A $\chi$-analogue of a formula of Ramanujan for $\zeta(1/2)$

by

SHIGEKI EGAMI (Toyama)

To the memory of Professor Norikata Nakagoshi

In his famous Notebooks ([4]) Ramanujan stated the following formula for $\zeta(1/2)$: For $\tau > 0$,

$$
\sum_{n=1}^{\infty} \frac{1}{e^{\pi n^2/\tau} - 1} = \frac{1}{6\tau} + \frac{1}{2} \sqrt{\frac{\pi}{\tau}} \zeta\left(\frac{1}{2}\right) + \frac{1}{4} + \frac{1}{2} \sqrt{\frac{\pi}{\tau}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left( \frac{\sinh(2\pi \sqrt{\pi n/\tau}) - \sin(2\pi \sqrt{\pi n/\tau})}{\cosh(2\pi \sqrt{\pi n/\tau}) - \cos(2\pi \sqrt{\pi n/\tau})} - 1 \right).
$$

Berndt and Evans ([2], see also [1]) gave a proof of this formula by using the Poisson summation formula. The purpose of this paper is to show a similar formula for the value $L(1/2, \chi)$ of Dirichlet $L$-functions. Our proof based on the Mellin transform is substantially different from [2].

The motivation for this work came from a discussion with Masanori Katsurada. The author would like to thank him.

Let $q$ be a positive integer, $\chi$ a primitive Dirichlet character modulo $q$, and $L(s, \chi)$ the Dirichlet $L$-function for $\chi$. Furthermore, we will use the following standard notation:

$$
E(\chi) = \begin{cases} 1 & \text{if } \chi \text{ is principal}, \\ 0 & \text{otherwise}, \end{cases}
$$

$$
W(\chi) = \begin{cases} \sqrt{q}g(\chi)^{-1} & \text{for } \chi(-1) = 1, \\ i\sqrt{q}g(\chi)^{-1} & \text{for } \chi(-1) = -1, \end{cases}
$$

where

$$
g(\chi) = \sum_{a \mod q} \chi(a)e^{2\pi ia/q}.
$$

Then our result can be stated as follows:
Theorem. For $\tau > 0$,

$$
\sum_{k \mod q} \chi(k) \sum_{n=1}^{\infty} \frac{e^{-kn^2\tau}}{1 - e^{-qnm^2\tau}} = \frac{E(\chi)\pi^2}{6\tau} - \frac{1}{2} L(0, \chi) + \frac{1}{2} \sqrt{\frac{\pi}{\tau}} L \left( \frac{1}{2}, \chi \right) + \frac{1}{2} \sqrt{\frac{\pi}{\tau}} W(\chi)
$$

\times \sum_{n=1}^{\infty} \frac{\chi(n)}{\sqrt{n}} \left( \frac{\sin(2\pi \sqrt{\frac{\pi n}{(q\tau)}}) - \chi(-1) \sin(2\pi \sqrt{\frac{\pi n}{(q\tau)}})}{\cosh(2\pi \sqrt{\frac{\pi n}{(q\tau)}}) - \cos(2\pi \sqrt{\frac{\pi n}{(q\tau)}})} - 1 \right).

Proof. First we express the left hand side of the above equation by the inverse Mellin integral:

$$
I(\tau, \chi) = \sum_{k \mod q} \chi(k) \sum_{n=1}^{\infty} \frac{e^{-kn^2\tau}}{1 - e^{-qnm^2\tau}} = \sum_{k \mod q} \chi(k) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} e^{-\left( kn^2\tau + qnm^2\tau \right)}
$$

$$
= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \chi(m)e^{-\tau n^2m} = \frac{1}{2\pi i} \int_{(c)} \Gamma(s)\zeta(2s)L(s, \chi)\tau^{-s} ds,
$$

where $c > 1$ and $\int_{(c)}$ denotes the integral along the line $\Re s = c$. Shifting the line of integration to $\Re s = 1/2 - c$ and changing the variable $s \leftrightarrow 1/2 - s$ we have

$$
I(\tau, \chi) = \frac{1}{2\pi i} \int_{(c)} \Gamma \left( \frac{1}{2} - s \right) \zeta(1-2s)L \left( \frac{1}{2} - s, \chi \right) \tau^{-1/2+s} ds + R(\tau, \chi),
$$

where $R(\tau, \chi)$ denotes the sum of the residues at $s = 1, 1/2,$ and 0,

$$
R(\tau, \chi) = \frac{E(\chi)\pi^2}{6\tau} + \frac{1}{2} \sqrt{\frac{\pi}{\tau}} L \left( \frac{1}{2}, \chi \right) - L(0, \chi).
$$

Using the functional equations for $\zeta(s)$ and $L(s, \chi)$ (see e.g. [3], p. 59 and p. 71) we have

$$
I(\tau, \chi) - R(\tau, \chi) = \frac{W(\chi)}{\pi i} \int_{(c)} \Gamma(2s)\zeta(2s)L \left( \frac{1}{2} + s, \chi \right)
$$

$$
\times \left( \cos \left( \frac{\pi s}{2} \right) - \chi(-1) \sin \left( \frac{\pi s}{2} \right) \right) \left( \frac{q\tau}{(2\pi)^3} \right)^s ds
$$

$$
= \frac{W(\chi)}{2\pi i} \sum_{m,n=1}^{\infty} \int_{(2c)} \Gamma(s)\left( \sqrt{\frac{q\tau}{(2\pi)^3m^2n}} \right)^s
$$

$$
\times \left( \cos \left( \frac{\pi s}{4} \right) - \chi(-1) \sin \left( \frac{\pi s}{4} \right) \right) ds.
$$
In order to calculate each integral in the above double series we note that
\[ \frac{1}{2\pi i} \int_{(c)} \Gamma(s) \left\{ \frac{\cos(\pi s/4)}{\sin(\pi s/4)} \right\} x^{-s} \, ds = e^{-x/\sqrt{2}} \left\{ \frac{\cos(x/\sqrt{2})}{\sin(x/\sqrt{2})} \right\}, \]
which can easily be obtained from the well known formula:
\[ e^{-(x+iy)} = \frac{1}{2\pi i} \int_{(c)} \Gamma(s)(x + iy)^{-s} \, ds \quad (x, c > 0). \]

Then we observe
\[ I(\tau, \chi) - R(\tau, \chi) = W(\chi) \sum_{n=1}^{\infty} \frac{\chi(n)}{\sqrt{n}} \left( \sum_{m=1}^{\infty} e^{-2\pi \sqrt{\frac{n}{q \tau}}} \left( \cos \left( 2\pi m \sqrt{\frac{n}{q \tau}} \right) - \chi(-1) \sin \left( 2\pi m \sqrt{\frac{n}{q \tau}} \right) \right) - \frac{\sinh(2\pi \sqrt{\frac{n}{q \tau}})}{\cosh(2\pi \sqrt{\frac{n}{q \tau}}) - \cos(2\pi \sqrt{\frac{n}{q \tau}})} - 1 \right), \]
which completes the proof of the Theorem.

References


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