi_m(x) = \sum_{s=1}^{m} b_{ms}\omega_s(x), \quad b_{mm} = \left( \frac{\Delta_m}{\Delta_{m-1}} \right)^{1/2},
\]

where \( \Delta_k \) is the Gram determinant of the system of functions \( \{\phi_i(x)\}_{i=1}^k \), \( \Delta_0 = 1 \).

We consider integrals of the type

\[
\int_a^b u(x)|Q_n(x)|^p dx, \quad p \geq 1,
\]

where \( Q_n(x) \) is a non-trivial generalized polynomial, i.e. a function of the

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[217]
form
\[ Q_n(x) = \sum_{k=1}^{n} \alpha_k \phi_k(x) \]
with coefficients \( \alpha_1, \ldots, \alpha_n \) not simultaneously zero.

We prove the following general theorem:

**Theorem 1.** There exists a non-trivial generalized polynomial \( Q_n(x) \) with rational integral coefficients such that

\[ I_n = \int_{a}^{b} u(x)|Q_n(x)|^p \, dx \leq n^{p-1} \Delta_n^{p/(2n)} \sum_{s=1}^{n} A_s, \]

where \( \Delta_n \) is the Gram determinant of the system \( \{\phi_k(x)\}_{k=1}^{n} \) with respect to the weight function \( p(x) \), \( A_s = \int_{a}^{b} u(x)|\omega_s(x)|^p \, dx \) and \( \{\omega_k(x)\} \) is the orthonormal system with weight \( p(x) \) that is obtained by the orthogonalization of the system \( \{\phi_k(x)\} \).

As applications of Theorem 1 we obtain bounds of the values of the integral (3) for integral polynomials \( Q_n(x) = \sum_{k=0}^{n} \alpha_k x^k \) on certain intervals and for several weight functions \( p(x) \) and \( u(x) \).

(i) In [12], Theorem 1 was proved for \( \{\phi_k(x)\} \subset C([a, b]) \) and \( p(x) = u(x) = 1 \). The case \( p = 2 \) was proved by E. Aparicio [2, 3].

(ii) Concerning the existence of polynomials with rational integral coefficients on intervals of length less than 4 and with arbitrarily small norms (see [9, 6, 8, 14, 4, 5]), D. Hilbert [9] proved the following theorem: If \( b-a < 4 \), then for all \( 0 < \delta < 1 \), there exists a polynomial \( P_n(x) \) with rational integral coefficients, not simultaneously zero, such that \( \int_{a}^{b} P_n^2(x) \, dx < \delta < 1 \).

In the case of uniform norm a similar theorem was proved by Fekete [6], see also [4]. The importance of these polynomials may be seen in [7].

**2. Proof of Theorem 1.** We consider an integral of type (3). Substituting in (3) the expressions (2) for the functions \( \phi_m(x) \), we obtain

\[ I_n = \int_{a}^{b} u(x) \left| \sum_{k=1}^{n} \alpha_k \sum_{s=1}^{k} b_{ks} \omega_s(x) \right|^p \, dx \]

and by changing the order of summation we get

\[ I_n = \int_{a}^{b} u(x) \left| \sum_{s=1}^{n} \left[ \sum_{k=s}^{n} b_{ks} \alpha_k \right] \omega_s(x) \right|^p \, dx \]
and hence

\[ I_n \leq \int_a^b u(x) \left[ \sum_{s=1}^n |L_s| |\omega_s(x)| \right]^p \, dx, \]

where

\[ L_s = \sum_{k=s}^n b_{ks} \alpha_k \quad (s = 1, \ldots, n). \]

By Minkowski’s Linear Forms Theorem [13], there exists a system of rational integers \( \alpha_1, \ldots, \alpha_n \), not simultaneously zero, such that

\[ |L_s| \leq \Delta^{1/n} \quad (s = 1, \ldots, n), \]

where \( \Delta \) is the determinant of the system (7).

By (2), \( b_{kk} = \int_a^b \phi_k(x) \omega_k(x) p(x) \, dx \) and the determinant \( \Delta = b_{11} \cdots b_{nn} \) becomes \( \Delta = \Delta_1^{1/2} \), and therefore,

\[ |L_s| \leq \Delta_1^{1/(2n)} \quad (s = 1, \ldots, n). \]

From (6) and (9) and taking into account the inequality

\[ \left( \sum_{s=1}^n |a_s| \right)^p \leq n^{p-1} \sum_{s=1}^n |a_s|^p, \]

(4) follows.

Remark 1. If \( p = 2 \) and \( p(x) = u(x) \), since the system \( \{\omega_k(x)\} \) is orthonormal, from (5) and (9) we can obtain (see [2, 3])

\[ I_n = \sum_{s=1}^n \left( \sum_{k=s}^n b_{ks} \alpha_k \right)^2 \leq n \Delta_1^{1/n}. \]

Remark 2. If the functions \( u(x) \) and \( \{\phi_k(x)\} \) belong to \( C([a, b]) \), then

\[ J_n = \max_{a \leq x \leq b} u(x) \left| \sum_{k=1}^n \alpha_k \phi_k(x) \right| \leq \Delta_1^{1/(2n)} \max_{a \leq x \leq b} \left( \sum_{s=1}^n |u(x) \omega_s(x)| \right) \leq n M_n \Delta_1^{1/(2n)}, \]

where

\[ M_n = \max_{1 \leq s \leq n} \max_{a \leq x \leq b} |u(x) \omega_s(x)|. \]

On the other hand, for a fixed natural number \( n \), we may consider \( \sigma_n \) defined by

\[ \sigma_n^{-np} = \inf_{Q_n} \frac{b}{a} \int_a^b u(x) |Q_n(x)|^p \, dx, \]
where the infimum is over all non-trivial generalized polynomials with rational integral coefficients.

We then have the following result:

**Corollary 1.** The inequality

\[(13) \quad \sigma = \lim_{n \to \infty} \sigma_n \geq \lim_{n \to \infty} \Delta_n \left( \sum_{s=1}^{n} A_s \right)^{-1/(pn)}\]

holds if the limits exist.

**Remark 3.** If \(p(x) = u(x) = (1 - x^2)^{-1/2}\), estimate (13) is optimal.

Consider the system \(\{\phi_k(x)\} = \{\hat{T}_{k-1}(x)\}, k = 1, \ldots, n,\) of normalized orthogonal Chebyshev polynomials with positive leading coefficient (as usual, we shall denote by \(\hat{R}_n(x)\) a polynomial of degree \(n\) normalized so that its leading coefficient is 1). Then

\[\sigma_n^{-np} = \inf_{\alpha_k \in \mathbb{Z}} \int_{-1}^{1} (1 - x^2)^{-1/2} \left| \sum_{k=1}^{n} \alpha_k \hat{T}_{k-1}(x) \right|^p dx\]

\[\geq \|\hat{T}_{n-1}(x)\|_{2,p(x)}^{-p} \inf_{0 \neq \alpha_n \in \mathbb{Z}} |\alpha_n|^p\]

\[\times \inf_{c_k \in \mathbb{R}} \int_{-1}^{1} (1 - x^2)^{-1/2} |\hat{T}_{n-1}(x) + c_{n-2} \hat{T}_{n-2}(x) + \ldots|^p dx\]

\[\geq \|\hat{T}_{n-1}(x)\|_{2,p(x)}^{-p} \int_{-1}^{1} (1 - x^2)^{-1/2} |\hat{T}_{n-1}(x)|^p dx.\]

This last inequality follows by Rivlin [11, p. 81]. Here \(\| \cdot \|_{p,v(x)}\) is the \(L_p\)-norm with weight \(v(x)\). In view of Achiessier [1, p. 251],

\[\sigma_n^{-np} \geq \left( \frac{2}{\pi} \right)^{p/2} \frac{\Gamma(\frac{1}{2}) \Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2} + 1)} (n \geq 2)\]

and hence

\[\sigma = \lim_{n \to \infty} \sigma_n \leq 1.\]

But for this case \(\Delta_n = 1\), and

\[A_n = \int_{-1}^{1} (1 - x^2)^{-1/2} |\hat{T}_{n-1}(x)|^p dx = \left( \frac{2}{\pi} \right)^{p/2} \frac{\Gamma(\frac{1}{2}) \Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2} + 1)},\]

\[A_1 = \pi^{1-p/2}.\]

Therefore the limit as \(n \to \infty\) of the right-hand side of (13) is also 1.

**3. Theorems of Hilbert’s type.** Let \(u(x) = p(x) = [(x-a)(b-x)]^{-1/2}\) and consider the system \(\{\phi_k(x) = x^k\}, k = 0, 1, \ldots,\) on the interval \([a,b]\).
Then the polynomials \( \{ \omega_k(x) \} \) which form an orthonormal system are the Chebyshev polynomials \( \{ \hat{T}_k(x) \} \), \( k = 0, 1, \ldots \) (see [10, 11]). Since

\[
 b_{kk} = \int_a^b [(x-a)(b-x)]^{-1/2} x^k \hat{T}_k(x) \, dx = \left( \frac{b-a}{4} \right)^k (2\pi)^{1/2},
\]

\[
 b_{00} = \pi^{1/2},
\]
it is clear that

\[
 \Delta_{n+1} = \prod_{k=0}^n b_{kk}^2 = \pi^{n+1} 2^n \left( \frac{b-a}{4} \right)^{n(n+1)}.
\]

Moreover,

\[
 A_s = \left( \frac{2}{\pi} \right)^{p/2} \frac{\Gamma\left( \frac{1}{2} \right) \Gamma\left( \frac{p+1}{2} \right)}{\Gamma\left( \frac{p}{2} + 1 \right)} \Gamma\left( \frac{p}{2} + 1 \right), \quad A_0 = \pi^{1-p/2}.
\]

Applying the inequality (4), we then have the following result:

**Theorem 2.** For every natural number \( n \), there exists a non-trivial polynomial \( Q_n(x) = \sum_{k=0}^n \alpha_k x^k \) with rational integral coefficients such that

\[
 \int_a^b [(x-a)(b-x)]^{-1/2} |Q_n(x)|^p \, dx \leq \pi \left( \frac{b-a}{2} \right) \left( \frac{n+1}{n} \right) \left( \frac{b-a}{4} \right)^n.
\]

We note that the limit as \( n \to \infty \) of the right-hand side is zero if \( b-a < 4 \). Thus we can reword Theorem 2 in the following way (see (ii)):

**Theorem 3.** If \( b-a < 4 \), then for all \( 0 < \delta < 1 \), there exists a polynomial \( Q_n(x) = \sum_{k=0}^n \alpha_k x^k \) with rational integral coefficients, not simultaneously zero, such that

\[
 \int_a^b [(x-a)(b-x)]^{-1/2} |Q_n(x)|^p \, dx \leq \delta < 1 \quad (p = 1, 2, \ldots).
\]

It is clear that in the case \( p = 2 \) by (10) we can get

\[
 \int_a^b [(x-a)(b-x)]^{-1/2} Q_n^2(x) \, dx \leq \pi (n+1) 2^{n/(n+1)} \left( \frac{b-a}{4} \right)^n.
\]

**Theorem 4.** For every natural number \( n \), there exists a non-trivial polynomial \( Q_n(x) \) with rational integral coefficients, of degree \( \leq n \), such that

\[
 I_{n+1} = \int_a^b |Q_n(x)| \, dx \leq 2 \left( \frac{b-a}{2} \right) (n+1) \left( \frac{b-a}{4} \right)^{n/2}.
\]
Proof. Consider the Chebyshev polynomials \( \{ \hat{U}_k(x) \} \) of the second kind which form an orthonormal system with weight \( p(x) = [(x - a)(b - x)]^{1/2} \) on the interval \( [a, b] \) (see [10, 11]).

From Theorem 1 it follows that

\[
I_{n+1} \leq \Delta_{n+1}^{1/(2n+2)} \sum_{s=0}^{n} A_s.
\]

Since

\[
b_{kk} = \int_a^b [(x - a)(b - x)]^{1/2} x^k \hat{U}_k(x) \, dx
\]

\[
= (\pi/2)^{1/2} \left( \frac{b-a}{2} \right)^k \left( \frac{b-a}{4} \right)^{k}, \quad k = 0, 1, \ldots,
\]

it follows that

\[
\Delta_{n+1} = (\pi/2)^{n+1} \left( \frac{b-a}{2} \right)^{2(n+1)} \left( \frac{b-a}{4} \right)^{n(n+1)}.
\]

Moreover,

\[
A_s = \int_a^b |\hat{U}_s(x)| \, dx = 2(2/\pi)^{1/2}, \quad s = 0, 1, \ldots,
\]

and therefore

\[
\sum_{s=0}^{n} A_s = 2(2/\pi)^{1/2}(n + 1).
\]

From (14)–(16) the theorem follows. \( \blacksquare \)

We next turn to the least-squares approximation problem on an interval:

**Theorem 5.** For every natural number \( n \), there exists a non-trivial polynomial \( Q_n(x) = \sum_{k=0}^{n} \alpha_k x^k \) with rational integral coefficients such that

\[
I_{n+1} = \int_a^b [(x - a)(b - x)]^{1/2} Q_n^2(x) \, dx \leq \frac{\pi}{2} \left( \frac{b-a}{2} \right)^2 (n + 1) \left( \frac{b-a}{4} \right)^n.
\]

**Proof.** By Remark 1 we have

\[
I_{n+1} \leq (n + 1) \Delta_{n+1}^{1/(n+1)}.
\]

Let \( \{ \hat{U}_k(x) \} \) be the orthonormal system obtained by the orthogonalization of \( \{x^k\} \) with weight \( [(x - a)(b - x)]^{1/2} \). Since

\[
b_{kk} = (\pi/2)^{1/2} \left( \frac{b-a}{2} \right)^k \left( \frac{b-a}{4} \right)^k, \quad k = 0, 1, \ldots,
\]
it follows that
\[ \Delta_{n+1} = (\pi/2)^{n+1} \left( \frac{b-a}{2} \right)^{2(n+1)} \left( \frac{b-a}{4} \right)^{n(n+1)}. \]

From (17) and (18) the theorem follows.

**Theorem 6.** For every natural number \( n \), there exists a non-trivial polynomial \( Q_n(x) = \sum_{k=0}^{n} \alpha_k x^k \) with rational integral coefficients such that
\[ I_{n+1} \leq \pi \left( \frac{b-a}{2} \right)^{n+1} \left( \frac{b-a}{4} \right)^n \left( \frac{b-a}{2} \right)^n. \]

**Proof.** In this case we consider the polynomials \( \{ \hat{W}_k(x) \} \) which form an orthonormal system on \([a, b]\) with weight \((b-x)/(x-a)\)^{1/2} (see [10, 11]). But now
\[ b_{kk} = \int_a^b \left( \frac{b-x}{x-a} \right)^{1/2} x^k \hat{W}_k(x) \, dx = \pi^{1/2} \left( \frac{b-a}{2} \right)^{1/2} \left( \frac{b-a}{4} \right)^k, \]
and therefore
\[ \Delta_{n+1} = \pi^{n+1} \left( \frac{b-a}{2} \right)^{n+1} \left( \frac{b-a}{4} \right)^{n(n+1)}, \]
so that (10) becomes
\[ I_{n+1} \leq \pi \left( \frac{b-a}{2} \right)^{(n+1)} \left( \frac{b-a}{4} \right)^n. \]

Following the notation used by Achieser [1, pp. 249–254], let
\[ \omega(x) = \left( 1 - \frac{x}{a_1} \right) \left( 1 - \frac{x}{a_2} \right) \ldots \left( 1 - \frac{x}{a_{2q}} \right) \]
be a polynomial which is positive in \((-1, 1)\), and can have simple roots at one or both ends of \((-1, 1)\). The polynomial \( \omega(x) \) is of degree \( 2q - 1 \) if \( a_{2q} = \infty \) and \(|a_k| < \infty, k = 1, \ldots, 2q - 1 \). Set
\[ x = \frac{1}{2} \left( v + \frac{1}{v} \right) \quad (|v| \leq 1), \]
\[ a_k = \frac{1}{2} \left( c_k + \frac{1}{c_k} \right) \quad (|c_k| \leq 1, k = 1, \ldots, 2q), \]
\[ \Omega(v) = \prod_{k=1}^{2q} \sqrt{v - c_k}, \]
monic polynomials by the orthogonalization of

is an orthonormal system with weight $p$.

Let $\{\omega_k(x)\}_{k=0}^n$ be the orthonormal system with weight $p(x)$ that is obtained by the orthogonalization of $\{x^k\}_{k=0}^n$. By Akhieser [1, p. 251], the system of monic polynomials $\{U_m(x; \omega)\}_{m \geq q}$ of degree $m$ in $x$,

$$U_m(x; \omega) = L_{m+1} \left\{ v^{m+1-q} \frac{\Omega(1/v)}{\Omega(v)} - v^{m+1-2q} \frac{\Omega(v)}{\Omega(1/v)} \right\} \frac{\sqrt{\omega(x)}}{1/v - v},$$

is orthogonal on $[-1, 1]$ with weight function $p(x)$. Hence

$$\{\omega_0(x), \omega_1(x), \ldots, \omega_{q-1}(x), \hat{U}_q(x; \omega), \ldots, \hat{U}_n(x; \omega)\}$$

is an orthonormal system with weight $p(x)$ on $[-1, 1]$.

Since

$$b_{kk} = \int_{-1}^{1} \frac{(1 - x^2)^{1/2}}{\omega(x)} x^k \omega_k(x) \, dx = \|\tilde{\omega}_k\|_{2,p(x)}, \quad k = 0, 1, \ldots, q - 1,$$

and

$$b_{kk} = \int_{-1}^{1} \frac{(1 - x^2)^{1/2}}{\omega(x)} x^k \hat{U}_k(x; \omega) \, dx = \left( \frac{\pi}{2} \right)^{1/2} L_{k+1}, \quad k = q, \ldots, n,$$

it follows that

$$\Delta_{n+1} = \left( \prod_{k=0}^{q-1} \|\tilde{\omega}_k\|_{2,p(x)}^2 \right) \left( \frac{\pi}{2} \right)^{n-q+1} 2^{-(n+q)(n-q+1)} \prod_{k=1}^{2q} (1 + c_k^2)^{n-q+1}.$$

On the other hand,

$$A_s = \int_{-1}^{1} \frac{(1 - x^2)^{(p-1)/2}}{|\omega(x)|^{p/2}} |\hat{U}_s(x; \omega)|^p \, dx$$

$$= \left( \frac{\pi}{2} \right)^{-p/2} L_{s+1} \int_{-1}^{1} \frac{|(1 - x^2)^{1/2} \hat{U}_s(x; \omega)|^p}{|\omega(x)|^{1/2}} \, dx \frac{dx}{(1 - x^2)^{1/2}}$$

$$= \left( \frac{\pi}{2} \right)^{-p/2} \Gamma\left( \frac{1}{2} \right) \Gamma\left( \frac{s+1}{2} \right) \frac{\Gamma\left( \frac{s+1}{2} \right)}{\Gamma\left( \frac{s+1}{2} \right)} (s \geq q).$$
Therefore

\[
\sum_{s=0}^{n} A_s = - \sum_{s=0}^{q-1} A_s + \left( \pi \right)^{-p/2} \frac{\Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{p+1}{2} \right)}{\Gamma \left( \frac{p}{2} + 1 \right)} (n-q+1).
\]

From (4), (20) and (21), we deduce the following result:

**Theorem 7.** Suppose that

\[
\omega(x) = \left( 1 - \frac{x}{a_1} \right) \left( 1 - \frac{x}{a_2} \right) \ldots \left( 1 - \frac{x}{a_{2q}} \right) > 0
\]
in (-1, 1), and \( \omega(x) \) can have simple roots at one or both ends of the interval (-1, 1). For every natural number \( n \geq q \), there exists a non-trivial polynomial \( Q_n(x) = \sum_{k=0}^{n} \alpha_k x_k \) with rational integral coefficients such that

\[
\int_{-1}^{1} \left| \frac{(1-x^2)^{1/2}}{|\omega(x)|^{1/2}} Q_n(x) \right|^p \frac{dx}{(1-x^2)^{1/2}} \leq (n+1)^{p-1} \left\{ \left( \prod_{k=0}^{q-1} \| \tilde{\omega}_k \|_{2,p(x)} \right) \left( \frac{\pi}{2} \right)^{n-q+1} 2^{-(n+1)(n-q+1)} \right. \]

\[
\times \left( \prod_{k=1}^{2q} (1 + c_k^2)^n \right)^{n-q+1} \left[ \sum_{s=0}^{q-1} \| \omega_s \|_{p,u(x)} + \left( \frac{\pi}{2} \right)^{-p/2} \frac{\Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{p+1}{2} \right)}{\Gamma \left( \frac{p}{2} + 1 \right)} (n-q+1) \right].
\]

4. **Theorem of Fekete’s type.** As before we follow the notation used by Achieser [1, p. 249]. Given a polynomial

\[
\omega(x) = \left( 1 - \frac{x}{a_1} \right) \left( 1 - \frac{x}{a_2} \right) \ldots \left( 1 - \frac{x}{a_{2q}} \right)
\]
which is positive in [-1, 1]. The degree of \( \omega(x) \) is \( 2q - 1 \) if \( a_{2q} = \infty \) and \( |a_k| < \infty \) \((k = 1, \ldots , 2q - 1)\). We use the notation (19).

Let \( \{ \omega_k(x) \} \) be the orthonormal system with weight

\[
p(x) = \frac{1}{\omega(x)(1-x^2)^{1/2}}
\]

obtained by the orthogonalization of the system \( \{ x^k \} \). By Achieser [1, p. 250] it is known that the system of monic polynomials \( \{ \tilde{T}_m(x; \omega) \} \), \( m \geq q \),

\[
\tilde{T}_m(x; \omega) = \frac{L_m}{2} \left\{ v^{2q-m} \Omega(1/v) \frac{\Omega'(v)}{\Omega(v)} + v^{m-2q} \Omega(v) \frac{\Omega'(1/v)}{\Omega(1/v)} \right\} \}
\]
is orthogonal on [-1, 1] with weight \( p(x) \).
Consider the orthonormal system

\[ \{ \omega_0(x), \omega_1(x), \ldots, \omega_{q-1}(x), \hat{T}_q(x; \omega), \ldots, \hat{T}_n(x; \omega) \} \]

with weight \( p(x) \).

By Remark 2,

\[ \max_{-1 \leq x \leq 1} \left| \frac{1}{\sqrt{\omega(x)}} \sum_{k=0}^{n} \alpha_k x^k \right| \leq (n + 1) M_{n+1} \Delta_{n+1}^{1/(2n+2)}, \tag{22} \]

where

\[ M_{n+1} = \max \left\{ \max_{-1 \leq x \leq 1} \left| \frac{1}{\sqrt{\omega(x)}} \omega_s(x) \right|, \max_{0 \leq s \leq q-1} \left| \frac{1}{\sqrt{\omega(x)}} \hat{T}_s(x; \omega) \right| \right\} \tag{23} \]

and \( \Delta_{n+1} \) is the Gram determinant of the system \( \{ x^k \}_{k=0}^{n} \) with weight \( p(x) \).

Since

\[ b_{kk} = \int_{-1}^{1} \omega_k(x) x^k p(x) \, dx = \| \tilde{\omega}_k \|_{2,p(x)}, \quad k = 0, \ldots, q - 1, \]

and

\[ b_{kk} = \int_{-1}^{1} \hat{T}_k(x; \omega) \frac{x^k}{\omega(x)(1 - x^2)^{1/2}} \, dx = (\pi L_k L_{k+1})^{1/2}, \quad k \geq q, \]

we have

\[ \Delta_{n+1} = \left( \prod_{k=0}^{q-1} \| \tilde{\omega}_k \|_{2,p(x)}^2 \right) \pi^{n-q+1} 2^{-n^2 + (q-1)^2} \frac{1}{1 + c_1 \ldots c_{2q}} \prod_{k=1}^{2q} \left( 1 + c_k^2 \right)^{n-q+1}. \tag{24} \]

From (22)–(24), we have thus proved the following:

**Theorem 8.** Suppose that

\[ \omega(x) = \left( 1 - \frac{x}{a_1} \right) \left( 1 - \frac{x}{a_2} \right) \ldots \left( 1 - \frac{x}{a_2} \right) \]

is positive in \([-1, 1]\). For every natural number \( n \geq q \), there exists a non-trivial polynomial \( Q_n(x) = \sum_{k=0}^{n} \alpha_k x^k \) with rational integral coefficients such that
$$\max_{-1 \leq x \leq 1} \left| \frac{1}{\sqrt{\omega(x)}} \sum_{k=0}^{n} \alpha_k x^k \right|$$

$$\leq (n + 1) \max \left\{ \max_{-1 \leq x \leq 1} \left| \frac{1}{\sqrt{\omega(x)}} \omega_k(x) \right|, \right.$$

$$\left( \frac{2}{\pi} \right)^{1/2} \left( 1 + c_1 \ldots c_{2q} \right)^{-1/2}, \left( \frac{2}{\pi} \right)^{1/2} \right\}$$

$$\times \left\{ \left( \prod_{k=0}^{q-1} \| \tilde{\omega}_k \|_{L_2, p(x)}^2 \right)^{n-q+1} 2^{-n^2 + (q-1)^2} \right.$$ 

$$\times \left. \frac{1}{1 + c_1 \ldots c_{2q}} \prod_{k=1}^{2q} (1 + c_k^2)^{n-q+1} \right\}^{1/(2n+2)}.$$

References