Sums of coefficients of Hecke series

by

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1. Introduction and statement of results. Let

$$(1.1) \qquad H_j(s) = \sum_{n=1}^{\infty} t_j(n) n^{-s} \qquad (\operatorname{Re} s > 1)$$

be the Hecke series attached to the Maass wave form $\varphi_j(z)$ corresponding to the eigenvalue $\lambda_j = \kappa_j^2 + 1/4$ ($\kappa_j > 0$, $j \geq 1$) of the non-Euclidean Laplacian acting on the space of all non-holomorphic automorphic functions with respect to $\mathrm{SL}(2, \mathbb{Z})$. The set $\{\varphi_j\}$ forms an orthonormal base of the subspace spanned by all cusp forms. The $t_j(n)$'s are real-valued and are the eigenvalues of the Hecke operator $T(n)$, which means that, for $\operatorname{Im} z > 0$,

$$(T(n)\varphi_j)(z) = n^{-1/2} \sum_{ad=n} \sum_{b=1}^{d} \varphi_j\left(\frac{az+b}{d}\right) = t_j(n)\varphi_j(z).$$

The series in (1.1) converges absolutely only for $\operatorname{Re} s > 1$, but the function $H_j(s)$ can be continued analytically to an entire function over the s-plane. It satisfies the functional equation

$$(1.2) \qquad H_j(s) = X_j(s) H_j(1-s)$$

with

$$(1.3) \quad X_j(s) = 2^{2s-1} \pi^{2s-2} \Gamma(1-s+i\kappa_j) \Gamma(1-s-i\kappa_j)(\varepsilon_j \operatorname{ch}(\pi\kappa_j) - \cos(\pi s)).$$

Here ε_j is the parity sign of the Maass wave form φ_j, namely

$$\varepsilon_j = \begin{cases} 1 & \text{if } \varphi_j(-x+iy) = \varphi_j(x+iy), \\ -1 & \text{if } \varphi_j(-x+iy) = -\varphi_j(x+iy). \end{cases}$$

Starting with the pioneering works of N. V. Kuznetsov [17]–[19] and H. Iwaniec [11], spectral theory and Hecke series have become increasingly important in analytic number theory. Recently Y. Motohashi made heavy use of this theory (see [23], [24], [10] and Ch. 5 of [9]) to establish precise

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formulas for the fourth moment of $|\zeta(1/2 + it)|$. A comprehensive account on spectral theory and Maass wave forms is found in the works of H. Iwaniec [12], [13].

The aim of this work is to study the partial sum of the coefficients in (1.1), namely

$$(1.4) \qquad T_j(x) = {\sum_{n \leq x}}' t_j(n),$$

its mean square integral, and the summatory function of $t_j^2(n)$. In (1.4) the dash denotes that the last term in the sum is to be halved if x is an integer. The accent in our investigations is on the fact, often encountered in applications, that κ_j does not have to be fixed, but may vary with x. Naturally this causes difficulties, since all estimates must then be uniform in κ_j. The results that we obtain (see Theorems 1–3 below) are the sharpest ones hitherto.

If κ_j is fixed, then it was shown by Hafner–Ivić [7] that

$$(1.5) \qquad T_j(x) \ll x^{2/5}.$$

Such a result follows from a general theorem of Chandrasekharan–Narasimhan [3] on Dirichlet series with a functional equation containing gammafactors. The second author proved (Corollary to Theorem 1 in [20]) that

$$(1.6) \qquad T_j(x) - H_j(0) \ll_\varepsilon x^{(1+\alpha)/3+\varepsilon} \qquad (\kappa_j \leq x^{1/2}),$$

where as usual ε will denote arbitrarily small, positive constants, not necessarily the same at each occurrence. In fact, [20] treats the more general case when $t_j(n)$ in (1.4) is multiplied by an exponential factor $\exp(2\pi ihn/k)$. In (1.6), $\alpha \geq 0$ is a constant for which

$$(1.7) \qquad t_j(n) \ll_\varepsilon n^{\alpha+\varepsilon}$$

uniformly in κ_j. The famous Ramanujan–Petersson conjecture for Maass wave forms asserts that $\alpha = 0$. It is known that $\alpha \leq 1/5$ from the work of J.-P. Serre (see [22] and [25]), and recently D. Bump *et al.* [2] proved that $\alpha \leq 5/28$. In bounding $T_j(x)$ it will be enough to consider the range $x^{1/2} < \kappa_j \leq x$. Namely, Deshouillers–Iwaniec [5] proved that

$$(1.8) \qquad \sum_{n \leq x} |\varrho_j(n)|^2 \ll \mathrm{ch}(\pi\kappa_j)(x + \kappa_j)\log(2 + x/\kappa_j)$$

uniformly in κ_j, where $\varrho_j(n)$ denotes the nth Fourier coefficient of the Maass wave form $\varphi_j(z)$. Also for the quantity

$$(1.9) \qquad \alpha_j = |\varrho_j(1)|^2/\mathrm{ch}(\pi\kappa_j)$$

H. Iwaniec [14] proved the lower bound

(1.10) $$\alpha_j \gg_\varepsilon \kappa_j^{-\varepsilon}.$$

Now if one uses

(1.11) $$\varrho_j(n) = \varrho_j(1)t_j(n)$$

and the multiplicative property

(1.12) $$t_j(m)t_j(n) = \sum_{d|(m,n)} t_j(mn/d^2) \quad (m \geq 1,\ n \geq 1),$$

then it follows from (1.8) and (1.10) (see H. Iwaniec [15]) that, uniformly in κ_j,

(1.13) $$\sum_{n \leq x} t_j^2(n) \ll_\varepsilon \kappa_j^\varepsilon x.$$

Hence from (1.13) one obtains, by using the Cauchy–Schwarz inequality, that uniformly in κ_j,

(1.14) $$\sum_{n \leq x} t_j(n) \ll_\varepsilon \kappa_j^\varepsilon x.$$

Our first result improves (1.14) for $\kappa_j \leq x$:

THEOREM 1. *For any given $\varepsilon > 0$ and $x^{1/2} < \kappa_j \leq x$ we have*

(1.15) $$T_j(x) = {\sum_{n \leq x}}' t_j(n) \ll_\varepsilon \kappa_j^{1+\varepsilon}.$$

This is compatible with (1.6), since by the functional equation (1.2) we have $|H_j(0)| = |X_j(0)H_j(1)|$, and by (1.3),

$$|X_j(0)| = \frac{\kappa_j}{2\pi}\left|\frac{\varepsilon_j \,\mathrm{ch}(\pi\kappa_j) - 1}{\mathrm{sh}(\pi\kappa_j)}\right| \asymp \kappa_j,$$

while on using (1.6) and (1.14) one finds that

$$H_j(1) = \sum_{n=1}^\infty t_j(n)n^{-1} = \int\limits_{1-0}^\infty u^{-2}\Big(\sum_{n \leq u} t_j(n)\Big)\, du = \int\limits_{1-0}^{\kappa_j^4} + \int\limits_{\kappa_j^4}^\infty$$

$$= O_\varepsilon(\kappa_j^\varepsilon) + H_j(0)\kappa_j^{-2} = O_\varepsilon(\kappa_j^\varepsilon) + X_j(0)H_j(1)\kappa_j^{-2}.$$

Thus $H_j(1) \ll_\varepsilon \kappa_j^\varepsilon$ and consequently $\kappa_j|H_j(1)| \ll |H_j(0)| \ll_\varepsilon \kappa_j^{1+\varepsilon}$. This shows that $H_j(0)$ may be large if κ_j is not fixed. Therefore, apart from ε, the bound in (1.15) is in a certain sense the best one can hope for.

Our mean square result on $T_j(x)$ is contained in

THEOREM 2. *If $X < Y \leq 2X$, $\kappa_j \leq X^{1/2}$ and $h(\kappa_j)$ $(\ll \kappa_j^{\varepsilon-1/2})$ is given by (4.6), then for any given $\varepsilon > 0$ we have, uniformly in κ_j,*

$$(1.16) \quad \int_X^Y (T_j(x) - H_j(0) - h(\kappa_j))^2 \, dx$$

$$= \frac{1}{4\pi^2} \sum_{n=1}^{\infty} t_j^2(n) \int_X^Y x^{1/2} \left(n + \frac{1}{x}\left(\frac{\kappa_j}{2\pi}\right)^2 \right)^{-3/2} dx + O_\varepsilon(X^{1+\varepsilon}).$$

The analogue of Theorem 2 holds for holomorphic cusp forms. This is due to M. Jutila [16] (see Th. 1.2 and Notes to Ch. 1).

From (1.16) we obtain without difficulty the following

COROLLARY 1. *For $\kappa_j \leq X^{1/2}$ and any given $\varepsilon > 0$ we have, uniformly in κ_j,*

$$(1.17) \quad \int_X^{2X} T_j^2(x) \, dx = \left\{ \frac{\sqrt{8}-1}{6\pi^2} \sum_{n=1}^{\infty} t_j^2(n) n^{-3/2} \right\} X^{3/2} + H_j^2(0)X$$

$$+ O_\varepsilon\{(\kappa_j^2 X^{1/2} + \kappa_j^{1/2}X + \kappa_j X^{3/4})X^\varepsilon\}.$$

If κ_j is fixed, then

$$(1.18) \quad \int_1^X T_j^2(x) \, dx = \left\{ \frac{1}{6\pi^2} \sum_{n=1}^{\infty} t_j^2(n) n^{-3/2} \right\} X^{3/2} + O_\varepsilon(X^{1+\varepsilon}).$$

Note that, for $Y = 2X$, the main term on the right-hand side of (1.16) is

$$\geq \frac{1}{4\pi^2} t_j^2(1) \int_X^{2X} x^{1/2} \, dx = \frac{\sqrt{8}-1}{6\pi^2} X^{3/2}.$$

Suppose now that $T_j(x) = o(x^{1/4})$ as $x \to \infty$. Since $H_j(0) \ll_\varepsilon \kappa_j^{1+\varepsilon}$, for $\kappa_j \leq x^{1/4-\varepsilon_1}$ and any given $\varepsilon_1 > 0$ we deduce that the left-hand side of (1.16) is $o(X^{3/2})$, which is a contradiction. Thus we obtain

COROLLARY 2. *If $\kappa_j \leq x^{1/4-\varepsilon_1}$ for any given $\varepsilon_1 > 0$, then $T_j(x) = \Omega(x^{1/4})$.*

To assess the strength of Theorem 2 it may be remarked that (1.18) is analogous to the mean square formula

$$(1.19) \quad \int_1^X \Delta^2(x) \, dx = \left\{ \frac{1}{6\pi^2} \sum_{n=1}^{\infty} d^2(n) n^{-3/2} \right\} X^{3/2} + O(X \log^4 X),$$

where

$$\Delta(x) = \sum_{n \leq x}{}' d(n) - x(\log x + 2\gamma - 1) - 1/4$$

is the error term in the classical divisor problem. Here $d(n)$ is the number of divisors of n, and γ is Euler's constant. The omega-result $T_j(x) = \Omega(x^{1/4})$ is the analogue of the classical omega-result $\Delta(x) = \Omega(x^{1/4})$, which follows from (1.19). For $\Delta(x)$ even sharper omega-results are known, but their analogues for $T_j(x)$ are hard to obtain, since little is known about the arithmetical structure of the $t_j(n)$'s.

The error term in (1.19) is the sharpest one known. It is due to E. Preissmann [26], while K.-C. Tong [30] and T. Meurman [21] had $O(X \log^5 X)$. The method of [21] was a substantial simplification of Tong's work, and yielded also the analogue of (1.19) for the important function

$$E(T) := \int_0^T |\zeta(1/2 + it)|^2 \, dt - T(\log(T/(2\pi)) + 2\gamma - 1).$$

It can clearly be used for general mean square formulas, for example the ones considered by Chandrasekharan–Narasimhan [4]. A variant of this method will be used for the proof of Theorem 2. Since the function $\Delta(x)$ is less difficult to handle than $T_j(x)$, the error term in (1.16) appears to be the limit of the present methods.

To deal with the summatory function of $t_j^2(n)$ we introduce the arithmetical function

$$(1.20) \qquad c_j(n) := \sum_{d^2 \mid n} |\varrho_j(n/d^2)|^2.$$

In terms of the generating Dirichlet series, (1.20) is equivalent to

$$(1.21) \qquad C_j(s) = \zeta(2s)L_j(s),$$

where

$$(1.22) \qquad L_j(s) := \sum_{n=1}^{\infty} |\varrho_j(n)|^2 n^{-s}, \qquad C_j(s) := \sum_{n=1}^{\infty} c_j(n) n^{-s}.$$

From (1.11) and (1.13) we obtain, uniformly in κ_j,

$$(1.23) \qquad \sum_{n \le x} |\varrho_j(n)|^2 \ll_\varepsilon \alpha_j \kappa_j^\varepsilon \operatorname{ch}(\pi\kappa_j) x, \qquad \sum_{n \le x} c_j(n) \ll_\varepsilon \alpha_j \kappa_j^\varepsilon \operatorname{ch}(\pi\kappa_j) x,$$

so that both series in (1.22) converge absolutely for $\operatorname{Re} s > 1$. In fact, $C_j(s)$ possesses meromorphic continuation having only a simple pole at $s = 1$ with residue $2\operatorname{ch}(\pi\kappa_j)$. It satisfies the functional equation (see N. V. Proskurin [27] for a proof of these facts)

$$(1.24) \qquad \pi^{-2s} C_j(s)\theta_j(s) = \pi^{-2(1-s)} C_j(1-s)\theta_j(1-s),$$

where

$$(1.25) \qquad \theta_j(s) = \Gamma(s/2 + i\kappa_j)\Gamma(s/2 - i\kappa_j)\Gamma^2(s/2).$$

The proof of (1.24)–(1.25) is obtained by the so-called Rankin–Selberg convolution method (see D. Bump [1] for an extensive account).

Suppose now that κ_j is fixed. As in the proof of (1.5) we can appeal to the method of Chandrasekharan–Narasimhan [3] to evaluate the summatory function of $c_j(n)$. In their notation $q = r = \delta = 1$, $A = 2$, $u = 3/8$. We use their Remark (5.5), since (e.g. by the Wiener–Ikehara–Tauberian theorem) $\sum_{n\leq x} c_j(n) = (1 + o(1))2x \operatorname{ch}(\pi\kappa_j)$ as $x \to \infty$, so that $a = b = 1$. Thus (4.2) of [3] yields

$$(1.26) \qquad \sum_{n\leq x} c_j(n) = 2x \operatorname{ch}(\pi\kappa_j) + O(x^{3/8+3\eta/2}) + O(x^{3/4-\eta})$$

$$= 2x \operatorname{ch}(\pi\kappa_j) + O(x^{3/5})$$

if $\eta = 3/20$. In view of (1.9), (1.11) and (1.20) it is seen that from (1.26) one obtains by a convolution argument

$$(1.27) \qquad \sum_{n\leq x} t_j^2(n) = \frac{12x}{\pi^2\alpha_j} + O(x^{3/5}).$$

This result corresponds to R. A. Rankin's classical theorem [28] for holomorphic cusp forms. H. Iwaniec [12], [13] states (1.27) with the incorrect constant $6/(\pi^2\alpha_j)$ in the main term, as is also done on p. 244 of [9].

When κ_j is not fixed, we naturally cannot expect to obtain formulas for the above sum with error term as good as $O(x^{3/5})$. What we shall prove is

THEOREM 3. If α is the constant defined by (1.7) and $\kappa_j \leq x^{1-\alpha}$ then for any given $\varepsilon > 0$ we have, uniformly in κ_j,

$$(1.28) \quad \sum_{n\leq x} t_j^2(n) = \frac{12x}{\pi^2\alpha_j} + O_\varepsilon(x^\varepsilon(\kappa_j^{1/(2-2\alpha)}x^{1/2}$$

$$+ \min(\kappa_j^{(1+10\alpha)/(3+6\alpha)}x^{1/2} + x^{(3+6\alpha)/5}, \kappa_j^{-1}x^{1+2\alpha}))).$$

The asymptotic formula (1.28) shows that, under the truth of the Ramanujan–Petersson conjecture $\alpha = 0$, we can extend (1.27) (with an additional ε-factor) to the range $\kappa_j \leq x^{1/5}$, namely

$$(1.29) \qquad \sum_{n\leq x} t_j^2(n) = \frac{12x}{\pi^2\alpha_j} + O_\varepsilon(x^{3/5+\varepsilon}) \quad (\kappa_j \leq x^{1/5}).$$

If $1/10 < \alpha < 1/2$ and $\kappa_j \leq x^{(3+6\alpha)/8}$ then the error terms in (1.28) are

$$O_\varepsilon(x^\varepsilon(\kappa_j^{(1+10\alpha)/(3+6\alpha)}x^{1/2} + x^{(3+6\alpha)/5})).$$

Note that (1.28) improves the result of N. V. Proskurin [27] (in the special case when $\kappa_{j_1} = \kappa_{j_2} = \kappa_j$), who investigated the summatory function of $\varrho_{j_1}(n)\overline{\varrho_{j_2}(n)}$.

2. Proof of Theorem 1. For the proof of Theorem 1 we shall need the following

LEMMA 1. *Let $s = \sigma + it$, $-\varepsilon \leq \sigma \leq 1 + \varepsilon$ for a small, fixed $\varepsilon > 0$, and $t \geq 1$. Then*

$$(2.1) \qquad H_j(s) \ll_\varepsilon (\kappa_j + t)^{1-\sigma+2\varepsilon},$$

and if $|t - \kappa_j| \geq 2$, then

$$(2.2) \quad X_j(s) = \frac{H_j(s)}{H_j(1-s)}$$

$$= \pi^{2s-1} c_t \left(\tfrac{1}{4}|t^2 - \kappa_j^2|\right)^{1/2-s} \left|\frac{t - \kappa_j}{t + \kappa_j}\right|^{i\kappa_j} e^{2it} \left(1 + O\left(\frac{1}{|t - \kappa_j|}\right)\right),$$

where

$$c_t = \begin{cases} \varepsilon_j & \text{if } t < \kappa_j - 2, \\ i & \text{if } t > \kappa_j + 2. \end{cases}$$

Proof. By using standard properties of the gamma-function one can rewrite the functional equation (1.2) as

$$\theta_j(s) H_j(s) = \theta_j(1-s) H_j(1-s) \qquad \text{if } \varepsilon_j = 1,$$
$$\theta_j(1+s) H_j(s) = -\theta_j(2-s) H_j(1-s) \quad \text{if } \varepsilon_j = -1,$$

where

$$\theta_j(s) = \pi^{-s} \Gamma\left(\frac{s}{2} + \frac{i}{2}\kappa_j\right) \Gamma\left(\frac{s}{2} - \frac{i}{2}\kappa_j\right).$$

Hence by using Stirling's formula in the form

$$\Gamma(s) = \sqrt{2\pi}|t|^{\sigma-1/2} \exp\left\{-\frac{\pi}{2}|t| + i\left(t \log|t| - t + \frac{\pi t}{2|t|}\left(\sigma - \frac{1}{2}\right)\right)\right\}$$
$$\times \left(1 + O\left(\frac{1}{|t|}\right)\right) \qquad (|t| \geq 1),$$

one obtains (2.2). By (1.13) we have

$$|H_j(1 + \varepsilon + it)| \leq \sum_{n=1}^{\infty} |t_j(n)| n^{-1-\varepsilon} \ll_\varepsilon \kappa_j^\varepsilon,$$

and by (2.2),

$$|H_j(-\varepsilon + it)| = |H_j(1 + \varepsilon + it) X_j(-\varepsilon + it)| \ll_\varepsilon \kappa_j^\varepsilon (\kappa_j + t)^{1+2\varepsilon}.$$

Therefore (2.1) follows by the Phragmén–Lindelöf principle (see (A.36) of [8]).

We now pass to the proof of Theorem 1, supposing that $x^{1/2} < \kappa_j \leq x$ and starting from the Perron inversion formula for Dirichlet series (see e.g.

(A.10) of [8]). If $x^{\varepsilon_1} \le T \le x^{1-\varepsilon_1}$ and α is the constant defined by (1.7), then by the residue theorem we have

$$(2.3) \quad \sum_{n \le x} t_j(n)$$

$$= \frac{1}{2\pi i} \int_{1+\varepsilon-iT}^{1+\varepsilon+iT} H_j(s) x^s \frac{ds}{s} + O_\varepsilon(x^{1+\alpha+\varepsilon}T^{-1})$$

$$= \frac{1}{2\pi i} \int_{-\varepsilon-iT}^{-\varepsilon+iT} H_j(s) x^s \frac{ds}{s} + H_j(0)$$

$$+ O_\varepsilon\left(x^{1+\alpha+\varepsilon}T^{-1} + \max_{-\varepsilon \le \sigma \le 1+\varepsilon}(\kappa_j + T)^{1-\sigma+2\varepsilon}x^\sigma T^{-1}\right)$$

$$= \frac{1}{2\pi i} \int_{-\varepsilon-iT}^{-\varepsilon+iT} H_j(1-s) X_j(s) x^s \frac{ds}{s}$$

$$+ O_\varepsilon(\kappa_j^{1+\varepsilon} + x^{1+\alpha+\varepsilon}T^{-1})$$

$$= \sum_{n \le \kappa_j^2/x} t_j(n)\left(\frac{1}{2\pi i} \int_{-\varepsilon-iT}^{-\varepsilon+iT} n^{s-1} X_j(s) x^s \frac{ds}{s}\right)$$

$$+ \sum_{n > \kappa_j^2/x} t_j(n) n^{-\varepsilon-1} \operatorname{Re}\left\{\frac{1}{\pi x^\varepsilon} \int_1^T (xn)^{it} X_j(-\varepsilon + it) \frac{dt}{-\varepsilon + it}\right\}$$

$$+ O_\varepsilon(\kappa_j^{1+\varepsilon} + x^{1+\alpha+\varepsilon}T^{-1})$$

$$= \sum_1 + \sum_2 + O_\varepsilon(\kappa_j^{1+\varepsilon} + x^{1+\alpha+\varepsilon}T^{-1}),$$

say. Here we used (2.1), the functional equation (1.2), and exchanged the order of summation and integration in view of the absolute convergence of $H_j(1-s)$ for $\operatorname{Re} s < 0$. In the integrals in $\sum_1$, which is a non-empty sum since $\kappa_j > x^{1/2}$, we replace the segment of integration $[-\varepsilon - iT, -\varepsilon + iT]$ by the segment $[1/2 - iT, 1/2 + iT]$. By the residue theorem, (1.3) and (2.2) we obtain

$$\frac{1}{2\pi i} \int_{-\varepsilon-iT}^{-\varepsilon+iT} n^s X_j(s) x^s \frac{ds}{s}$$

$$= -X_j(0) + O\{(nx)^{1/2} \log T + \max_{-\varepsilon \le \sigma \le 1/2}(nx)^\sigma T^{-1}(\kappa_j + T)^{1-2\sigma}\}$$

$$\ll \kappa_j + (nx)^{1/2} \log T.$$

Therefore by using (1.13) we obtain

$$(2.4) \qquad \sum_1 \ll \kappa_j \sum_{n \leq \kappa_j^2/x} |t_j(n)| n^{-1} + x^{1/2} \log T \sum_{n \leq \kappa_j^2/x} |t_j(n)| n^{-1/2}$$

$$\ll_\varepsilon \kappa_j^{1+2\varepsilon}.$$

To estimate $\sum_2$ we use (2.2) to write

$$(2.5) \qquad \int_1^T (xn)^{it} X_j(-\varepsilon + it) \frac{dt}{-\varepsilon + it}$$

$$= \int_1^{T*} (xn)^{it} c_t \pi^{-1-2\varepsilon+2it} \left(\tfrac{1}{4}|t^2 - \kappa_j^2|\right)^{1/2+\varepsilon-it} \left|\frac{t - \kappa_j}{t + \kappa_j}\right|^{i\kappa_j} e^{2it} \frac{dt}{it}$$

$$+ O_\varepsilon(\kappa_j^{1+2\varepsilon})$$

$$= \int_1^{T*} d_t |t^2 - \kappa_j^2|^{1/2+\varepsilon} e^{iF(t)} \frac{dt}{t} + O_\varepsilon(\kappa_j^{1+2\varepsilon}),$$

where d_t is piecewise constant,

$$F(t) = F(t; \kappa_j, x, n) := 2t + \kappa_j \log\left|\frac{t - \kappa_j}{t + \kappa_j}\right| - t \log|t^2 - \kappa_j^2| + t \log(4\pi^2 xn),$$

and $*$ indicates that the interval $[\kappa_j - 2, \kappa_j + 2]$ is excluded from integration if it intersects $[1, T]$ (its contribution is trivially $\ll 1$). The total error coming from the error term in (2.2) and from replacing $-\varepsilon + it$ by it is $\ll_\varepsilon \kappa_j^{1+2\varepsilon}$. We divide the interval of integration in the last integral in $O(\log T)$ subintervals of the form $[T_1, T_2]$, where $1 \ll T_1 < T_2 \leq 2T_1 \leq T$, and note that

$$F'(t) = \log \frac{4\pi^2 xn}{|t^2 - \kappa_j^2|}, \qquad F''(t) = \frac{-2t}{t^2 - \kappa_j^2}.$$

In the subinterval of $[T_1, T_2]$ in which

$$(2.6) \qquad\qquad\qquad |F''(t)| \geq 2/t$$

we estimate the corresponding integral by the second derivative test (Lemma 2.2 of [8]) as

$$(2.7) \qquad \ll_\varepsilon (T_1 + \kappa_j)^{1+2\varepsilon} T_1^{-1/2} \ll_\varepsilon T^{1/2+2\varepsilon} + \kappa_j^{1+2\varepsilon},$$

providing that the subinterval in question is not empty. If (2.6) fails to hold, then $t^2 < \tfrac{1}{2}\kappa_j^2$ holds. Hence $|t^2 - \kappa_j^2| < \tfrac{3}{2}\kappa_j^2$ and

$$F'(t) \geq \log \frac{4\pi^2 \kappa_j^2}{|t^2 - \kappa_j^2|} > \log \frac{8\pi^2}{3} > 3.$$

Therefore, by the first derivative test (Lemma 2.1 of [8]), the bound in (2.7) certainly holds for the corresponding portion of the integral over $[T_1, T_2]$. Hence we obtain

$$(2.8) \quad \sum_2 \ll_\varepsilon (T^{1/2+2\varepsilon} + \kappa_j^{1+2\varepsilon}) \sum_{n > \kappa_j^2/x} |t_j(n)| n^{-\varepsilon-1} \ll_\varepsilon (T^{1/2} + \kappa_j) x^{3\varepsilon}.$$

From (2.3), (2.4) and (2.8) we obtain

$$(2.9) \quad \sum_{n \leq x} t_j(n) \ll_\varepsilon (\kappa_j + x^{1+\alpha} T^{-1} + T^{1/2}) x^{3\varepsilon} \ll_\varepsilon (\kappa_j + x^{(1+\alpha)/3}) x^{3\varepsilon}$$

if $T = x^{(2+2\alpha)/3}$. Since $x^{(1+\alpha)/3} \leq \kappa_j^{(2+2\alpha)/3} \leq \kappa_j^{5/6}$, Theorem 1 follows from (2.9).

3. Some auxiliary results. In this section we lay the groundwork for the proof of Theorem 2, and state five lemmas. In the next section we shall obtain a suitable explicit formula for $T_j(x)$ that will be squared and integrated termwise to produce Theorem 2. It would be possible to treat also the case $\kappa_j > X^{1/2}$, but the difficulties involving the transitional region of the K-Bessel function terms would be overwhelming, and for this reason we shall be working with the condition $\kappa_j \leq X^{1/2}$. Such a condition, however, is not necessary for this section, except in Lemma 2.

LEMMA 2. *Let $1 \ll N \ll x$, $\kappa_j \leq x^{1/2}$. Then for any given $\varepsilon > 0$ we have uniformly*

$$(3.1) \quad T_j(x) = \frac{x^{1/4}}{\pi\sqrt{2}} \sum_{n \leq N} t_j(n) \left(n + \frac{\kappa_j^2}{4\pi^2 x} \right)^{-3/4} \sin G_{\kappa_j}(4\pi\sqrt{nx})$$

$$+ H_j(0) + O_\varepsilon(x^{1/2+\alpha+\varepsilon} N^{-1/2}),$$

where α is the constant in (1.7) and

$$(3.2) \qquad G_\kappa(u) := (u^2 + 4\kappa^2)^{1/2} - 2\kappa \operatorname{arsh}\left(\frac{2\kappa}{u}\right) + \frac{\pi}{4},$$

$$\operatorname{arsh} u = \log(u + (u^2 + 1)^{1/2}).$$

LEMMA 3. *For $x > 0$ we have*

$$(3.3) \quad T_j(x) = \frac{\pi i}{\operatorname{sh}(\pi\kappa_j)} \sum_{n=1}^{\infty} t_j(n) \int_0^x (J_{2i\kappa_j}(4\pi\sqrt{nv}) - J_{-2i\kappa_j}(4\pi\sqrt{nv}))\, dv$$

$$+ 4\varepsilon_j \operatorname{ch}(\pi\kappa_j) \sum_{n=1}^{\infty} t_j(n) \int_0^x K_{2i\kappa_j}(4\pi\sqrt{nv})\, dv.$$

The series occurring in (3.3) are boundedly convergent for x lying in any fixed interval $[x_1, x_2] \subset (0, \infty)$.

The proofs of Lemmas 2 and 3 are found in [20]. For the definitions and relevant properties of the J- and K-Bessel functions the reader is referred to the monograph of G. N. Watson [31].

LEMMA 4. *For $x > 0$ and $\kappa \gg 1$ we have*

$$(3.4) \quad \frac{-J_{2i\kappa}(x) + J_{-2i\kappa}(x)}{\sin(i\pi\kappa)}$$

$$= 2\pi^{-1/2}\left(\frac{x^2}{4} + \kappa^2\right)^{-1/4} \cos G_\kappa(x)$$

$$+ \left(\frac{x^2}{4} + \kappa^2\right)^{-3/4} (a^+(x,\kappa)e^{iG_\kappa(x)} + a^-(x,\kappa)e^{-iG_\kappa(x)})$$

$$+ O((x^2 + \kappa^2)^{-5/4}),$$

where $G_\kappa(x)$ is given by (3.2) and

$$(3.5) \qquad a^\pm(x,\kappa) \ll 1, \quad \frac{\partial}{\partial x}a^\pm(x,\kappa) \ll 1/x.$$

This is deduced from the well-known asymptotic formulas for the J-Bessel function (see A. Erdélyi *et al.* [6] or T. Meurman [20]). We shall also need two lemmas on integrals with the J-Bessel function. Henceforth we let for brevity

$$(3.6) \qquad J := \frac{-J_{2i\kappa}(u) + J_{-2i\kappa}(u)}{\sin(i\pi\kappa)}, \quad K := \frac{u^2}{4} + \kappa^2.$$

Then we have

LEMMA 5. *For $\kappa \gg 1$ the integral*

$$I := \int\limits_0^\infty \{uJ - \pi^{-1/2}(u^2 K^{-3/4} \sin G_\kappa(u))'\}\,du$$

is convergent, and moreover,

$$(3.7) \qquad\qquad I \ll \kappa^{-1/2}.$$

Proof. We have

$$(3.8) \quad (u^2 K^{-3/4} \sin G_\kappa(u))' = 2uK^{-3/4} \sin G_\kappa(u) + u^2(K^{-3/4} \sin G_\kappa(u))',$$

$$(3.9) \qquad\qquad G_\kappa'(u) = 2u^{-1}K^{1/2}.$$

Therefore by the first derivative test $\int_0^\infty uK^{-3/4} \sin G_\kappa(u)\,du \ll \kappa^{-1/2}$, and it remains to show that $I = I(1)$ converges and that $I(1) \ll \kappa^{-1/2}$, where we set

$$I(s) := \int\limits_0^\infty u^s\{J - \pi^{-1/2}u(K^{-3/4} \sin G_\kappa(u))'\}\,du.$$

In fact, $I(s)$ is holomorphic for $0 < \operatorname{Re} s \leq 1$. Namely, by using (3.9) one can write $I(s) = I_1(s) + I_2(s)$, where

$$I_1(s) = \frac{3}{8\sqrt{\pi}} \int_0^\infty u^{s+2} K^{-7/4} \sin G_\kappa(u)\, du,$$

$$I_2(s) = \int_0^\infty u^s \{J - 2\pi^{-1/2} K^{-1/4} \cos G_\kappa(u)\}\, du.$$

The holomorphy of $I_1(s)$ for $0 < \operatorname{Re} s \leq 1$ follows then by the first derivative test, and that of $I_2(s)$ by Lemma 4.

Let now $0 < \operatorname{Re} s < 1/2$. By using the Mellin transform formula

$$\int_0^\infty J_\nu(x) x^{s-1}\, dx = \frac{2^{s-1} \Gamma((s+\nu)/2)}{\Gamma((\nu-s)/2+1)} \qquad (-\operatorname{Re}\nu < \operatorname{Re} s < 3/2)$$

it follows that

$$(3.10) \qquad \int_0^\infty u^s J\, du = -\pi^{-1} 2^{s+1} \sin\left(\frac{\pi s}{2}\right) \Gamma\left(\frac{s+1}{2} + i\kappa\right) \Gamma\left(\frac{s+1}{2} - i\kappa\right).$$

Integration by parts yields

$$(3.11) \qquad \int_0^\infty u^{s+1} (K^{-3/4} \sin G_\kappa(u))'\, du = -(s+1) \int_0^\infty u^s K^{-3/4} \sin G_\kappa(u)\, du,$$

and the integral on the right-hand side of (3.11) is, by the first derivative test, holomorphic for $0 < \operatorname{Re} s \leq 1$. Hence by analytic continuation we infer from (3.10) and (3.11) that

$$I(1) = -4\pi^{-1} |\Gamma(1 + i\kappa)|^2 + 2\pi^{-1/2} \int_0^\infty u K^{-3/4} \sin G_\kappa(u)\, du.$$

By the first derivative test it is seen that the above integral converges and is $\ll \kappa^{-1/2}$. This proves the lemma.

LEMMA 6. *For $\kappa \gg 1$,*

$$(3.12) \qquad \int_U^\infty \{uJ - \pi^{-1/2}(u^2 K^{-3/4} \sin G_\kappa(u))'\}\, du \ll (U^2 + \kappa^2)^{-1/4}.$$

P r o o f. We note that by (3.8) and (3.9),

$$(u^2 K^{-3/4} \sin G_\kappa(u))' = 2uK^{-3/4} \sin G_\kappa(u)$$
$$- \tfrac{3}{8} u^3 K^{-7/4} \sin G_\kappa(u) + 2uK^{-1/4} \cos G_\kappa(u),$$

and we insert this expression in (3.12). The third term is cancelled by the

main term given by Lemma 4. Hence it remains to show that

$$(3.13) \quad \left| \int_U^\infty uK^{-3/4} \sin G_\kappa(u) \, du \right| + \left| \int_U^\infty u^3 K^{-7/4} \sin G_\kappa(u) \, du \right|$$

$$\ll (U^2 + \kappa^2)^{-1/4},$$

$$(3.14) \quad \int_U^\infty uK^{-3/4} a^\pm(u, \kappa) e^{\pm i G_\kappa(u)} \, du \ll (U^2 + \kappa^2)^{-1/4}$$

and

$$(3.15) \quad \int_U^\infty uK^{-5/4} \, du \ll (U^2 + \kappa^2)^{-1/4}.$$

The bound in (3.13) follows from the first derivative test, from the one in (3.14) by an integration by parts and from (3.5), and the one given by (3.15) is trivial.

4. The explicit formula for $T_j(x)$. In this section we derive an explicit formula for $T_j(x)$, which is suitable for the proof of Theorem 2. We start from (3.3) of Lemma 3.

By using Lemmas 5 and 6 ($G_\kappa(u)$ is defined by (3.2)) we find that

$$(4.1) \quad \frac{\pi i}{\mathrm{sh}(\pi\kappa)} \int_0^x \left\{ J_{2i\kappa}(4\pi\sqrt{nv}) - J_{-2i\kappa}(4\pi\sqrt{nv}) \right\} dv$$

$$= 2\sqrt{\pi}x(4\pi^2 nx + \kappa^2)^{-3/4} \sin G_\kappa(4\pi\sqrt{nx})$$

$$+ \frac{1}{8\pi n} \int_0^{4\pi\sqrt{nx}} \left\{ u \frac{J_{2i\kappa}(u) - J_{-2i\kappa}(u)}{-i\,\mathrm{sh}(\pi\kappa)} \right.$$

$$\left. - \frac{1}{\sqrt{\pi}} \left(u^2 \left(\frac{u^2}{4} + \kappa^2 \right)^{-3/4} \sin G_\kappa(u) \right)' \right\} du$$

$$= 2\sqrt{\pi}x(4\pi^2 nx + \kappa^2)^{-3/4} \sin G_\kappa(4\pi\sqrt{nx})$$

$$+ \frac{1}{8\pi n} \int_0^\infty \left\{ u \frac{J_{2i\kappa}(u) - J_{-2i\kappa}(u)}{-i\,\mathrm{sh}(\pi\kappa)} \right.$$

$$\left. - \frac{1}{\sqrt{\pi}} \left(u^2 \left(\frac{u^2}{4} + \kappa^2 \right)^{-3/4} \sin G_\kappa(u) \right)' \right\} du$$

$$+ O(n^{-1}(nx + \kappa^2)^{-1/4}).$$

To transform the second series in (3.3) note first that

$$\int\limits_0^\infty K_{2i\kappa}(4\pi\sqrt{nv})\,dv = \frac{|\Gamma(1+i\kappa)|^2}{8\pi^2 n}.$$

Hence

$$(4.2) \quad 4\varepsilon_j\,\mathrm{ch}(\pi\kappa_j)\int\limits_0^x K_{2i\kappa_j}(4\pi\sqrt{nv})\,dv$$

$$= -4\varepsilon_j\,\mathrm{ch}(\pi\kappa_j)\int\limits_x^\infty K_{2i\kappa_j}(4\pi\sqrt{nv})\,dv + \frac{1}{2\pi^2 n}\varepsilon_j\,\mathrm{ch}(\pi\kappa_j)|\Gamma(1+i\kappa_j)|^2$$

$$= -4\varepsilon_j\,\mathrm{ch}(\pi\kappa_j)\int\limits_x^\infty K_{2i\kappa_j}(4\pi\sqrt{nv})\,dv + \left\{X_j(0) + \frac{1}{2\pi^2}|\Gamma(1+i\kappa_j)|^2\right\}\frac{1}{n},$$

where $X_j(s)$ is given by (1.3). At this point we recall that (see G. N. Watson [31])

$$K_{2i\kappa_j}(x) = \frac{1}{2\,\mathrm{ch}(\pi\kappa_j)}\int\limits_{-\infty}^\infty \cos(x\,\mathrm{sh}\,t)\cos(2\kappa_j t)\,dt.$$

In the above integral write

$$\cos(x\,\mathrm{sh}\,t) = \mathrm{Re}(e^{ix\,\mathrm{sh}\,t})$$

and shift the line of integration to $\mathrm{Im}\,t = \pi/2$. Since

$$\mathrm{sh}(t + \pi i/2) = i\,\mathrm{ch}\,t$$

we obtain

$$(4.3) \qquad K_{2i\kappa_j}(x) \ll \int\limits_0^\infty e^{-x\,\mathrm{ch}\,t}\,dt \le \int\limits_0^\infty e^{-x(1+t^2/2)}\,dt = \left(\frac{\pi}{2x}\right)^{1/2}e^{-x}$$

uniformly in κ_j. At this point we suppose that $X \le x \le Y \le 2X$, and make use of the condition $\kappa_j \le X^{1/2}$, required for Theorem 2. Then by using (4.3) we have

$$\mathrm{ch}(\pi\kappa_j)\int\limits_x^\infty K_{2i\kappa_j}(4\pi\sqrt{nv})\,dv \ll e^{\pi\kappa_j}\int\limits_x^\infty (nv)^{-1/4}e^{-4\pi\sqrt{nv}}\,dv$$

$$\ll e^{\pi\kappa_j}n^{-1}\int\limits_{4\pi\sqrt{nx}}^\infty u^{1/2}e^{-u}\,du$$

$$\ll n^{-1}(nx)^{1/4}e^{\pi\kappa_j - 4\pi\sqrt{nx}}$$

$$\ll n^{-1}(nx + \kappa_j^2)^{-1/4},$$

which is the same as the error term in (4.1). Thus (4.2) gives

$$(4.4) \quad 4\varepsilon_j \operatorname{ch}(\pi\kappa_j) \int\limits_0^x K_{2i\kappa_j}(4\pi\sqrt{nv})\, dv$$

$$= \left\{ X_j(0) + \frac{1}{2\pi^2}|\Gamma(1+i\kappa_j)|^2 \right\} \frac{1}{n} + O\left(\frac{1}{n}(nx + \kappa_j^2)^{-1/4} \right).$$

We insert (4.1) and (4.4) in (3.3), noting that by the functional equation (1.2),

$$X_j(0) \sum_{n=1}^{\infty} t_j(n)n^{-1} = X_j(0)H_j(1) = H_j(0),$$

and using (see (1.13))

$$\sum_{n=1}^{\infty} |t_j(n)|n^{-1}(nx + \kappa_j^2)^{-1/4} \ll_{\varepsilon} \kappa_j^{\varepsilon} x^{-1/4}.$$

The resulting formula is contained in

LEMMA 7. *Suppose* $\kappa_j \leq X^{1/2}$ *and* $X \leq x \leq Y \leq 2X$, *let* $G_\kappa(u)$ *be given by* (3.2), *and let*

$$(4.5) \quad S_j(x) := \frac{x^{1/4}}{\pi\sqrt{2}} \sum_{n=1}^{\infty} t_j(n) \left(n + \frac{1}{x}\left(\frac{\kappa_j^2}{2\pi} \right) \right)^{-3/4} \sin G_{\kappa_j}(4\pi\sqrt{nx}),$$

$$(4.6) \quad h(\kappa) := \left\{ \frac{|\Gamma(1+i\kappa)|^2}{2\pi^2} + \frac{1}{8\pi} \int\limits_0^{\infty} \left(u\frac{J_{2i\kappa}(u) - J_{-2i\kappa}(u)}{-i\operatorname{sh}(\pi\kappa)} \right. \right.$$

$$\left. \left. - \pi^{-1/2}\left(u^2\left(\frac{u^2}{4} + \kappa^2 \right)^{-3/4} \sin G_\kappa(u) \right)' \right) du \right\} H_j(1).$$

Then we have, uniformly in κ_j,

$$(4.7) \quad T_j(x) = S_j(x) + H_j(0) + h(\kappa_j) + O_\varepsilon(\kappa_j^\varepsilon X^{-1/4}).$$

We shall now further transform the above formula for $T_j(x)$, but we wish to note here that by Lemma 5,

$$(4.8) \quad h(\kappa_j) \ll_\varepsilon \kappa_j^{\varepsilon-1/2},$$

since $H_j(1) \ll_\varepsilon \kappa_j^\varepsilon$. Thus $h(\kappa_j)$ is the "small" constant appearing in (4.7), whereas $H_j(0)$ may be considered "large", in view of

$$|H_j(0)| = |H_j(1)X_j(0)| \gg \kappa_j|H_j(1)|.$$

Our goal is to truncate the series $S_j(x)$ and estimate the resulting error.

Let for brevity $q = (\kappa_j/(2\pi))^2$ and consider

$$(4.9) \quad V = V(x; M, \kappa_j) := x^{1/4} \sum_{n>M} t_j(n) \left(n + \frac{q}{x} \right)^{-3/4} \sin G_{\kappa_j}(4\pi\sqrt{nx}),$$

where $M \geq x^3$, so that $\kappa_j \leq X^{1/2} \leq x^{1/2} \leq M^{1/6}$. By partial summation and (1.6) it follows that

$$
\begin{aligned}
V = & - x^{1/4} \int_M^\infty \left(\sum_{n \leq y} t_j(n) \right) \{ (y + q/x)^{-3/4} \sin G_{\kappa_j}(4\pi\sqrt{yx}) \}' \, dy \\
& - x^{1/4} \left(\sum_{n \leq M} t_j(n) \right) (M + q/x)^{-3/4} \sin G_{\kappa_j}(4\pi\sqrt{Mx}) \\
= & - 2\pi x^{3/4} \int_M^\infty \left(\sum_{n \leq y} t_j(n) \right) y^{-1} (y + q/x)^{-1/4} \cos G_{\kappa_j}(4\pi\sqrt{yx}) \, dy \\
& + O_\varepsilon(x^{1/4} M^{-1/4}) \\
= & - 2\pi x^{3/4} \sum_{k=0}^\infty I(2^k M) + O_\varepsilon(x^{1/4} M^{-1/4}),
\end{aligned}
$$

where

$$(4.10) \quad I(W) := \int_W^{2W} \left(\sum_{n \leq y} t_j(n) \right) y^{-1} (y + q/x)^{-1/4} \cos G_{\kappa_j}(4\pi\sqrt{yx}) \, dy.$$

We replace the sum in (4.10) by the expression given by Lemma 2, namely

$$
\sum_{n \leq y} t_j(n) = \frac{y^{1/4}}{\pi\sqrt{2}} \sum_{n \leq W} t_j(n)(n + q/y)^{-3/4} \sin G_{\kappa_j}(4\pi\sqrt{ny})
$$
$$
+ H_j(0) + O_\varepsilon(W^{\alpha+\varepsilon}) \quad (W \leq y \leq 2W),
$$

where $\alpha \leq 1/5$. Since $H_j(0) \ll_\varepsilon \kappa_j^{1+\varepsilon}$, we obtain

$$
I(W) = \frac{1}{\pi\sqrt{2}} \sum_{n \leq W} t_j(n) \int_W^{2W} (y + q/x)^{-1/4} (ny + q)^{-3/4}
$$
$$
\times \cos G_{\kappa_j}(4\pi\sqrt{xy}) \sin G_{\kappa_j}(4\pi\sqrt{ny}) \, dy
$$
$$
+ O_\varepsilon((\kappa_j^{1+\varepsilon} + W^{\alpha+\varepsilon}) W^{-1/4}).
$$

To evaluate the above integral we let

$$
\Psi(y) := G_{\kappa_j}(4\pi\sqrt{yx}) \pm G_{\kappa_j}(4\pi\sqrt{yn}),
$$

so that

$$|\Psi'(y)| = \left| \frac{4\pi^2(x-n)}{(4\pi^2 yx + \kappa_j^2)^{1/2} \mp (4\pi^2 yn + \kappa_j^2)^{1/2}} \right|$$

$$\gg \begin{cases} (x/y)^{1/2} & \text{if } n \le x/2, \\ |x-n|(xy)^{-1/2} & \text{if } x/2 < n \le 2x, \\ (n/y)^{1/2} & \text{if } n > 2x. \end{cases}$$

Hence by the first derivative test

$$\int_W^{2W} (y+q/x)^{-1/4}(ny+q)^{-3/4} e^{i\Psi(y)} \, dy$$

$$\ll \begin{cases} n^{-3/4} W^{-1/2} x^{-1/2} & \text{if } n \le x/2, \\ x^{-1/4} W^{-1/2}|x-n|^{-1} & \text{if } x/2 < n \le 2x, \\ n^{-5/4} W^{-1/2} & \text{if } n > 2x. \end{cases}$$

This gives

$$I(W) \ll \sum_{n \le x/2} |t_j(n)| n^{-3/4} W^{-1/2} x^{-1/2}$$

$$+ \sum_{x/2 < n \le 2x, |n-x| \ge 1/2} |t_j(n)| x^{-1/4} W^{-1/2} |x-n|^{-1}$$

$$+ \sum_{n \ge 2x} |t_j(n)| n^{-5/4} W^{-1/2}$$

$$+ \min(x^{-1/4} W^{-1/2} \|x\|^{-1}, x^{-3/4}) |t_j(n_x)|$$

$$+ (\kappa_j^{1+\varepsilon} + W^{\alpha+\varepsilon}) W^{-1/4},$$

where $\|u\|$ is the distance from u to the nearest integer, and n_x is the integer satisfying $|n_x - x| < 1/2$, if it exists. All three sums above are easily seen to be $\ll W^{-1/2}$, and moreover,

$$\min(x^{-1/4} W^{-1/2} \|x\|^{-1}, x^{-3/4}) |t_j(n_x)| \ll \min(W^{-1/2} \|x\|^{-1}, 1),$$
$$(\kappa_j^{1+\varepsilon} + W^{\alpha+\varepsilon}) W^{-1/4} \le (M^{1/6+\varepsilon} + W^{\alpha+\varepsilon}) W^{-1/4} \le 2W^{\alpha+\varepsilon-1/4} \le W^{-1/25}$$

for $W \ge W_0(\varepsilon)$ if $0 < \varepsilon < 1/100$. Altogether we certainly have, for $W \ge M$,

$$(4.11) \qquad\qquad I(W) \ll \min(W^{-1/25} \|x\|^{-1}, 1).$$

Let now

$$k_0 := \max\left(0, \frac{\log(M^{-1}\|x\|^{-25})}{\log 2}\right).$$

By using (4.11) we have, since $M^{-1/4} \leq x^{-3/4}$,

$$V \ll x^{3/4} \sum_{0 \leq k \leq k_0} |I(2^k M)| + x^{3/4} \sum_{k \geq k_0} |I(2^k M)| + x^{1/4} M^{-1/4}$$

$$\ll k_0 x^{3/4} + x^{3/4} \|x\|^{-1} \sum_{k \geq k_0} (2^k M)^{-1/25} + x^{-1/2}$$

$$\ll k_0 x^{3/4} + x^{3/4} \|x\|^{-1} (2^{k_0} M)^{-1/25} + x^{-1/4}.$$

It follows that

$$(4.12) \qquad V \ll \begin{cases} x^{-1/4} & \text{if } \|x\| > x M^{-1/25}, \\ x^{3/4} \log(1/\|x\|) & \text{if } \|x\| \leq x M^{-1/25}. \end{cases}$$

Indeed, in the first case $k_0 = 0$ and we get $V \ll x^{-1/4}$. But in any case

$$\frac{\log(M^{-1} \|x\|^{-25})}{\log 2} \leq k_0 \ll \log \frac{1}{\|x\|}$$

so that

$$V \ll x^{3/4} \log \frac{1}{\|x\|} + x^{3/4} \|x\|^{-1} (M^{-1} \|x\|^{-25} M)^{-1/25} + x^{-1/4} \ll x^{3/4} \log \frac{1}{\|x\|}.$$

Thus finally from Lemma 7, (4.9) and (4.12) we obtain the desired expression for $T_j(x)$. This is

LEMMA 8. *Suppose* $\kappa_j \leq X^{1/2}$, $M \geq x^3$ *and* $X \leq x \leq Y \leq 2X$, *and let* $G_\kappa(u)$ *be given by* (3.2) *and* $h(\kappa)$ *by* (4.6). *Then*

$$(4.13) \quad T_j(x) = \frac{x^{1/4}}{\pi \sqrt{2}} \sum_{n \leq M} t_j(n) \left(n + \frac{1}{x} \left(\frac{\kappa_j}{2\pi} \right)^2 \right)^{-3/4} \sin G_{\kappa_j}(4\pi \sqrt{nx})$$

$$+ H_j(0) + h(\kappa_j) + R_j(x),$$

where uniformly

$$(4.14) \qquad R_j(x) \ll_\varepsilon \begin{cases} \kappa_j^\varepsilon x^{-1/4} & \text{if } \|x\| > x M^{-1/25}, \\ x^{3/4} \log(1/\|x\|) & \text{if } \|x\| \leq x M^{-1/25}. \end{cases}$$

5. Proof of Theorem 2. We can now use the fundamental Lemma 8 to prove Theorem 2. Suppose $X < Y \leq 2X$, $\kappa_j \leq X^{1/2}$, choose $M = X^{100}$ in (4.13), (4.14), and consider

$$F_j = F_j(X, Y) := \int_X^Y (T_j(x) - H_j(0) - h(\kappa_j))^2 \, dx.$$

Setting

$$W_j(x) := \frac{x^{1/4}}{\pi \sqrt{2}} \sum_{n \leq M} t_j(n) \left(n + \frac{1}{x} \left(\frac{\kappa_j}{2\pi} \right)^2 \right)^{-3/4} \sin G_{\kappa_j}(4\pi \sqrt{nx})$$

we have

$$(5.1) \qquad F_j = \int\limits_X^Y W_j^2(x)\,dx + 2 \int\limits_X^Y W_j(x) R_j(x)\,dx + \int\limits_X^Y R_j^2(x)\,dx.$$

We have by (4.14),
(5.2)
$$\int\limits_X^Y R_j^2(x)\,dx \ll_\varepsilon \kappa_j^\varepsilon X^{1/2} + X^{3/2} \int\limits_{\|x\|\ll X^{-3}} \left(\log\frac{1}{\|x\|}\right)^2 dx \ll_\varepsilon X^{1/2+\varepsilon}.$$

We write

$$(5.3) \qquad\qquad \int\limits_X^Y W_j^2(x)\,dx = F_{j1} + F_{j2},$$

say, where

$$F_{j1} = \frac{1}{2\pi^2} \sum_{m,n\le M;\, m\ne n} t_j(m) t_j(n) \int\limits_X^Y x^{1/2} \left(m + \frac{1}{x}\left(\frac{\kappa_j}{2\pi}\right)^2\right)^{-3/4}$$

$$\times \left(n + \frac{1}{x}\left(\frac{\kappa_j}{2\pi}\right)^2\right)^{-3/4} \sin G_{\kappa_j}(4\pi\sqrt{mx}) \sin G_{\kappa_j}(4\pi\sqrt{nx})\,dx,$$

$$F_{j2} = \frac{1}{2\pi^2} \sum_{n\le M} t_j^2(n) \int\limits_X^Y x^{1/2} \left(n + \frac{1}{x}\left(\frac{\kappa_j}{2\pi}\right)^2\right)^{-3/2} \sin^2 G_{\kappa_j}(4\pi\sqrt{nx})\,dx.$$

Consider first F_{j1}. In view of (3.9) we have

$$\left|\frac{\partial}{\partial x}\left(G_{\kappa_j}(4\pi\sqrt{mx}) \pm G_{\kappa_j}(4\pi\sqrt{nx})\right)\right| \gg \frac{|m-n|}{\sqrt{x}(\sqrt{m}+\sqrt{n})},$$

and consequently by the first derivative test

$$\int\limits_X^Y \ll X(mn)^{-3/4}(m^{1/2} + n^{1/2})|m-n|^{-1}.$$

Hence

$$F_{j1} \ll X \sum_{m<n\le M} |t_j(m) t_j(n)| m^{-3/4} n^{-1/4}(n-m)^{-1}$$

$$= X \sum_{n\le M} \sum_{m\le n/2} |t_j(m) t_j(n)| \frac{m^{-3/4} n^{-1/4}}{n-m}$$

$$+ X \sum_{n\le M} \sum_{n/2<m<n} |t_j(m) t_j(n)| \frac{m^{-3/4} n^{-1/4}}{n-m}.$$

Trivially we have

$$\sum_{n \le M} \sum_{m \le n/2} |t_j(m)t_j(n)| \frac{m^{-3/4}n^{-1/4}}{n-m}$$

$$\ll \sum_{n \le M} |t_j(n)|n^{-5/4} \sum_{m \le n/2} |t_j(m)|m^{-3/4} \ll_\varepsilon X^\varepsilon.$$

By writing $m = n - k$ and using $|ab| \le \frac{1}{2}(a^2 + b^2)$ we obtain

$$\sum_{n \le M} \sum_{n/2 < m < n} |t_j(m)t_j(n)| \frac{m^{-3/4}n^{-1/4}}{n-m}$$

$$\ll \sum_{n \le M} |t_j(n)|n^{-1} \sum_{1 \le k < n/2} |t_j(n-k)|k^{-1}$$

$$\ll \log M \cdot \max_{M_1 \le M} M_1^{-1} \sum_{k \le M_1/2} k^{-1} \sum_{\max(2k, M_1/2) < n \le M_1} |t_j(n)t_j(n-k)|$$

$$\ll \log M \cdot \max_{M_1 \le M} M_1^{-1} \sum_{k \le M_1/2} k^{-1} \sum_{n \le M_1} t_j^2(n) \ll_\varepsilon X^\varepsilon,$$

where (1.13) was used. Thus

(5.4) $F_{j1} \ll_\varepsilon X^{1+\varepsilon}.$

It remains to consider F_{j2}. We use

$$\sin^2 G_{\kappa_j}(4\pi\sqrt{nx}) = \tfrac{1}{2}(1 - \cos 2G_{\kappa_j}(4\pi\sqrt{nx}))$$

and estimate the cosine terms by the first derivative test. This gives

$$F_{j2} = \frac{1}{4\pi^2} \sum_{n \le M} t_j^2(n) \int_X^Y x^{1/2}\left(n + \frac{1}{x}\left(\frac{\kappa_j}{2\pi}\right)^2\right)^{-3/2} dx + O_\varepsilon(X^{1+\varepsilon}).$$

Since $M = X^{100}$ we obtain

$$\sum_{n > M} t_j^2(n) \int_X^Y x^{1/2}\left(n + \frac{1}{x}\left(\frac{\kappa_j}{2\pi}\right)^2\right)^{-3/2} dx \ll X^{3/2} \sum_{n > M} t_j^2(n)n^{-3/2}$$

$$\ll_\varepsilon X^{3/2}\kappa_j^\varepsilon M^{-1/2} \ll 1,$$

and consequently

(5.5) $$F_{j2} = \frac{1}{4\pi^2} \sum_{n=1}^\infty t_j^2(n) \int_X^Y x^{1/2}\left(n + \frac{1}{x}\left(\frac{\kappa_j}{2\pi}\right)^2\right)^{-3/2} dx + O_\varepsilon(X^{1+\varepsilon}).$$

Thus from (5.3)–(5.5) it follows that

$$(5.6) \quad \int_X^Y W_j^2(x)\,dx$$

$$= \frac{1}{4\pi^2} \sum_{n=1}^\infty t_j^2(n) \int_X^Y x^{1/2}\left(n + \frac{1}{x}\left(\frac{\kappa_j}{2\pi}\right)^2\right)^{-3/2} dx + O_\varepsilon(X^{1+\varepsilon}),$$

and the main term on the right-hand side of (5.6) is $\ll_\varepsilon \kappa_j^\varepsilon X^{3/2} \ll_\varepsilon X^{3/2+\varepsilon}$. By (5.2), (5.6) and the Cauchy–Schwarz inequality we obtain

$$\int_X^Y W_j(x)R_j(x)\,dx \ll_\varepsilon X^{1+\varepsilon},$$

and (5.1) gives then (1.16).

To obtain (1.17) from (1.16) write first

$$(5.7) \quad \int_X^{2X} T_j^2(x)\,dx = \int_X^{2X} (T_j(x) - H_j(0) - h(\kappa_j))^2\,dx$$

$$+ (H_j(0) + h(\kappa_j))^2 X$$

$$+ 2(H_j(0) + h(\kappa_j)) \int_X^{2X} (T_j(x) - H_j(0) - h(\kappa_j))\,dx.$$

The first integral on the right-hand side of (5.7) is evaluated by (1.16), if we note that, for $\kappa_j \le X^{1/2}$ and $X \le x \le 2X$,

$$\left(n + \frac{1}{x}\left(\frac{\kappa_j}{2\pi}\right)^2\right)^{-3/2} = n^{-3/2}\left(1 + O\left(\frac{\kappa_j^2}{nX}\right)\right),$$

and the contribution of the error term is

$$\sum_{n=1}^\infty t_j^2(n) n^{-5/2} \kappa_j^2 \int_X^{2X} x^{1/2} X^{-1}\,dx \ll_\varepsilon \kappa_j^2 X^{1/2+\varepsilon}.$$

By Lemma 7 and the first derivative test we obtain

$$(5.8) \quad \int_X^{2X} (T_j(x) - H_j(0) - h(\kappa_j))\,dx \ll_\varepsilon X^{3/4+\varepsilon}.$$

Since $H_j(0) \ll_\varepsilon \kappa_j^{1+\varepsilon}$ and $h(\kappa_j) \ll_\varepsilon \kappa_j^{\varepsilon-1/2}$, we then obtain (1.17) from (5.7) and (5.8).

6. Proof of Theorem 3. As stated in Section 1, N. V. Proskurin [27] investigated the summatory function of $\varrho_{j_1}(n)\overline{\varrho_{j_2}(n)}$, where κ_{j_1} and κ_{j_2} are not necessarily fixed. If $\kappa_{j_1} = \kappa_{j_2} = \kappa_j$, then his results are

$$(6.1) \qquad \sum_{n \leq x} t_j^2(n) = \frac{12}{\pi^2 \alpha_j} + O_\varepsilon(x^{7/6+\varepsilon} + x^{1+\varepsilon}\kappa_j^{1/2} + x^{1/2}\kappa_j^{1+\varepsilon}),$$

$$(6.2) \qquad \sum_{n \leq x} c_j(n) = 2x\,\mathrm{ch}(\pi\kappa_j) + O_\varepsilon(\alpha_j\,\mathrm{ch}(\pi\kappa_j)(\kappa_j^{3/2+\varepsilon} + x^{19/18+\varepsilon}$$
$$+ x^{7/10+\varepsilon}\kappa_j^{4/5} + x^{-1/2}\kappa_j^{2+\varepsilon})),$$

$$(6.3) \qquad \sum_{n \leq x} t_j^2(n) = \frac{12x}{\pi^2 \alpha_j} + O_\varepsilon(x^{3/4+\varepsilon} + x^{2/3+\varepsilon}\kappa_j^{1/3} + x^{1/2}\kappa_j^{1/2+\varepsilon}),$$

$$(6.4) \qquad \sum_{n \leq x} c_j(n) = 2x\,\mathrm{ch}(\pi\kappa_j)$$
$$+ O_\varepsilon(\alpha_j\,\mathrm{ch}(\pi\kappa_j)(\kappa_j^{1+\varepsilon} + x^{3/5+\varepsilon} + x^{1/3}\kappa_j^{2/3+\varepsilon})),$$

where α_j is defined by (1.9) and $c_j(n)$ by (1.20). In the proofs Proskurin used the complex integration method and the bound (1.7) with $\alpha = 1/4$ for (6.1)–(6.2), and with $\alpha = 0$ for (6.3)–(6.4). Since $\alpha = 0$ is a form of the Ramanujan–Petersson conjecture for Maass wave forms (Proskurin states, erroneously, that it was proved by N. V. Kuznetsov), it follows that (6.3) and (6.4) are conditional results. Note that the error terms in (6.1) and (6.2) are very large.

The starting point in our proof is, as in [27], the Perron inversion formula for Dirichlet series. For $x^\varepsilon \leq T \leq x$ it gives

$$(6.5) \qquad \sum_{n \leq x} c_j(n) = \frac{1}{2\pi i} \int_{1+\varepsilon-iT}^{1+\varepsilon+iT} C_j(s)\frac{x^s}{s}\,ds + O_\varepsilon(|\varrho_j(1)|^2 T^{-1} x^{1+2\alpha+\varepsilon}),$$

where $C_j(s)$ is defined by (1.22). It is in the nature of Perron's inversion formula that the exponent α must appear in (6.5). By using the method of Chandrasekharan–Narasimhan [3] this could be avoided, but on the other hand, considerable difficulties would arise since κ_j is not necessarily fixed. Thus we found it expedient to use (6.5), but unlike Proskurin, who used Perron's formula also to evaluate the sum in (6.1), we find that for this sum the convolution method seems to yield better results. In the course of the proof we shall use (1.23), which was not available to Proskurin, and proceed somewhat differently in treating the integral in (6.5). In fact, it does not seem clear how Proskurin estimates the saddle-point exponential integrals occurring in his proof.

We apply the residue theorem and the fact that $C_j(0) = 0$ to obtain

$$\frac{1}{2\pi i} \int\limits_{1+\varepsilon-iT}^{1+\varepsilon+iT} C_j(s)\frac{x^s}{s}\,ds = 2x\,\mathrm{ch}(\pi\kappa_j) + I_1 + I_2 + I_3 + I_4 + I_5,$$

where $2x\,\mathrm{ch}(\pi\kappa_j)$ comes from the residue of the integrand at $s = 1$, and where we set

$$I_1 = \frac{1}{2\pi i} \int\limits_{-\varepsilon-2i}^{-\varepsilon+2i} C_j(s)\frac{x^s}{s}\,ds,$$

$$I_2 = \frac{1}{2\pi i} \int\limits_{-\varepsilon-iT}^{-\varepsilon-2i} C_j(s)\frac{x^s}{s}\,ds, \qquad I_3 = \frac{1}{2\pi i} \int\limits_{-\varepsilon+2i}^{-\varepsilon+iT} C_j(s)\frac{x^s}{s}\,ds,$$

$$I_4 = \frac{1}{2\pi i} \int\limits_{-\varepsilon+iT}^{1+\varepsilon+iT} C_j(s)\frac{x^s}{s}\,ds, \qquad I_5 = \frac{1}{2\pi i} \int\limits_{1+\varepsilon-iT}^{-\varepsilon-iT} C_j(s)\frac{x^s}{s}\,ds.$$

We may write the functional equation (1.24) as

(6.6) $$C_j(s) = \gamma_j(s)C_j(1-s),$$

where

$$\gamma_j(s) = \pi^{4s-2}\frac{\Gamma\left(\dfrac{1-s}{2} + i\kappa_j\right)\Gamma\left(\dfrac{1-s}{2} - i\kappa_j\right)\Gamma^2\left(\dfrac{1-s}{2}\right)}{\Gamma\left(\dfrac{s}{2} + i\kappa_j\right)\Gamma\left(\dfrac{s}{2} - i\kappa_j\right)\Gamma^2\left(\dfrac{s}{2}\right)}.$$

Hence, by Stirling's formula, we have

(6.7) $\quad \gamma_j(-\varepsilon + it)$

$$\ll_\varepsilon (|t| + 1)^{1+2\varepsilon}(|\kappa_j - t/2| + 1)^{1/2+\varepsilon}(|\kappa_j + t/2| + 1)^{1/2+\varepsilon}.$$

In view of (1.23) we have $C_j(1 + \varepsilon + it) \ll_\varepsilon \alpha_j\,\mathrm{ch}(\pi\kappa_j)\kappa_j^\varepsilon$, hence by (6.6) and (6.7) we have, for $|t| \leq 2$,

(6.8) $$C_j(-\varepsilon + it) \ll_\varepsilon \alpha_j\kappa_j^{1+\varepsilon}\,\mathrm{ch}(\pi\kappa_j),$$

and the same upper bound holds for I_1. By applying convexity (the Phragmén–Lindelöf principle) to $C_j(s)$ it follows that

(6.9) $$I_4 + I_5 \ll T^{-1} \int\limits_{-\varepsilon}^{1+\varepsilon} |C_j(\sigma + iT)|x^\sigma\,d\sigma$$

$$\ll_\varepsilon (\kappa_j x)^\varepsilon \alpha_j\,\mathrm{ch}(\pi\kappa_j)(x/T + T + \kappa_j).$$

We note that $I_2 = \bar{I}_3$, and use the functional equation (6.6) in I_3. In view

of the absolute convergence of the series for $C_j(1-s)$ we obtain

$$(6.10) \qquad I_3 = \frac{1}{2\pi x^\varepsilon} \sum_{n=1}^{\infty} c_j(n) n^{-\varepsilon-1} \int_2^T \gamma_j(-\varepsilon+it)(xn)^{it} \frac{dt}{-\varepsilon+it}.$$

Stirling's formula gives, for $2 \le t \le T$ and $|\kappa_j - t/2| \ge 2$,

$$\gamma_j(-\varepsilon+it)$$
$$= A(\kappa_j, \varepsilon, t) t^{1+2\varepsilon} |\kappa_j^2 - t^2/4|^{1/2+\varepsilon} e^{iG(t)} \left(1 + O\left(\frac{1}{t}\right) + O\left(\frac{1}{|\kappa_j - t/2|}\right)\right).$$

Here

$$A(\kappa_j, \varepsilon, t) = C_\varepsilon \exp\left(\frac{i\pi}{4} \operatorname{sgn}\left(\frac{t}{2} - \kappa_j\right)\right),$$

and

$$(6.11) \quad G(t) = G(t, \kappa_j)$$
$$:= 4t \log \pi - (t - 2\kappa_j) \log|t/2 - \kappa_j| + t - 2\kappa_j$$
$$- (t + 2\kappa_j) \log(t/2 + \kappa_j) + t + 2\kappa_j - 2t \log(t/2) + 2t.$$

The integral in (6.10) is estimated by the technique similar to the one used for the proof of Theorem 1. The portion for which $|\kappa_j - t/2| < 2$, if it is non-empty, is estimated trivially as $\ll_\varepsilon \kappa_j^{1/2+\varepsilon}$, and its contribution will be $\ll_\varepsilon \alpha_j \kappa_j^{1/2+\varepsilon} \operatorname{ch}(\pi\kappa_j)$. The error terms in the formula for $\gamma_j(-\varepsilon+it)$ contribute a total $\ll_\varepsilon (\kappa_j x)^\varepsilon \alpha_j \operatorname{ch}(\pi\kappa_j)(\kappa_j + T)$. The integral coming from the main term is split into $O(\log T)$ subintegrals of the form

$$(6.12) \qquad \int_{T_1}^{T_2} \varphi(\kappa_j, \varepsilon, t) e^{ih(t)} \, dt \quad (T_1 < T_2 \le 2T_1 \le T),$$

where φ is piecewise monotonic and satisfies

$$\varphi(\kappa_j, \varepsilon, t) \ll_\varepsilon T_1^{2\varepsilon} \max(\kappa_j, T_1)^{1+2\varepsilon} \quad (T_1 \le t \le T_2),$$

while $h(t) := t \log(xn) + G(t)$ with $G(t)$ given by (6.11). Hence

$$h'(t) = \log(\pi^4 xn) - \log|t/2 - \kappa_j| - \log(t/2 + \kappa_j) - 2\log(t/2),$$
$$h''(t) = \frac{8\kappa_j^2 - 4t^2}{t(t^2 - 4\kappa_j^2)}, \qquad h'''(t) = 4\frac{t^4 - 2\kappa_j^2 t^2 + 8\kappa_j^4}{t^2(t^2 - 4\kappa_j^2)^2}.$$

Let us fix a small, positive δ. For the values of t for which

$$(6.13) \qquad\qquad\qquad |h''(t)| \ge \delta/t$$

we estimate the corresponding portion of the integral in (6.12) as

$$\ll_\varepsilon T_1^{1/2+2\varepsilon} \max(\kappa_j, T_1)^{1+2\varepsilon} \ll_\varepsilon T^{1/2+2\varepsilon} \max(\kappa_j, T)^{1+2\varepsilon}.$$

The values of t for which (6.13) fails to hold are those which lie in the interval

$$(6.14) \qquad 2\left(\frac{2-\delta}{4-\delta}\right)^{1/2}\kappa_j < t < 2\left(\frac{2+\delta}{4+\delta}\right)^{1/2}\kappa_j.$$

Hence if δ is sufficiently small and (6.14) holds, then $h'''(t) \geq \kappa_j^{-2}$. Thus the portion of the integral in I_3 over the interval in (6.14) (which we denote by J) can be estimated by the third derivative test (p. 90 of [29]) as

$$\ll_\varepsilon \kappa_j^{1+4\varepsilon}\max_{t\in J}|h'''(t)|^{-1/3} \ll_\varepsilon \kappa_j^{5/3+4\varepsilon}$$

uniformly in n. Note that this contribution can exist only if $T \geq \kappa_j$. In this case we thus obtain

$$(6.15) \quad \sum_{n\leq x} c_j(n) = 2x\,\mathrm{ch}(\pi\kappa_j)$$

$$+ O_\varepsilon(\alpha_j\,\mathrm{ch}(\pi\kappa_j)(\kappa_j x)^\varepsilon(x^\beta T^{-1} + \kappa_j^{5/3} + T^{1/2}\max(\kappa_j,T))),$$

where we have set $\beta = 1 + 2\alpha$ for brevity. If $T < \kappa_j$ then in (6.15) the term $\kappa_j^{5/3}$ on the right-hand side may be omitted. The optimal T in the range $T \geq \kappa_j$ is

$$T = \max(\kappa_j, x^{2\beta/5}),$$

and we choose this T if $\kappa_j \leq x^{3\beta/8}$. The optimal T in the range $T < \kappa_j$ is

$$T = \tfrac{1}{2}\min(\kappa_j, \kappa_j^{-2/3}x^{2\beta/3}),$$

and we choose this T if $\kappa_j > x^{3\beta/8}$. In either case $x^\varepsilon \leq T \leq x$, as required. Hence, from (6.15), it follows that

$$(6.16) \quad \sum_{n\leq x} c_j(n) = 2x\,\mathrm{ch}(\pi\kappa_j)$$

$$+ O_\varepsilon(\alpha_j\,\mathrm{ch}(\pi\kappa_j)(\kappa_j x)^\varepsilon\min(\kappa_j^{5/3} + x^{3\beta/5}, \kappa_j^{-1}x^\beta + \kappa_j^{2/3}x^{\beta/3})).$$

Finally, we deduce (1.28) from (6.16). By the Möbius inversion formula, (1.20) gives

$$|\varrho_j(n)|^2 = \sum_{d^2|n}\mu(d)c_j(n/d^2).$$

In view of $\varrho_j(n) = \varrho_j(1)t_j(n)$ and (1.9) it follows that

$$\Sigma := \sum_{n\leq x} t_j^2(n) = (\alpha_j\,\mathrm{ch}(\pi\kappa_j))^{-1}\sum_{d^2m\leq x}\mu(d)c_j(m).$$

Suppose now that $\max(1, x\kappa_j^{-8/(3\beta)}) \leq y \leq x$. (Values of y in the range $y < x\kappa_j^{-8/(3\beta)}$ give weaker estimates.) Using (6.16) we obtain

$$\Sigma = (\alpha_j \operatorname{ch}(\pi\kappa_j))^{-1}$$
$$\times \Big(\sum_{d \leq \sqrt{y}} \mu(d) \sum_{m \leq x/d^2} c_j(m) + \sum_{m \leq x/y} c_j(m) \sum_{\sqrt{y} < d \leq \sqrt{x/m}} \mu(d) \Big)$$
$$= \sum_{d \leq \sqrt{y}} \mu(d) \frac{2x}{\alpha_j d^2} + O_\varepsilon \Big(\sum_{d \leq x^{1/2} \kappa_j^{-4/(3\beta)}} (\kappa_j x)^\varepsilon (\kappa_j^{5/3} + x^{3\beta/5} d^{-6\beta/5}) \Big)$$
$$+ O_\varepsilon \Big(\sum_{x^{1/2} \kappa_j^{-4/(3\beta)} < d \leq \sqrt{y}} (\kappa_j x)^\varepsilon (\kappa_j^{-1} x^\beta d^{-2\beta} + \kappa_j^{2/3} x^{\beta/3} d^{-2\beta/3}) \Big)$$
$$+ O_\varepsilon \Big((\alpha_j \operatorname{ch}(\pi\kappa_j))^{-1} \sum_{m \leq x/y} c_j(m) (x/m)^{1/2} \Big).$$

Here the explicit term and the third error term together give
$$\frac{12x}{\pi^2 \alpha_j} + O_\varepsilon(\kappa_j^\varepsilon x y^{-1/2}),$$
by (1.10) and (1.23). The first and the second error terms together give
$$O_\varepsilon \big((\kappa_j x)^\varepsilon (\min(x^{1/2} \kappa_j^{(5\beta-4)/(3\beta)} + x^{3\beta/5}, \kappa_j^{-1} x^\beta) + \kappa_j^{2/3} x^{\beta/3} y^{(3-2\beta)/6}) \big),$$
by considering the cases $\kappa_j \leq x^{3\beta/8}$ and $\kappa_j > x^{3\beta/8}$ separately. Now the optimal y is $y = x \kappa_j^{-2/(3-\beta)}$ and this satisfies the asumptions on y if $\kappa_j \leq x^{(3-\beta)/2}$. Thus we obtain
$$\Sigma = 12x/(\pi^2 \alpha_j)$$
$$+ O_\varepsilon \big(x^\varepsilon (\min(x^{1/2} \kappa_j^{(5\beta-4)/(3\beta)} + x^{3\beta/5}, \kappa_j^{-1} x^\beta) + x^{1/2} \kappa_j^{1/(3-\beta)}) \big).$$
Then (1.28) follows, since $\beta = 1 + 2\alpha$.

The asymptotic formula (1.28) with $\alpha = 1/4$ ($\alpha = 5/28$ is known by [2]) improves (6.1), while (1.28) with $\alpha = 0$ improves (6.3).

In conclusion, it may be remarked that in the course of the proof we used the trivial bound $\sum_{n \leq x} \mu(n) \ll x$. However, if we assume the Riemann hypothesis, which is equivalent to the statement (see Ch. 14 of [29]) that
$$\sum_{n \leq x} \mu(n) \ll_\varepsilon x^{1/2+\varepsilon},$$
then a conditional improvement of Theorem 3 may be obtained.

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