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A χ-analogue of a formula of Ramanujan for $\zeta(1/2)$

by

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To the memory of Professor Norikata Nakagoshi

In his famous *Notebooks* ([4]) Ramanujan stated the following formula for $\zeta(1/2)$: For $\tau > 0$,

$$\sum_{n=1}^{\infty} \frac{1}{e^{\tau n^2}-1} = \frac{1}{6\tau} + \frac{1}{2}\sqrt{\frac{\pi}{\tau}}\zeta\left(\frac{1}{2}\right) + \frac{1}{4} + \frac{1}{2}\sqrt{\frac{\pi}{\tau}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left(\frac{\sinh(2\pi\sqrt{\pi n/\tau}) - \sin(2\pi\sqrt{\pi n/\tau})}{\cosh(2\pi\sqrt{\pi n/\tau}) - \cos(2\pi\sqrt{\pi n/\tau})} - 1 \right).$$

Berndt and Evans ([2], see also [1]) gave a proof of this formula by using the Poisson summation formula. The purpose of this paper is to show a similar formula for the value $L(1/2, \chi)$ of Dirichlet L-functions. Our proof based on the Mellin transform is substantially different from [2].

The motivation for this work came from a discussion with Masanori Katsurada. The author would like to thank him.

Let q be a positive integer, χ a primitive Dirichlet character modulo q, and $L(s,\chi)$ the Dirichlet L-function for χ. Furthermore, we will use the following standard notation:

$$E(\chi) = \begin{cases} 1 & \text{if } \chi \text{ is principal,} \\ 0 & \text{otherwise,} \end{cases}$$

$$W(\chi) = \begin{cases} \sqrt{q}g(\chi)^{-1} & \text{for } \chi(-1) = 1, \\ i\sqrt{q}g(\chi)^{-1} & \text{for } \chi(-1) = -1, \end{cases}$$

where

$$g(\chi) = \sum_{a \bmod q} \chi(a) e^{2\pi i a/q}.$$

Then our result can be stated as follows:

Theorem. *For* $\tau > 0$,

$$\sum_{k \bmod q} \chi(k) \sum_{n=1}^{\infty} \frac{e^{-kn^2\tau}}{1-e^{-qn^2\tau}}$$
$$= \frac{E(\chi)\pi^2}{6\tau} - \frac{1}{2}L(0,\chi) + \frac{1}{2}\sqrt{\frac{\pi}{\tau}}L\left(\frac{1}{2},\chi\right) + \frac{1}{2}\sqrt{\frac{\pi}{\tau}}W(\overline{\chi})$$
$$\times \sum_{n=1}^{\infty} \frac{\overline{\chi}(n)}{\sqrt{n}} \left(\frac{\sinh(2\pi\sqrt{\pi n/(q\tau)}) - \chi(-1)\sin(2\pi\sqrt{\pi n/(q\tau)})}{\cosh(2\pi\sqrt{\pi n/(q\tau)}) - \cos(2\pi\sqrt{\pi n/(q\tau)})} - 1 \right).$$

Proof. First we express the left hand side of the above equation by the inverse Mellin integral:

$$I(\tau,\chi) = \sum_{k \bmod q} \chi(k) \sum_{n=1}^{\infty} \frac{e^{-kn^2\tau}}{1-e^{-qn^2\tau}} = \sum_{k \bmod q} \chi(k) \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} e^{-(kn^2\tau + qmn^2\tau)}$$
$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \chi(m) e^{-\tau n^2 m} = \frac{1}{2\pi i} \int_{(c)} \Gamma(s)\zeta(2s)L(s,\chi)\tau^{-s}\, ds,$$

where $c > 1$ and $\int_{(c)}$ denotes the integral along the line $\Re s = c$. Shifting the line of integration to $\Re s = 1/2 - c$ and changing the variable $s \leftrightarrow 1/2 - s$ we have

$$I(\tau,\chi) = \frac{1}{2\pi i} \int_{(c)} \Gamma\left(\frac{1}{2}-s\right)\zeta(1-2s)L\left(\frac{1}{2}-s,\chi\right)\tau^{-1/2+s}\, ds + R(\tau,\chi),$$

where $R(\tau,\chi)$ denotes the sum of the residues at $s = 1, 1/2$, and 0,

$$R(\tau,\chi) = \frac{E(\chi)\pi^2}{6\tau} + \frac{1}{2}\sqrt{\frac{\pi}{\tau}}L\left(\frac{1}{2},\chi\right) - L(0,\chi).$$

Using the functional equations for $\zeta(s)$ and $L(s,\chi)$ (see e.g. [3], p. 59 and p. 71) we have

$$I(\tau,\chi) - R(\tau,\chi) = \frac{W(\overline{\chi})}{\pi i} \int_{(c)} \Gamma(2s)\zeta(2s)L\left(\frac{1}{2}+s,\overline{\chi}\right)$$
$$\times \left(\cos\left(\frac{\pi s}{2}\right) - \chi(-1)\sin\left(\frac{\pi s}{2}\right)\right)\left(\frac{q\tau}{(2\pi)^3}\right)^s ds$$
$$= \frac{W(\overline{\chi})}{2\pi i} \sum_{m,n=1}^{\infty} \int_{(2c)} \Gamma(s)\left(\sqrt{\frac{q\tau}{(2\pi)^3 m^2 n}}\right)^s$$
$$\times \left(\cos\left(\frac{\pi s}{4}\right) - \chi(-1)\sin\left(\frac{\pi s}{4}\right)\right) ds.$$

In order to calculate each integral in the above double series we note that

$$\frac{1}{2\pi i}\int_{(c)} \Gamma(s)\begin{Bmatrix}\cos(\pi s/4)\\ \sin(\pi s/4)\end{Bmatrix} x^{-s}\,ds = e^{-x/\sqrt{2}}\begin{Bmatrix}\cos(x/\sqrt{2})\\ \sin(x/\sqrt{2})\end{Bmatrix},$$

which can easily be obtained from the well known formula:

$$e^{-(x+iy)} = \frac{1}{2\pi i}\int_{(c)} \Gamma(s)(x+iy)^{-s}\,ds \qquad (x, c > 0).$$

Then we observe

$$\begin{aligned} I(\tau,\chi) &- R(\tau,\chi)\\ &= W(\overline{\chi})\sum_{n=1}^{\infty}\frac{\overline{\chi}(n)}{\sqrt{n}}\sum_{m=1}^{\infty} e^{-2\pi\sqrt{\frac{\pi n}{q\tau}}}\\ &\quad\times\left(\cos\left(2\pi m\sqrt{\frac{\pi n}{q\tau}}\right) - \chi(-1)\sin\left(2\pi m\sqrt{\frac{\pi n}{q\tau}}\right)\right)\\ &= W(\overline{\chi})\sum_{n=1}^{\infty}\frac{\overline{\chi}(n)}{\sqrt{n}}\left(\frac{\sinh(2\pi\sqrt{\pi n/(q\tau)}) - \chi(-1)\sin(2\pi\sqrt{\pi n/(q\tau)})}{\cosh(2\pi\sqrt{\pi n/(q\tau)}) - \cos(2\pi\sqrt{\pi n/(q\tau)})} - 1\right), \end{aligned}$$

which completes the proof of the Theorem.

References

[1] B. C. Berndt, *Ramanujan's Notebooks*, *Part II*, Springer, 1989.

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[3] H. Davenport, *Multiplicative Number Theory*, 2nd ed., Graduate Texts in Math. 74, Springer, 1980.

[4] S. Ramanujan, *Notebooks*, 2 Vols., Tata Institute, Bombay, 1957.

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