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Zeros of quadratic zeta-functions on the critical line

by

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Dedicated to the sixtieth birthday of Professor K. Ramachandra

1. Introduction. We follow the usual practice of writing $s = \sigma + it$. It is a well-known theorem of Hardy that the Riemann zeta-function $\zeta(s)$ has an infinity of zeros on the critical line $\sigma = 1/2$. In fact, Hardy's proof gives that if $1/2 + i\gamma_n$ ($\gamma_n \geq 0$) is the nth zero of $\zeta(s)$ on $\sigma = 1/2$ (see [12]), then

$$\gamma_{n+1} - \gamma_n \ll \gamma_n^{1/4+\varepsilon}. \tag{1.1}$$

The result (1.1) was improved by R. Balasubramanian (see [1]), namely

$$\gamma_{n+1} - \gamma_n \ll \gamma_n^{1/6+\varepsilon}. \tag{1.2}$$

In this paper, we consider the zeros of quadratic zeta-functions on the critical line. We begin by explaining the term quadratic zeta-functions. By this, we mean either the Epstein zeta-function associated with a positive definite binary quadratic form or the zeta-function of an ideal class in a quadratic field. One common feature of these things is that each of them has a functional equation of certain type (see §2, in particular (2.9)).

The main result of this paper is an analogue of (1.1). If $1/2+i\gamma_n^*$ ($\gamma_n^* \geq 0$) is the nth zero of any of the quadratic zeta-functions mentioned above, we prove

$$\gamma_{n+1}^* - \gamma_n^* \ll \gamma_n^{*1/2} \log(\gamma_n^* + 10). \tag{1.3}$$

Some important ground work in this direction has already been built up by H. S. A. Potter and E. C. Titchmarsh [9], E. Hecke [5], K. Chandrasekharan and Raghavan Narasimhan [4], and B. C. Berndt [3]. The main difference between the earlier papers and the present one is that while they argue on the line $\sigma = 1 + \delta$, we argue on the line $\sigma = 1$. For precise results of the earlier authors see (1.8).

Let

$$(1.4)\qquad Z(s)=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\frac{1}{(am^2+bmn+cn^2)^s}=\sum\sum{}'\frac{1}{(\varphi(m,n))^s}$$

in $\sigma>1$. Here a,b and c are real numbers with $a>0, c>0$ and $\Delta=4ac-b^2>0$, so that $\varphi(m,n)$ is a positive definite quadratic form. The dash indicates that the summation is taken over all values of m and n except $m=n=0$. In what follows we take a,b and c are integers with $a>0, c>0$ and $\Delta=4ac-b^2>0$. If $\mathbb{K}$ is a quadratic field and $\mathcal{C}$ is an ideal class in $\mathbb{K}$, then the Dedekind zeta-function of the class $\mathcal{C}$ in $\mathbb{K}$ is defined by the Dirichlet series

$$(1.5)\qquad \zeta_{\mathbb{K}}(s,\mathcal{C})=\sum_{0\neq\mathcal{A}\in\mathcal{C}}\frac{1}{(N\mathcal{A})^s}$$

in $\sigma>1$. Here $N\mathcal{A}$ means the norm of the ideal $\mathcal{A}\in\mathcal{C}$. We note that we can write

$$(1.6)\qquad \zeta_{\mathbb{K}}(s,\mathcal{C})=\sum_{m=1}^{\infty}\frac{a_m}{m^s}\qquad\text{in } \sigma>1,$$

where

$$(1.7)\qquad a_m=\sum_{0\neq\mathcal{A}\in\mathcal{C}}\frac{1}{(N\mathcal{A})^s}.$$

In fact, in [9] H. S. A. Potter and E. C. Titchmarsh proved that

$$(1.8)\qquad \gamma^*_{n+1}-\gamma^*_n\ll\gamma_n^{*1/2+\varepsilon}$$

for $Z(s)$. In [4], K. Chandrasekharan and Raghavan Narasimhan proved that *there are infinitely many zeros on* $\sigma=1/2$ *for* $\zeta_{\mathbb{K}}(s,\mathcal{C})$. In [3], Bruce C. Berndt proved (1.8) for $\zeta_{\mathbb{K}}(s,\mathcal{C})$.

Remark. If $\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_r$ are ideal classes in a field $\mathbb{K}=\mathbb{Q}(\sqrt{\pm d})$, then we can prove an analogue of the inequality (1.3) of the same form for the function $\sum_{j=1}^{r}d_j\zeta_{\mathbb{K}}(s,\mathcal{C}_j)$ where the coefficients d_j are real constants.

2. Notation and preliminaries. $C_1,C_2,\ldots,A_1,A_2,\ldots$ denote positive constants unless it is specified. We write $f(x)\ll g(x)$ to mean $|f(x)|<C_1g(x)$ (sometimes, we use the O-notation to mean the same). We write $s=\sigma+it$, $s_0=1+it$ and $w=u+iv$. Let $\lambda=|d|$ and $\Delta=4ac-b^2>0$. All the constants $C_1,C_2,\ldots,A_1,A_2,\ldots$ are effective. The implied constants from $\ll$ and O also are effective. In any fixed strip $\alpha\le\sigma\le\beta$, as $t\to\infty$, we have

$$(2.1)\qquad \Gamma(\sigma+it)=t^{\sigma+it-1/2}e^{-\pi t/2-it+(i\pi/2)(\sigma-1/2)}\sqrt{2\pi}(1+O(1/t)).$$

$Z(s)$ satisfies the functional equation (see [6] or [11]):

$$\left(\frac{\sqrt{\Delta}}{2\pi}\right)^{s}\Gamma(s)Z(s)=\left(\frac{\sqrt{\Delta}}{2\pi}\right)^{1-s}\Gamma(1-s)Z(1-s),\tag{2.2}$$

and $\zeta_{\mathbb{K}}(s,\mathcal{C})$ satisfies the functional equation (see [6])

$$\left(\frac{\sqrt{\lambda}}{2\pi}\right)^{s}\Gamma(s)\zeta_{\mathbb{K}}(s,\mathcal{C})=\left(\frac{\sqrt{\lambda}}{2\pi}\right)^{1-s}\Gamma(1-s)\zeta_{\mathbb{K}}(1-s,\mathcal{C})\quad\text{if } d<0\tag{2.3}$$

and

$$\left(\frac{\sqrt{\lambda}}{\pi}\right)^{s}\Gamma^2\left(\frac{s}{2}\right)\zeta_{\mathbb{K}}(s,\mathcal{C})=\left(\frac{\sqrt{\lambda}}{\pi}\right)^{1-s}\Gamma^2\left(\frac{1-s}{2}\right)\zeta_{\mathbb{K}}(1-s,\mathcal{C})\quad\text{if } d>0.\tag{2.4}$$

If we write

$$Z(s)=\chi_1(s)Z(1-s),\tag{2.5}$$

$$\zeta_{\mathbb{K}}(s,\mathcal{C})=\chi_2(s)\zeta_{\mathbb{K}}(1-s,\mathcal{C})\quad\text{if } d<0,\tag{2.6}$$

$$\zeta_{\mathbb{K}}(s,\mathcal{C})=\chi_3(s)\zeta_{\mathbb{K}}(1-s,\mathcal{C})\quad\text{if } d>0,\tag{2.7}$$

from (2.2)–(2.4), we get $|\chi_j(1/2+it)|=1$ for $j=1,2,3$, since $\chi_j(s)$ is real for real s. Since $\Gamma(s)$ has no zeros and only real poles, the function $\{\chi_j(s)\}^{-1}$ has a square root $(\chi_j(s))^{-1/2}$ in the simply connected region $t\ge t_0$ (t_0 large enough). We define

$$W_j(t)=f_j(1/2+it),\qquad f_j(s)=G(s)/\sqrt{\chi_j(s)}\tag{2.8}$$

where

$$G(s)=\begin{cases}Z(s) & \text{if } j=1 \text{ (defined by (2.2))},\\ \zeta_{\mathbb{K}}(s,\mathcal{C}); d<0 & \text{if } j=2 \text{ (defined by (2.3))},\\ \zeta_{\mathbb{K}}(s,\mathcal{C}); d>0 & \text{if } j=3 \text{ (defined by (2.4))}.\end{cases}$$

We note that $f_j(s)=f_j(1-s)$ for $j=1,2,3$ and hence $W_j(t)$ is real for real t. The zeros of $Z(s)$, $\zeta_{\mathbb{K}}(s,\mathcal{C})$ (with $\mathbb{K}=\mathbb{Q}(\sqrt{d})$, $d<0$), $\zeta_{\mathbb{K}}(s,\mathcal{C})$ (with $\mathbb{K}=\mathbb{Q}(\sqrt{d})$, $d>0$) on $\sigma=1/2$ respectively correspond to the real zeros of $W_1(t)$, $W_2(t)$ and $W_3(t)$. From (2.1)–(2.4), it follows that, for $1/2\le\sigma\le 1$, we have

$$(\chi_j(s))^{-1/2}=\left(\frac{M_j}{2\pi}\right)^{\sigma-1/2}t^{\sigma-1/2}\left(\frac{tM_j}{2\pi e}\right)^{it}e^{(i\pi/2)(\sigma-1/2)}(1+O(1/t))\tag{2.9}$$

for $j=1,2,3$ where $M_1=\sqrt{\Delta}, M_2=M_3=\sqrt{\lambda}$.

Let $T \ge T_0$ (T_0 is a large positive constant) and let $T \le T' \le 2T$. For $\mu > 0$, we define

$$J = \int_T^{T'} t^{\mu} \left(\frac{t}{e\xi} \right)^{it} dt, \qquad \xi > 0. \tag{2.10}$$

3. Some lemmas

Lemma 3.1. *For $\mu > 0$, we have*

$$J = O(T^{\mu}/\log(T/\xi)) \qquad \text{if } \xi < T, \tag{3.1.1}$$

$$J = O(T^{\mu}/\log(\xi/T')) \qquad \text{if } \xi > T', \tag{3.1.2}$$

$$\begin{aligned} J = (2\pi)^{1/2}\xi^{\mu+1/2}e^{i\pi/4-\xi} + O(T^{\mu+2/5}) + O(T^{\mu}/\log(\xi/T)) \\ + O(T^{\mu}/\log(T'/\xi)) \qquad \text{if } T < \xi < T' \end{aligned} \tag{3.1.3}$$

and

$$J = O(T^{\mu+1/2}) \qquad \text{for all } \xi > C_2. \tag{3.1.4}$$

Proof. (3.1.1) and (3.1.2) follow by using the first derivative test. (3.1.3) follows by the saddle point method and (3.1.4) follows on using the second derivative test. For example see [9].

Remark. For a more general version of Lemma 3.1, we refer to [4]. The estimate (3.1.4) with $\mu > 0$ is due to Landau.

Lemma 3.2. *If $R(x)$ is the number of lattice points inside or on the ellipse*

$$a_1(u-u_0)^2 + b_1(u-u_0)(v-v_0) + c_1(v-v_0)^2 = x$$

where a_1, b_1, c_1, u_0, v_0 are fixed, then

$$R(x) = 2\pi(4a_1c_1 - b_1^2)^{-1/2}x + O(x^{1/2}).$$

Proof. See for example VII. Teil, Kap. 7 of [7]. It is given for a circle and it is applicable for the ellipse also.

Lemma 3.3. *If $l(j)$ denotes the number of representations of j as $j = am^2 + bmn + cn^2$, then*

(i) $\sum_{j \le x} l(j) = C_3 x + O(x^{1/2})$,

(ii) $\sum_{m \le x} a_m = C_4 x + O(x^{1/2})$,

where a_m is as defined in (1.7).

Proof. (i) and (ii) follow from Lemma 3.2.

Lemma 3.4. *For $t \ge C_5$, we have*

(i) $Z(1+it) \ll \log t$,

(ii) $\zeta_{\mathbb{K}}(1+it, \mathcal{C}) \ll \log t$.

P r o o f. First we note that $Z(s)$ is of finite order (see [8]). Hence

$$Z(\sigma+it) \ll t^{C_6}, \tag{3.4.1}$$

where $C_6 \ge 5$, uniformly for $1/2 \le \sigma \le 3$. By Mellin's inverse transform, we have

$$\begin{aligned}\sum_{\varphi} \frac{e^{-\varphi/X_1}}{\varphi^{s_0}} &= \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} Z(s_0+w)\Gamma(w)X_1^w\,dw \\ &= \frac{1}{2\pi i} \int\limits_{\substack{u=2\\ |v|\le(\log t)^2}} Z(s_0+w)\Gamma(w)X_1^w\,dw + O(X_1^2 e^{-C_7(\log t)^2}).\end{aligned} \tag{3.4.2}$$

Note that $s_0 = 1+it$. In the integral of the right hand side of (3.4.2), we move the line of integration to $u=-1/2$. The pole $w=0$ contributes $Z(s_0)$. The horizontal portions contribute an error which is $O(t^{C_6}e^{-C_8(\log t)^2}X_1^2)$. We notice that

$$\frac{1}{2\pi i} \int\limits_{\substack{u=-1/2\\ |v|\le(\log t)^2}} Z(s_0+w)\Gamma(w)X_1^w\,dw = O(t^{C_6}X_1^{-1/2}). \tag{3.4.3}$$

Hence, from (3.4.2) we obtain

$$\sum_{\varphi} \frac{e^{-\varphi/X_1}}{\varphi^{s_0}} = Z(s_0) + O(t^{C_6}X_1^{-1/2}) + O(t^{C_6}X_1^2 e^{-C_9(\log t)^2}), \tag{3.4.4}$$

$$\sum_{\varphi} \frac{e^{-\varphi/X_1}}{\varphi^{s_0}} = O\Big(\sum_{\varphi\le X_1}\frac{1}{\varphi}\Big) + O\Big(X_1 \sum_{\varphi> X_1}\frac{1}{\varphi^2}\Big). \tag{3.4.5}$$

Since φ is a positive definite quadratic form, from Lemma 3.3(i) we obtain

$$\sum_{\varphi\le X_1}\frac{1}{\varphi} = \sum_{n\le X_1}\frac{l(n)}{n} \ll \log X_1, \tag{3.4.6}$$

where $l(n)$ is the number of representations of n as $n=\varphi(x,y)$. Also,

$$X_1 \sum_{\varphi> X_1}\frac{1}{\varphi^2} = X_1 \sum_{n> X_1}\frac{l(n)}{n^2} \ll 1. \tag{3.4.7}$$

We choose $X_1 = t^{2C_6}$. Hence (i) follows from (3.4.4)–(3.4.7). The proof of (ii) follows in a similar way.

LEMMA 3.5. *For* $t \ge 10$, *we have*

(i) $Z(\sigma+it) \ll t^{1-\sigma}\log t$,
(ii) $\zeta_{\mathbb{K}}(\sigma+it,\mathcal{C}) \ll t^{1-\sigma}\log t$

uniformly for $0 \le \sigma \le 1$.

P r o o f. (i) From Lemma 3.3, we have

$$Z(1+it) \ll \log t. \tag{3.5.1}$$

From the functional equation (2.2), and (3.5.1), we get

$$Z(it) \ll t \log t. \tag{3.5.2}$$

We apply the maximum-modulus principle to the function

$$F(w) = Z(w) e^{(w-s)^2} X_2^{w-s} \tag{3.5.3}$$

in the rectangle defined by the line segments joining the points $i(t-(\log t)^2)$, $1+i(t-(\log t)^2)$, $1+i(t+(\log t)^2)$, $i(t+(\log t)^2)$ and $i(t-(\log t)^2)$. Now,

$$|Z(s)| \ll V_1 + V_2 + H_1 + H_2, \tag{3.5.4}$$

where V_1, V_2 are the contributions from the vertical lines and H_1, H_2 are the contributions from the horizontal lines. We notice that $H_1 \ll 1$ and $H_2 \ll 1$. From (3.5.1)–(3.5.3), we obtain

$$|Z(s)| \ll t(\log t) X_2^{-\sigma} + (\log t) X_2^{1-\sigma} + 1. \tag{3.5.5}$$

Choosing $X_2 = t$, we obtain (i). (ii) follows in a similar way.

LEMMA 3.6. *Let $T \le t \le 2T$ and $X_3 = \sqrt{\Delta} T^4$. We have*

$$Z(s_0) = \sum_{\varphi} \frac{e^{-\varphi/X_3}}{\varphi^{s_0}} + O(T^{-3/2}(\log T)^3),$$

where $s_0 = 1 + it$.

P r o o f. As we did in Lemma 3.4, we obtain

$$\begin{aligned}
\sum_{\varphi} \frac{e^{-\varphi/X_3}}{\varphi^{s_0}} &= \frac{1}{2\pi i} \int\limits_{\substack{u=-1/2 \\ |v| \le (\log T)^2}} Z(s_0+w)\Gamma(w) X_3^w \, dw + Z(s_0) \\
&\quad + O(X_3^2 e^{-C_{10}(\log T)^2}) + O(T^{1/2}(\log T)^5 e^{-C_{11}(\log T)^2} X_3^2) \\
&= Z(s_0) + O(T^{1/2}(\log T)^3 X_3^{-1/2}) + O(X_3^2 e^{-C_{10}(\log T)^2}) \\
&\quad + O(T^{1/2}(\log T)^5 e^{-C_{11}(\log T)^2} X_3^2).
\end{aligned}$$

From our choice of $X_3 = \sqrt{\Delta} T^4$, the lemma follows.

LEMMA 3.7. *Let $T \le t \le 2T$ and $X_4 = \sqrt{\lambda} T^4$. We have*

$$\zeta_{\mathbb{K}}(s_0, \mathcal{C}) = \sum_{m=1}^{\infty} \frac{a_m}{m^{s_0}} e^{-m/X_4} + O(T^{-3/2}(\log T)^3).$$

P r o o f. This follows in a similar way to Lemma 3.6.

LEMMA 3.8. *If α_1 is irrational, then*
$$\sum_{n=N}^{N'} e^{2\pi i(\alpha_1 n^2+\beta_1 n)} = o(N'-N)$$
as $N'-N$ tends to infinity, uniformly with respect to β_1 and N.

Proof. See for example [8].

LEMMA 3.9. *For every irrational x and $H = H(T) \le T$ such that $H/\sqrt{T}$ tends to infinity with T we have*
$$\sum_{T\le m\le T+H} a_m e^{2\pi imx} = o(H),$$
where a_m is as defined in (1.7).

Proof. Let

(3.9.1) $$S(T,T+H) = \sum_{T\le m\le T+H} a_m e^{2\pi imx}.$$

For a given ideal class $\mathcal{C}$ and a non-zero integral ideal $\mathcal{A}\in\mathcal{C}$, we choose a non-zero integral ideal $\mathcal{B}\in\mathcal{C}^{-1}$ such that $\mathcal{AB} = (\alpha)$ for $\alpha\in\mathcal{B}$. We note that $(1, \omega = (d+\sqrt{d})/2)$ is a base of the ring of integers of $\mathbb{K} = \mathbb{Q}(\sqrt{d})$. We denote by $\omega' = (d-\sqrt{d})/2$ the conjugate of ω.

Case (i): $\mathbb{K} = \mathbb{Q}(\sqrt{d})$ with $d > 0$. Let α_0 be a generator of (α) and let α' be the conjugate of α. We have $\alpha = \pm\alpha_0\eta^r$ where η (>1) is the fundamental unit and $\eta\eta' = \pm 1$. (η' is the conjugate of η.) Now, by letting $L = |\alpha_0/\alpha_0'|$, we find that
$$|\alpha/\alpha'| = L\eta^{2r}.$$

We choose r to be the least integer such that $L\eta^{2r}\ge 1$. Hence, we get

(3.9.2) $$1 \le |\alpha/\alpha'| < \eta^2.$$

We can write $\alpha = k + l\omega$ with $k > 0$. We notice that for a given non-zero integral ideal $\mathcal{A}\in\mathcal{C}$, there exists one and only one $\alpha = k + l\omega$ with $k > 0$ such that $\mathcal{AB} = (\alpha)$ and satisfying the condition (3.9.2). For, if there are two, say $\alpha_{11} = k_1 + l_1\omega$ and $\alpha_{12} = \pm(k_2 + l_2\omega)\eta^r$ with $r\ge 1$, then
$$\eta^2 > \left|\frac{\alpha_{11}}{\alpha_{11}'}\right| = \left|\frac{\alpha_{12}}{\alpha_{12}'}\right|\eta^{2r} \ge \eta^{2r},$$
which is a contradiction. If, in (3.9.2), $k = 0$ for α then $\alpha = l\omega$ is unique if we specify that $l > 0$, for otherwise $(l_1\omega)$ and $(l_2\omega)$ are different ideals. Now,

(3.9.3) $$m = N\mathcal{A} = |(k+l\omega)(k+l\omega')|(N\mathcal{B})^{-1} = |P(k,l)|(N\mathcal{B})^{-1},$$

where

(3.9.4) $$P(k,l) = (k+l\omega)(k+l\omega') = k^2 + a_2kl + b_2l^2$$

with $4b_2 - a_2^2 < 0$. From (3.9.2), we have

$$1 \leq \frac{|k + l\omega|}{|k + l\omega'|} < \eta^2 \tag{3.9.5}$$

and hence,

$$|k + l\omega'|^2 \leq |P(k,l)| < \eta^2 |k + l\omega'|^2 \tag{3.9.6}$$

and

$$|P(k,l)| \leq |k + l\omega|^2 < \eta^2 |P(k,l)|. \tag{3.9.7}$$

From (3.9.3), we obtain

$$T(N\mathcal{B}) \leq |P(k,l)| \leq (T + H)(N\mathcal{B}). \tag{3.9.8}$$

From (3.9.6)–(3.9.8), we get

$$|k + l\omega| \leq \eta\sqrt{2(N\mathcal{B})T} \quad \text{and} \quad |k + l\omega'| \leq \sqrt{2(N\mathcal{B})T}. \tag{3.9.9}$$

Subtracting one from the other of (3.9.9), we get first $|l| \leq C_{12}\sqrt{T}$ and using this we obtain $k \leq C_{13}\sqrt{T}$. For a given $k > 0$, from the inequality (3.9.5), we get two intervals (say) J_1 and J_2 for l. From (3.9.4) and (3.9.8), we get

$$4b_2(N\mathcal{B})T \leq |4b_2k^2 + 4b_2a_2kl + 4b_2^2l^2| \leq 4b_2(N\mathcal{B})(T + H),$$

i.e.,

$$C_{14}T \leq |(2b_2l + a_2k)^2 + (4b_2 - a_2^2)k^2| \leq C_{14}(T + H). \tag{3.9.10}$$

The inequality (3.9.10) leads to four intervals (say) J_3, J_4, J_5 and J_6 and we notice that

$$\text{length of } J_r \ll H/\sqrt{T} \quad \text{for } r = 3, 4, 5, 6. \tag{3.9.11}$$

We define the set $S(k)$ for a fixed $k \geq 0$ to be

$$S(k) = \Big\{(k,l) \,\Big|\, l \in \Big(\bigcup_{r=3}^{6} J_r\Big) \cap (J_1 \cup J_2)\Big\}. \tag{3.9.12}$$

From (3.9.1), we have

$$S(T, T+H) = \frac{1}{2}\Big\{\sum_k \sideset{}{^*}\sum_{\substack{l \\ P(k,l)\equiv 0\,(N\mathcal{B})}} e^{2\pi i |P(k,l)| x (N\mathcal{B})^{-1}}\Big\}$$

(where $*$ indicates that l runs over $S(k)$ for fixed $k \geq 0$)

$$= \frac{1}{2}\Big\{\sum_k \sideset{}{^*}\sum_l \Big(\frac{1}{N\mathcal{B}} \sum_{j=1}^{N\mathcal{B}} e^{2\pi i |P(k,l)| (N\mathcal{B})^{-1} j}\Big) e^{2\pi i |P(k,l)| x (N\mathcal{B})^{-1}}\Big\}.$$

Therefore

$$|S(T,T+H)| \le \frac{1}{2}\max_{j}\Big\{\sum_{k}\Big|{\sum_{l}}^{*} e^{2\pi i|P(k,l)|y_j}\Big|\Big\}. \tag{3.9.13}$$

Since j runs over a finite set of positive integers and since x is irrational, y_j is irrational and hence $(P(k,l))y_j$ is a quadratic polynomial in l with the leading coefficient irrational. First, we note that $k=0$ trivially gives $o(H)$ to (3.9.13). So, it is enough to consider $k>0$. Let $M=(H/\sqrt{T})/\sqrt{H/\sqrt{T}}$. For fixed $k>0$, we see that

$${\sum_{l}}^{**} e^{2\pi i|P(k,l)|y_j} = O(M), \tag{3.9.14}$$

where $**$ indicates that l belongs to those intervals whose length is $\le M$. For fixed $k>0$, using Lemma 3.8, we obtain

$${\sum_{l}}^{***} e^{2\pi i|P(k,l)|y_j} = o(H/\sqrt{T}), \tag{3.9.15}$$

where $***$ indicates that l runs over those intervals whose length lies between M and $H/\sqrt{T}$. Therefore from (3.9.13)–(3.9.15), we get

$$S(T,T+H)=o(H),$$

since $k \ll \sqrt{T}$.

Case (ii): $\mathbb{K}=\mathbb{Q}(\sqrt{d})$ with $d<0$. When $d=-1$ and -3, the class number of the field $\mathbb{K}$ is 1 and hence $\zeta_{\mathbb{K}}(s,\mathcal{C})$ will contain a factor $\zeta(s)$. So, for the purpose of our paper, we can assume $d\neq -1$ and $\neq -3$. In this case, the class number of $\mathbb{K}$ is >1 and there are two units of $\mathbb{K}$. Hence, we get

$$P(k,l)=k^2+a_2kl+b_2l^2 \qquad \text{with } 4b_2-a_2^2>0$$

and $T(N\mathcal{B}) \le |P(k,l)| \le (T+H)(N\mathcal{B})$. Trivially, we get $0\le k \ll \sqrt{T}$ and now we can argue as we did in the case (i), and obtain the lemma.

Lemma 3.10. *If $a/\sqrt{\Delta}$ or $c/\sqrt{\Delta}$ is irrational (in particular, if a, b, c are integers and Δ is not a square) and $H=H(T)$ is such that $H\le T$ and $H/\sqrt{T}$ tends to infinity with T, then*

$$\int_{T}^{T+H} W_1(t)\,dt = o(H)+O(T^{1/2}\log T).$$

Proof. Recall $s_0=1+it$. First, we note that from (2.9),

$$(\chi_1(s_0))^{-1/2} = C_{15}t^{1/2}\left(\frac{t\sqrt{\Delta}}{2\pi e}\right)^{it}+O(t^{-1/2}), \tag{3.10.1}$$

where $C_{15} = e^{i\pi/4}(\sqrt{\Delta/(2\pi)})^{1/2}$. Now, we have

$$(3.10.2)\quad \int\limits_{T}^{T+H} W_1(t)\,dt = -i \int\limits_{1/2+iT}^{1/2+i(T+H)} f_1(s)\,ds$$
$$= -i\Big\{ \int\limits_{1/2+iT}^{1+iT} + \int\limits_{1+iT}^{1+i(T+H)} + \int\limits_{1+i(T+H)}^{1/2+i(T+H)} \Big\} f_1(s)\,ds$$
$$= L_1 + L_2 + L_3 \quad \text{(say)}.$$

From Lemma 3.5(i) and (2.9), we obtain

$$(3.10.3)\qquad L_1 \ll \int\limits_{1/2}^{1} T^{(1-\sigma)+\sigma-1/2}(\log T)\,d\sigma \ll T^{1/2}\log T.$$

Similarly, we obtain

$$(3.10.4)\qquad L_3 \ll T^{1/2}\log T.$$

From Lemma 3.6 and (3.10.1), we have

$$(3.10.5)\quad L_2 = \int\limits_{T}^{T+H} f_1(1+it)\,dt$$
$$= \int\limits_{T}^{T+H} \Big\{ C_{15} t^{1/2} \Big(\frac{t\sqrt{\Delta}}{2\pi e}\Big)^{it} + O(t^{-1/2}) \Big\}$$
$$\times \Big\{ \sum_{\varphi} \frac{e^{-\varphi/X_3}}{\varphi^{s_0}} + O(T^{-3/2}(\log T)^3) \Big\}\,dt$$
$$= C_{15} \sum_{\varphi} \frac{e^{-\varphi/X_3}}{\varphi} \int\limits_{T}^{T+H} t^{1/2} \Big(\frac{t\sqrt{\Delta}}{2\pi e\varphi}\Big)^{it}\,dt + O((\log T)^5)$$
$$+ O\Big(T^{1/2} \sum_{\varphi} \frac{e^{-\varphi/X_3}}{\varphi}\Big) + O(T^{-1}(\log T)^5).$$

We have

$$(3.10.6)\qquad \sum_{\varphi} \frac{e^{-\varphi/X_3}}{\varphi} \le \sum_{\varphi \le X_3} \frac{1}{\varphi} + \sum_{\varphi > X_3} \frac{e^{-\varphi/X_3}}{\varphi}$$
$$= \sum_{j \le X_3} \frac{l(j)}{j} + \sum_{j > X_3} \frac{l(j)e^{-j/X_3}}{j}$$
$$\ll \log X_3 + \int\limits_{1}^{\infty} e^{-v} v^{-1}\,dv \ll \log T$$

since $X_3 = \sqrt{\Delta} T^4$. Hence, we obtain
(3.10.7)

$$L_2 = C_{15} \sum_{\varphi} \frac{e^{-\varphi/X_3}}{\varphi} \int_T^{T+H} t^{1/2} \left(\frac{t\sqrt{\Delta}}{2\pi e \varphi} \right)^{it} dt + o(H) + O(T^{1/2} \log T).$$

To estimate the first term of (3.10.7), we divide the range of φ as follows, where $K_1 = \sqrt{\Delta}/(2\pi)$:

$$[0, K_1(\sqrt{T}-1)^2), \qquad [K_1(\sqrt{T}-1)^2, K_1(\sqrt{T}+1)^2),$$
$$[K_1(\sqrt{T}+1)^2, K_1(\sqrt{T+H}-1)^2),$$
$$[K_1(\sqrt{T+H}-1)^2, K_1(\sqrt{T+H}+1)^2),$$
$$[K_1(\sqrt{T+H}+1)^2, X_3^2), \qquad [X_3^2, \infty).$$

Let $\sum_1, \sum_2, \ldots, \sum_6$ be the corresponding parts of the above sum. Now,

$$\begin{aligned} \sum\nolimits_1 &= C_{15} \sum_{\varphi \le K_1(\sqrt{T}-1)^2} \frac{e^{-\varphi/X_3}}{\varphi} \int_T^{T+H} t^{1/2} \left(\frac{K_1 t}{e\varphi} \right)^{it} dt \\ &= C_{15} \sum_{r \le \sqrt{K_1}(\sqrt{T}-1)} \sum_{(r-1)^2 \le \varphi < r^2} \frac{e^{-\varphi/X_3}}{\varphi} \int_T^{T+H} t^{1/2} \left(\frac{K_1 t}{e\varphi} \right)^{it} dt \\ &= O\left(\sum_{r \le \sqrt{K_1}(\sqrt{T}-1)} \frac{1}{r} \cdot \frac{T^{1/2}}{|\log(K_1 T/\varphi)|} \right). \end{aligned}$$

(We have used (3.1.1) with $\mu = 1/2$ and the number of terms in the inner sum is $O(r)$.) Since

$$\left| \log \frac{m}{n} \right| \ge \frac{|m-n|}{m+n} \tag{3.10.8}$$

for any two positive numbers m, n, we have

$$\begin{aligned} \sum\nolimits_1 &= O\left(T^{1/2} \sum_{r \le \sqrt{K_1}(\sqrt{T}-1)} \frac{1}{r} \cdot \frac{r + \sqrt{K_1 T}}{|r - \sqrt{K_1 T}|} \right) \\ &= O\left(T^{1/2} \sum_{r \le \sqrt{K_1}(\sqrt{T}-1)} \left\{ \frac{1}{|r - \sqrt{K_1 T}|} + \frac{\sqrt{K_1 T}}{r|r - \sqrt{K_1 T}|} \right\} \right) \\ &= O(T^{1/2} \log T) \end{aligned} \tag{3.10.9}$$

since

$$\sum_{r \le \sqrt{K_1}(\sqrt{T}-1)} \frac{1}{|r - \sqrt{K_1 T}|} = O(\log T),$$

and

$$\sum_{r\le\sqrt{K_1}(\sqrt{T}-1)} \frac{\sqrt{K_1T}}{r|r-\sqrt{K_1T}|} = \sum_{r\le\sqrt{K_1T}/2} \frac{\sqrt{K_1T}}{r|r-\sqrt{K_1T}|} + \sum_{\sqrt{K_1T}/2<r\le\sqrt{K_1}(\sqrt{T}-1)} \frac{\sqrt{K_1T}}{r|r-\sqrt{K_1T}|} = O(\log T).$$

Now, using (3.1.4) with $\mu = 1/2$, we obtain

$$\sum\nolimits_2 = \sum_{K_1(\sqrt{T}-1)^2\le\varphi\le K_1(\sqrt{T}+1)^2} \frac{e^{-\varphi/X_3}}{\varphi} \int\limits_T^{T+H} t^{1/2}\left(\frac{K_1t}{e\varphi}\right)^{it} dt = O\left(\frac{1}{T}\cdot T(K_1(\sqrt{T}+1)^2 - K_1(\sqrt{T}-1)^2)\right) = O(T^{1/2}) \tag{3.10.10}$$

and similarly, we get

$$\sum\nolimits_4 = O(T^{1/2}). \tag{3.10.11}$$

We note that we can use (3.1.2) to estimate $\sum_5$ and $\sum_6$. Now,

$$\begin{aligned}\sum\nolimits_5 &= \sum_{K_1(\sqrt{T+H}+1)^2\le\varphi<X_3^2} \frac{e^{-\varphi/X_3}}{\varphi} \int\limits_T^{T+H} t^{1/2}\left(\frac{K_1t}{e\varphi}\right)^{it} dt \\ &\ll \sum_{\sqrt{K_1}(\sqrt{T+H}+1)\le r<X_3} \sum_{(r-1)^2\le\varphi<r^2} \frac{1}{\varphi}\cdot\frac{T^{1/2}}{\log\left(\frac{\varphi}{K_1(T+H)}\right)} \\ &\ll T^{1/2} \sum_{\sqrt{K_1}(\sqrt{T+H}+1)\le r<X_3} \frac{1}{r}\cdot\frac{1}{\log\left(\frac{r^2}{K_1(T+H)}\right)} \\ &\ll T^{1/2}\log T.\end{aligned} \tag{3.10.12}$$

Now,

$$\begin{aligned}\sum\nolimits_6 &\ll \sum_{\varphi\ge X_3^2} \frac{e^{-\varphi/X_3}}{\varphi}\cdot\frac{T^{1/2}}{\log\left(\frac{\varphi}{K_1(T+H)}\right)} \\ &\ll T^{1/2}e^{-X_3/2}X_3\frac{1}{X_3^2} \ll T^{-10}.\end{aligned} \tag{3.10.13}$$

It remains only to estimate $\sum_3$. Now,

$$\sum\nolimits_3 = C_{15} \sum_{K_1(\sqrt{T}+1)^2\le\varphi<K_1(\sqrt{T+H}-1)^2} \frac{e^{-\varphi/X_3}}{\varphi} \int\limits_T^{T+H} t^{1/2}\left(\frac{K_1t}{e\varphi}\right)^{it} dt. \tag{3.10.14}$$

Note that $K_1 = \sqrt{\Delta}/(2\pi)$. With φ in the range as in (3.10.14), using (3.1.3) with $\mu = 1/2$ and $\xi = \varphi/K_1$, we obtain

$$\begin{aligned}
\textstyle\sum_3 &= \sqrt{2\pi} C_{15} e^{i\pi/4} K_1^{-1} \sum_{\varphi} e^{-\varphi/X_3} e^{-2\pi i \varphi/\sqrt{\Delta}} \\
&\quad + O\Big(T^{9/10} \sum_{\varphi} \frac{e^{-\varphi/X_3}}{\varphi}\Big) + O\Big(T^{1/2} \sum_{\varphi} \frac{e^{-\varphi/X_3}}{\varphi \log\big(\frac{\varphi}{K_1 T}\big)}\Big) \\
&\quad + O\Big(T^{1/2} \sum_{\varphi} \frac{e^{-\varphi/X_3}}{\varphi \log\big(\frac{K_1(T+H)}{\varphi}\big)}\Big) \\
&= L_4 + L_5 + L_6 + L_7, \quad \text{say.}
\end{aligned} \tag{3.10.15}$$

Since $K_1(\sqrt{T}+1)^2 \le \varphi < K_1(\sqrt{T+H}-1)^2$ and $X_3 = K_1(\sqrt{T}-1)^8$, we note that

$$e^{-\varphi/X_3} = 1 + O(\varphi/X_3) = 1 + O(T^{-2}), \tag{3.10.16}$$

and we use (3.10.7). Also, note that the number of integers in $[K_1(\sqrt{T}+1)^2, K_1(\sqrt{T+H}-1)^2)$ is

$$C_{16} K_1 H + O(\sqrt{T}) = O(H). \tag{3.10.17}$$

From (3.10.17), we get

$$\begin{aligned}
L_5 &\ll T^{9/10} \sum_{K_1(\sqrt{T}+1)^2 \le j < K_1(\sqrt{T+H}-1)^2} \frac{l(j)}{j} e^{-j/X_3} \\
&\ll T^{9/10-1} H = o(H).
\end{aligned} \tag{3.10.18}$$

Using (3.10.7), we obtain,

$$\begin{aligned}
L_6 &\ll T^{1/2} \sum_{K_1(\sqrt{T}+1)^2 \le j < K_1(\sqrt{T+H}-1)^2} \frac{l(j)}{j} e^{-j/X_3} \frac{j + K_1 T}{|j - K_1 T|} \\
&\ll T^{1/2} \sum_{K_1(\sqrt{T}+1)^2 \le j < K_1(\sqrt{T+H}-1)^2} \frac{l(j)}{|j - K_1 T|} \\
&\ll T^{1/2} \log T
\end{aligned} \tag{3.10.19}$$

and

$$\begin{aligned}
L_7 &\ll T^{1/2} \\
&\quad \times \sum_{K_1(\sqrt{T}+1)^2 \le j < K_1(\sqrt{T+H}-1)^2} \frac{l(j)}{j} e^{-j/X_3} \frac{j + K_1(T+H)}{|j - K_1(T+H)|}
\end{aligned} \tag{3.10.20}$$

$$\ll T^{1/2} \sum_{K_1(\sqrt{T}+1)^2 \le j < K_1(\sqrt{T+H}-1)^2} \frac{l(j)}{|j - K_1(T+H)|}$$
$$+ \frac{T^{1/2}K_1(T+H)}{K_1 T} \sum_{\cdots} \frac{l(j)}{|j - K_1(T+H)|}$$
$$\ll T^{1/2} \log T.$$

Now,

$$L_4 = \sqrt{2\pi} C_{15} e^{i\pi/4} K_1^{-1} \times \sum_{K_1(\sqrt{T}+1)^2 \le \varphi < K_1(\sqrt{T+H}-1)^2} e^{-\varphi/X_3} e^{-2\pi i\varphi/\sqrt{\Delta}} \tag{3.10.21}$$
$$= \sqrt{2\pi} C_{15} e^{i\pi/4} K_1^{-1} \sum_{\cdots} e^{-2\pi i\varphi/\sqrt{\Delta}} + O\Big(\sum_{\cdots} \varphi/X_3\Big)$$
$$= \sqrt{2\pi} C_{15} e^{i\pi/4} K_1^{-1} \sum_{K_1(\sqrt{T}+1)^2 \le \varphi < K_1(\sqrt{T+H}-1)^2} e^{-2\pi i\varphi/\Delta}$$
$$+ O(T^{1+\varepsilon} H/X_3)$$
$$= \sqrt{2\pi} C_{15} e^{i\pi/4} K_1^{-1} \times \sum_{K_1(\sqrt{T}+1)^2 \le \varphi < K_1(\sqrt{T+H}-1)^2} e^{-2\pi i\varphi/\sqrt{\Delta}} + o(1)$$

since $X_3 \gg T^4$. Now, suppose that $c/\sqrt{\Delta}$ is irrational. Then

$$\Big|\sum e^{-2\pi i\varphi/\sqrt{\Delta}}\Big| = \Big|\sum_m \sum_n e^{-2\pi i(am^2+bmn+cn^2)/\sqrt{\Delta}}\Big|$$
$$\le \sum_m \Big|\sum_n e^{-2\pi i(am^2+bmn+cn^2)/\sqrt{\Delta}}\Big|.$$

Since $H/\sqrt{T}$ tends to infinity with T, the range of values of n consists of one or two intervals, the length of each of which tends to infinity. Hence by Lemma 3.8, we get

$$\sum_n e^{-2\pi i(am^2+bmn+cn^2)/\sqrt{\Delta}} = o\Big(\sum_n 1\Big)$$

and therefore from Lemma 3.2 we obtain

$$\sum_m \sum_n e^{-2\pi i(am^2+bmn+cn^2)/\sqrt{\Delta}} \tag{3.10.22}$$
$$= o\Big(\sum_m \sum_n 1\Big) = o(R(T+H) - R(T)) = o(H) + o(\sqrt{T}).$$

If $a/\sqrt{\Delta}$ is irrational, a similar argument holds with m and n interchanged. This proves the lemma.

LEMMA 3.11. *Let $H = H(T)$ be such that $H \le T$ and $H/\sqrt{T}$ tends to infinity with T. If λ is not a perfect square, then*

$$\text{(i)} \qquad \int_T^{T+H} W_2(t)\,dt = o(H) + O(T^{1/2}\log T),$$

$$\text{(ii)} \qquad \int_T^{T+H} W_3(t)\,dt = o(H) + O(T^{1/2}\log T).$$

Proof. We note that $\lambda = |d|$ is not a perfect square so that $\sqrt{\lambda}$ is irrational. Again, we notice that from (2.9),

$$(\chi_j(s_0))^{-1/2} = C_{16}\, t^{1/2}\left(\frac{t\sqrt{\lambda}}{2\pi e}\right)^{it} + O(t^{-1/2}) \qquad \text{for } j = 2,3,$$

where $C_{16} = e^{i\pi/4}(\sqrt{\lambda}/(2\pi))^{1/2}$. Instead of K_1 in the proof of Lemma 3.10, we take $K_2 = \sqrt{\lambda}/(2\pi)$. Now, the proof for (i) and (ii) is the same as for Lemma 3.10.

LEMMA 3.12 (see Theorem 1 of [10]). *Let $C_{17} \ge 1$ and $1 = \lambda_1 < \lambda_2 < \ldots$ be such that $1/C_{17} \le \lambda_{n+1} - \lambda_n \le C_{17}$. Let $1 = a_1, a_2, \ldots$ be a sequence of complex numbers with $|a_n| \le (nH)^{C_{17}}$ where $(\log T)^{10} \le H \le T$. Suppose $F(s) = \sum_{n=1}^{\infty} a_n\lambda_n^{-s}$ is analytically continuable in an infinite system of rectangles defined by $\{\sigma \ge 1/2, T \le t \le T+H\}$ and there $\max|F(s)| < T^{C_{17}}$. Then*

$$\int_T^{T+H} |F(1/2+it)|\,dt \gg H,$$

where the implied constant is effective.

Proof. First, we choose C_{18} large enough such that

$$(3.12.1) \qquad 1 \ll F(\sigma+it) \ll 1 \qquad \text{for } \sigma \ge 1/2 + C_{18}.$$

Consider the rectangle R_1 defined by the line segments joining the points $1/2+iT$, $1/2+2C_{18}+iT$, $1/2+2C_{18}+i(T+H)$, $1/2+i(T+H)$ and $1/2+iT$. Let $s_1 = 1/2+C_{18}+it_1$, where $T+H/10 \le t_1 \le T+H-H/10$. By the residue theorem, for $X > 0$ we have

$$(3.12.2) \qquad \begin{aligned} F(s_1) &= \frac{1}{2\pi i}\int_{R_1} \frac{F(s)X^{s-s_1}e^{(s-s_1)^2}}{s-s_1}\,ds \\ &= H_{11} + H_{12} + V_{11} + V_{12} \quad \text{(say)} \end{aligned}$$

where H_{11}, H_{12} are the horizontal lines contributions and V_{11}, V_{12} are the vertical lines contributions. We note that

$$H_{11} \ll \int_{1/2}^{1/2+2C_{18}} i\frac{|F(\sigma+iT)|X^{\sigma-1/2-C_{18}}e^{(\sigma-1/2-C_{18})^2-(T-t_1)^2}}{|\sigma-1/2-C_{18}+i(T-t_1)|}\,d\sigma \tag{3.12.3}$$

$$\ll T^{C_{17}}(X^{C_{18}}+X^{-C_{18}})e^{-C_{19}H^2}$$

and similarly,

$$H_{12} \ll T^{C_{17}}(X^{C_{18}}+X^{-C_{18}})e^{-C_{19}H^2}. \tag{3.12.4}$$

Now, from (3.12.1),

$$V_{12} = \frac{1}{2\pi}\int_{T}^{T+H} \frac{F(1/2+2C_{18}+it)X^{C_{18}+i(t-t_1)}e^{C_{18}^2-(t-t_1)^2+i(\ )}}{C_{18}+i(t-t_1)}\,dt \tag{3.12.5}$$

$$\ll X^{C_{18}} \int_{T}^{T+H} e^{-(t-t_1)^2}\,dt.$$

Also, we have

$$V_{11} = \frac{1}{2\pi}\int_{T}^{T+H} \frac{F(1/2+it)X^{-C_{18}+i(t-t_1)}e^{C_{18}^2-(t-t_1)^2+i(\ \ )}}{-C_{18}+i(t-t_1)}\,dt \tag{3.12.6}$$

$$\ll X^{-C_{18}} \int_{T}^{T+H} \frac{|F(1/2+it)|e^{-(t-t_1)^2}}{|-C_{18}+i(t-t_1)|}\,dt.$$

From (3.12.1)–(3.12.6), we obtain,

$$H \ll \int_{T+H/10}^{T+H-H/10} |F(s_1)|\,dt_1 \tag{3.12.7}$$

$$\ll X^{C_{18}}H + X^{-C_{18}} \int_{T+H/10}^{T+9H/10} dt_1 \int_{T}^{T+H} \frac{|F(1/2+it)|e^{-(t-t_1)^2}}{|-C_{18}+i(t-t_1)|}\,dt$$

$$\ll X^{C_{18}}H + X^{-C_{18}}\Big(\int_{T}^{T+H} |F(1/2+it)|\,dt\Big).$$

We choose X such that

$$X^{C_{18}}H = X^{-C_{18}}\Big(\sqrt{H} + \int_{T}^{T+H} |F(1/2+it)|\,dt\Big). \tag{3.12.8}$$

From (3.12.8), it clearly follows that $1/T^{C_{19}} \ll X \ll T^{C_{20}}$. First choose

$X = H^{-\varepsilon}$ to get

$$\int_T^{T+H} |F(1/2+it)|\,dt \gg H^{1-C_{18}\varepsilon};$$

then choose

$$X = H^{-1}\Big(\int_T^{T+H} |F(1/2+it)|\,dt\Big)^{1/2C_{18}}$$

to get

$$\int_T^{T+H} |F(1/2+it)|\,dt \gg H,$$

which proves the lemma.

Remark. For a more precise and general version of Lemma 3.12, we refer to [2].

4. Proof of the inequality (1.3). It is enough to prove (1.3) in the case of $Z(s)$. Others follow in the similar way. If $W_1(t)$ is of constant sign over the interval $[T, T+H]$, then we have

$$\Big|\int_T^{T+H} W_1(t)\,dt\Big| = \int_T^{T+H} |W_1(t)|\,dt. \tag{4.1}$$

By Lemmas 3.10 and 3.12, it follows that

$$o(H) + O(T^{1/2}\log T) = \Big|\int_T^{T+H} W_1(t)\,dt\Big| = \int_T^{T+H} |W_1(t)|\,dt > C_{21}H. \tag{4.2}$$

Since all the constants are effective, if we choose $H = C_{22}T^{1/2}\log T$ where C_{22} is a large effective positive constant, the inequality (4.2) is contradicted and hence $W_1(t)$ has a zero in every interval $(T, T + C_{22}T^{1/2}\log T)$ with T large enough. Now (1.3) follows.

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