ACTA ARITHMETICA
LXVIII.4 (1994)

Divisibility of the κ-fold iterated divisor function of n into n

by

Claudia Spiro-Silverman (Nashua, N.H.)

1. Introduction. Let $d(n)$ denote the number of positive integers dividing the positive integer n, and put

$$\omega(n) = \sum_{p|n} 1, \qquad s(n) = \prod_{p\|n} p.$$

For all integers $\kappa \ge 2$, define the κ-fold iterated divisor functions $d_\kappa(n)$ by the identity

$$\left(\sum_{n=1}^{\infty} \frac{1}{n^s}\right)^{\kappa} = \sum_{n=1}^{\infty} \frac{d_\kappa(n)}{n^s}, \qquad \operatorname{Re} s > 1.$$

For all positive real numbers x and all integers $\kappa \ge 2$, define

$$D_\kappa(x) = \#\{n \le x : d_\kappa(n) \mid n\},$$

and put $D(x) = D_2(x)$; $d(n) = d_2(n)$. Earlier, we established the following three results (cf. Theorems 1, 2, and 5 of [SP1]).

Theorem. (i) $D(x) = x/(\sqrt{\log x}(\log\log x)^{1+o(1)})$.

(ii) *Let $\kappa = p^\alpha$, where α is a positive integer, and p is a prime. Define $\{\xi(i,p)\}_{i=0}^{\infty}$ recursively by*

$$\xi(0,p) = 0, \qquad \xi(i,p) = \xi(i-1,p) + p^{\xi(i-1,p)} \quad \textit{for } i \ge 1.$$

Then there exists a constant $c(\kappa) > 0$ such that

$$\#\{n \le x : d_\kappa(n) \mid n\} \le c(\kappa) \frac{x\xi_p(x)}{(\log\log x)^{\kappa-1}} \left(\frac{\log\log\log\log x}{\log x}\right)^{1-1/\kappa}$$

for $x \ge 16$, where $\xi_p(x)$ is the number of subscripts i with $\xi(i,p)$ not exceeding x.

1991 *Mathematics Subject Classification*: Primary 11N25.

Further history of this and related problems is contained on p. 82 of [SP1]. Numbers n such that $d(n)$ divides n arise in connection with a variety of iteration problems, including the result of Section 3 of [SP2].

In the present paper, we obtain the following estimate.

Theorem 1. *There are positive constants* c_0 *and* c_1 *such that*

$$\frac{c_0 x (L_4 x)^{-3/4}}{(\log x)\log\log x} \sum_{\substack{i=1\\ i\, odd,\, squarefree}}^{\infty} \frac{\left(\frac{1}{2}\log\log x\right)^i}{i! 2^{\omega(i)}} \le D(x)$$
$$\le \frac{c_1 x (L_4 x)^{3/2}}{(\log x)\log\log x} \sum_{i=1}^{\infty} \frac{\left(\frac{1}{2}\log\log x\right)^i}{i! 2^{\omega(s(i))}}.$$

Here, $L_4 x$ denotes the four-fold iterated natural logarithm of x. In view of the facts that as $y = \frac{1}{2}\log\log x \to \infty$, the bulk of the contribution to the series $\sum_{n=0}^{\infty} y^n/n!$ for e^y occurs for n near y, and the normal order of both $\omega(i)$ and $\omega(s(i))$ is $\log\log i$, we can get a heuristic estimate by replacing each of the functions $\omega(i)$, $\omega(s(i))$ by $\log\log\left(\frac{1}{2}\log\log x\right)$ for all i. Thus, we have the approximations

$$\sum_{i=1}^{\infty} \frac{\left(\frac{1}{2}\log\log x\right)^i}{i! 2^{\omega(i)}} \approx \frac{e^{\frac{1}{2}\log\log x}}{2^{\log\log(\frac{1}{2}\log\log x)}},$$
$$\sum_{i=1}^{\infty} \frac{\left(\frac{1}{2}\log\log x\right)^i}{i! 2^{\omega(s(i))}} \approx \frac{e^{\frac{1}{2}\log\log x}}{2^{\log\log(\frac{1}{2}\log\log x)}}.$$

Simplifying and combining the results with Theorem 1 gives the following estimate.

Heuristic Estimate:
$$D(x) = \frac{x}{\sqrt{\log x}} \frac{(\log\log\log x)^{-\log 2 + o(1)}}{\log\log x}.$$

Note. An average-order estimate in place of a normal-order estimate would yield a somewhat different result.

The results of Theorem 1 generalize to $D_\kappa(x)$. As a corollary of that generalization, we obtain the following theorem.

Theorem 2. *For all integers* $\kappa \ge 2$ *we have*
$$D_\kappa(x) = \frac{x}{(\log x)^{1-1/\kappa}} (\log\log x)^{(1-\kappa)\omega(\kappa)+o(1)} \quad \text{as } x \to \infty.$$

2. Notation, basic definitions, and preliminary results. Throughout this paper, i, j, k, l, m, n, and t are reserved for integers (m, n, and t positive), p and q denote primes, and w, x, y, and z signify sufficiently large real numbers. A summation of the form $\sum_{n \le x}$ is assumed to extend over all positive integers n not exceeding x. A sum or product of the shape $\sum_p$ or $\prod_p$, respectively, signifies a sum or product over primes: Thus, for example, $\sum_{p|n} 1$ denotes the number of distinct primes dividing n. We summarize other notation commonly utilized in this paper in the following table.

Symbol	Meaning
$p^j \parallel n$	$p^j \mid n$, but $p^{j+1} \nmid n$
$\nu_p(n)$	The unique integer j satisfying $p^j \parallel n$
$\mathrm{odd}(n)$	The greatest odd integer dividing n
$\omega(n)$	The number of distinct prime divisors of n
$\log x$	The natural logarithm of x
$L_n x$	These functions are recursively defined for all integers $n \geq 2$ and for appropriate values of x by $L_2 x = \log\log x$; $L_{n+1} x = \log L_n x$
$f(x) = O(g(x))$	There exists a positive constant K for which $\lvert f(x)\rvert \leq K g(x)$ if x is sufficiently large
$f(x) \ll g(x)$	$f(x) = O(g(x))$
$f(x) \gg g(x)$	$g(x) \ll f(x)$
$f(x) = o(g(x))$	$f(x)/g(x)$ tends to 0 as x tends to ∞
$f(x) = O_{a,b,\ldots}(g(x))$	$f(x) = O(g(x))$. The implied constant possibly depends on $a, b, \ldots$
$f(x) \ll_{a,b,\ldots} g(x)$	$f(x) = O_{a,b,\ldots}(g(x))$
z-sufficiently large	Sufficiently large, possibly depending on z
$[a, b, \ldots]$-sufficiently large	Sufficiently large, possibly depending on $a, b, \ldots$
$c_0, c_1, c_2, \ldots$	Positive absolute constants
$\mathcal{S}$	The set of squarefree positive integers
$\mathcal{S}'$	The set of odd, squarefree positive integers
$r(n)$	The largest squarefull divisor of $\mathrm{odd}(n)$
$s(n)$	$(\mathrm{odd}(n))/r(n)$
$\pi(x)$	The number of primes not exceeding x
T	The set of squarefull numbers

Unless otherwise specified, all other notation will be identical to that of [SP1].

Definition. We term a positive integer t *squarefull* if p^2 divides t for every prime divisor p of t.

In both the upper and lower bound arguments of [SP1], the key place to search for an improvement of the resulting estimate is in lemmas of the following type (cf. the derivation of equation (15) and Lemma 7 of that paper).

Lemma. (i) *If* $d(n)\mid n$, *then* $\operatorname{odd}(1+\nu_2(n))\mid n$.

(ii) *Let* $\varepsilon>0$ *be given and let* $l\ge 2$ *be* ε*-sufficiently large. For every* $[\varepsilon,l]$*-sufficiently large positive integer* k *coprime to* $l!$, *there exists a multiple* k^* *of* k *for which* $k^*<k^{1+\varepsilon}$, $d(k^*)\mid 2^k k^*$, *and* $k^*\mid (l!k)^k$. (Thus, k^* has no large prime divisors which do not divide k.) In our applications, we take $k=\nu_2(n)$.

If we could make a better estimate of k^* than the bound $k^*<k^{1+\varepsilon}$, while retaining the other conclusions of (ii), then we could improve the lower bound for $D(x)$ given in [SP1]. In addition, if we could choose $l=2$ for every $\varepsilon>0$ when applying (ii), then we could squeeze the maximal amount of power from our lemma. For squarefree k, we can obtain the conclusion of (ii) with $k^*=k$, and $l=2$. The difficulty, then, becomes the problem of showing that

$$\sum_{k\,\mathrm{odd},\,\mathrm{squarefree}}\frac{y^k}{k!}\gg\sum_{j=0}^{\infty}\frac{y^j}{j!}=e^y \tag{1}$$

as y tends to infinity. Our method, an application of a squarefree sieve (cf. [ER]), is developed in the next three lemmas.

Lemma 1. *For all positive integers* k *and* h, *and all real* $y\ge 1$, *we have*

$$\Bigg|\sum_{\substack{l=0\\ k\mid l}}^{\infty}\frac{y^l}{l!}-\sum_{\substack{l=0\\ l\equiv h \bmod k}}^{\infty}\frac{y^l}{l!}\Bigg|\ll\frac{e^y}{\sqrt{y}}. \tag{2}$$

The implied constant is absolute.

Proof. For fixed y, the function $y^l/l!$ of l increases in the interval $0\le l\le y$, and decreases in the interval $y\le l$. Now, define the sequence $\{a(j)\}$ by

$$a(2j)=kj,\qquad a(2j+1)=kj+h,$$

for all nonnegative integers j. Then the left-hand side of (2) is

$$\Bigg|\sum_{j=0}^{\infty}\frac{(-1)^j y^{a(j)}}{a(j)!}\Bigg|=\Bigg|\sum_{\substack{j=0\\ a(j)\le y}}^{\infty}\frac{(-1)^j y^{a(j)}}{a(j)!}+\sum_{\substack{j=0\\ a(j)>y}}^{\infty}\frac{(-1)^j y^{a(j)}}{a(j)!}\Bigg|.$$

By applying the Alternating Series Inequality to each of the sums on the right, we obtain the upper bounds

$$\Bigg|\sum_{\substack{j=0\\ a(j)\le y}}^{\infty}\frac{(-1)^j y^{a(j)}}{a(j)!}\Bigg|\le\frac{y^A}{A!},\qquad\Bigg|\sum_{\substack{j=0\\ a(j)> y}}^{\infty}\frac{(-1)^j y^{a(j)}}{a(j)!}\Bigg|\le\frac{y^B}{B!},$$

where $A=\max\{j: a(j)\le y\}$, and $B=\min\{j: a(j)>y\}$. It therefore follows from the Triangle Inequality that the left-hand side of (2) is at most $y^A/A!+y^B/B!$. Now $y^A/A!=y^A/\Gamma(A+1)$. Moreover, $y^A/\Gamma(A+1)$ has its maximum, as a function of A, at a value A between $y-1$ and $y+1$. Thus, we can conclude from Stirling's Formula that $y^A/A!\le(1+o(1))e^y/\sqrt{2\pi y}$. Similarly, we deduce that $y^B/B!\le(1+o(1))e^y/\sqrt{2\pi y}$, and the lemma follows. ∎

Lemma 2.

$$\sum_{\substack{l=0\\ k\mid l}}^{\infty}\frac{y^l}{l!}=\frac{1}{k}e^y+O\Big(\frac{e^y}{\sqrt{y}}\Big).$$

The implied constant is absolute.

Proof. Partitioning the sum according to the residue class of l modulo k, and then applying Lemma 1, yields

$$\sum_{l=0}^{\infty}\frac{y^l}{l!}=\sum_{h \bmod k}\ \sum_{\substack{l=0\\ l\equiv h \bmod k}}^{\infty}\frac{y^l}{l!}=\sum_{h \bmod k}\Bigg(\sum_{\substack{l=0\\ k\mid l}}^{\infty}\frac{y^l}{l!}+O\Big(\frac{e^y}{\sqrt{y}}\Big)\Bigg).$$

Since the hth summand in the sum on the right is independent of h, we have

$$\sum_{l=0}^{\infty}\frac{y^l}{l!}=k\sum_{\substack{l=0\\ k\mid l}}^{\infty}\frac{y^l}{l!}+O\Big(\frac{ke^y}{\sqrt{y}}\Big).$$

Solving for the sum on the right now gives the lemma. ∎

Lemma 3. *Let $\mathcal{S}$ and $\mathcal{S}'$ respectively denote the sets of squarefree and odd squarefree numbers. Then*

$$\text{(i)}\quad \sum_{\substack{l=1\\ l\in\mathcal{S}'}}^{\infty}\frac{y^l}{l!}=\frac{2}{\pi^2}e^y+O\Big(\frac{e^y}{\log y}\Big);\tag{3}$$

$$\text{(ii)}\quad \sum_{\substack{l=1\\ l\in\mathcal{S}}}^{\infty}\frac{y^l}{l!}=\frac{6}{\pi^2}e^y+O\Big(\frac{e^y}{\log y}\Big).$$

Proof. The proof of (i) is very similar to the proof of (ii). Ergo, we present the proof of (ii) only, and leave the proof of (i) to the reader. The method is to sieve by squares of large primes first, and then to sieve by squares composed only of the remaining small prime divisors. For brevity of exposition, write $P(y)$ for the product of all primes not exceeding $\log y$, and $\mathcal{S}(y)$ for the left side of (3). By partitioning the last sum according to whether l exceeds $10y$, we obtain

$$\mathcal{S}(y) = \sum_{\substack{0\le l\le 10y\\ l \text{ squarefree}}} \frac{y^l}{l!} + O\Big(\sum_{l>10y} \frac{y^l}{l!}\Big).$$

Now the sequence $\{y^l/l!\}$ decreases geometrically in l for $l \ge 10y$. Indeed, $y^{l+1}/(l+1)! = (y^l/l!)(y/(l+1)) < .1y^l/l$. Thus, the error is of the order of the first term of the series. By Stirling's Formula,

$$\mathcal{S}(y) = \sum_{\substack{0\le l\le 10y\\ l \text{ squarefree}}} \frac{y^l}{l!} + O\Big(\Big(\frac{e}{10}\Big)^{10y}\frac{1}{\sqrt{y}}\Big). \tag{4}$$

Now the condition that l be squarefree is equivalent to the following pair of constraints: no prime $p > \log y$ satisfies $p^2 \mid l$, and no prime $p \le \log y$ satisfies $p^2 \mid l$. If we ignore the first condition, we make an error of

$$O\Big(\sum_{\substack{0\le l\le 10y\\ p^2\mid l \text{ for some } p>\log y}} \frac{y^l}{l!}\Big)$$

in the sum in (4). Now, we note that if l contributes to this error, then we have $p^2 \mid l$ for at least one prime p exceeding $\log y$. Hence,

$$\mathcal{S}(y) = \sum_{\substack{0\le l\le 10y\\ \text{if } p^2\mid l \text{ then } p>\log y}} \frac{y^l}{l!} + O\Big(\sum_{p>\log y}\ \sum_{0\le l\le 10y,\, p^2\mid l} \frac{y^l}{l!}\Big) + O\Big(\Big(\frac{e}{10}\Big)^{10y}\frac{1}{\sqrt{y}}\Big).$$

If the final sum on l is not void, then p cannot exceed $\sqrt{10y}$, since $p^2 \mid l$. Thus, that error is

$$O\Big(\sum_{\log y<p\le\sqrt{10y}}\ \sum_{0\le l\le 10y,\, p^2\mid l} \frac{y^l}{l!}\Big).$$

It therefore follows from Lemma 2 that

$$\begin{aligned}\mathcal{S}(y) = &\sum_{\substack{0\le l\le 10y\\ \text{if } p^2\mid l \text{ then } p>\log y}} \frac{y^l}{l!}\\ &+ O\Big(\sum_{\log y<p\le\sqrt{10y}} \Big(\frac{e^y}{p^2} + \frac{e^y}{\sqrt{y}}\Big)\Big) + O\Big(\Big(\frac{e}{10}\Big)^{10y}\frac{1}{\sqrt{y}}\Big).\end{aligned} \tag{5}$$

We can restate the last condition beneath the first sum as the condition that $P(y)$ be coprime to the maximal perfect square dividing l. So, the main term is

$$\sum_{0\le l\le 10y} \frac{y^l}{l!} \sum_{d\mid P(y),\, d^2\mid l} \mu(d).$$

Interchanging the order of summation gives the expression

$$\sum_{d\mid P(y)} \mu(d) \sum_{0\le l\le 10y,\, d^2\mid l} \frac{y^l}{l!}$$

for this quantity. We now apply the methods of the derivation of (4) to replace the inner sum by

$$\sum_{\substack{l=0\\ d^2\mid l}}^{\infty} \frac{y^l}{l!}.$$

Thus,

$$\sum_{0\le l\le 10y,\, d^2\mid l} \frac{y^l}{l!} = \sum_{\substack{l=0\\ d^2\mid l}}^{\infty} \frac{y^l}{l!} + O\biggl(\sum_{\substack{l>10y\\ d^2\mid l}} \frac{y^l}{l!}\biggr) = \sum_{\substack{l=0\\ d^2\mid l}}^{\infty} \frac{y^l}{l!} + O\biggl(\Bigl(\frac{e}{10}\Bigr)^{10y}\frac{1}{\sqrt{y}}\biggr).$$

Here, we have ignored the condition that d^2 divide l in the last sum. Hence, Lemma 2 implies that the main term in (5) is

$$\sum_{d\mid P(y)} \mu(d)\biggl(\frac{e^y}{d^2} + O\Bigl(\frac{e^y}{\sqrt{y}}\Bigr) + O\biggl(\Bigl(\frac{e}{10}\Bigr)^{10y}\frac{1}{\sqrt{y}}\biggr)\biggr).$$

The second error term can be absorbed into the first. Thus,

$$\begin{aligned}\mathcal{S}(y) = &\sum_{d\mid P(y)} \mu(d)\biggl(\frac{e^y}{d^2} + O\Bigl(\frac{e^y}{\sqrt{y}}\Bigr)\biggr)\\ &+ O\biggl(e^y \sum_{\log y<p\le\sqrt{10y}} \frac{1}{p^2} + \frac{e^y}{\sqrt{y}}\pi(\sqrt{10y}) + \Bigl(\frac{e}{10}\Bigr)^{10y}\frac{1}{\sqrt{y}}\biggr).\end{aligned}$$

According to the Chebyshev estimate for $\pi(x)$, we have

$$\begin{aligned}\mathcal{S}(y) = e^y \sum_{d\mid P(y)} \frac{\mu(d)}{d^2} &+ O\biggl(\frac{e^y}{\sqrt{y}} \sum_{d\mid P(y)} 1\biggr)\\ &+ O\biggl(e^y \sum_{\log y<p\le\sqrt{10y}} \frac{1}{p^2}\biggr) + O\Bigl(\frac{e^y}{\log y}\Bigr).\end{aligned}$$

Since $\sum_{p>z} p^{-2} < \sum_{n>z} n^{-2} \ll 1/\log z$, we can ignore the inequality

$p \le \sqrt{10y}$, and thus, arrive at the estimate

$$\mathcal{S}(y) = e^y \sum_{d \mid P(y)} \frac{\mu(d)}{d^2} + O\biggl(\frac{e^y}{\sqrt{y}} \sum_{d \mid P(y)} 1\biggr) + O\biggl(\frac{e^y}{\log y}\biggr).$$

Now $\mu(d)/d^2$ is a multiplicative function of d. Hence, the definition of $P(y)$ gives

$$\mathcal{S}(y) = e^y \prod_{p \le \log y} (1 - p^{-2}) + O\biggl(\frac{e^y}{\sqrt{y}} 2^{\pi(\log y)}\biggr) + O\biggl(\frac{e^y}{\log y}\biggr).$$

A second application of the Chebyshev estimate implies that the penultimate error term can be absorbed into the final remainder term. Now if the product were infinite, it would be the Euler product for $1/\zeta(2)$. Ergo,

$$\mathcal{S}(y) = \frac{6}{\pi^2} e^y \prod_{p > \log y} (1 - p^{-2})^{-1} + O\biggl(\frac{e^y}{\log y}\biggr).$$

And, from the inequality

$$\begin{aligned} 0 &< \prod_{p > \log y} (1 - p^{-2})^{-1} - 1 \\ &= \sum_{\substack{n=1 \\ \text{if } p \mid n \text{ then } p > \log y}}^{\infty} \frac{1}{n^2} - 1 \le \sum_{n > \log y} \frac{1}{n^2} \ll \frac{1}{\log y}, \end{aligned}$$

we conclude that

$$\prod_{p > \log y} (1 - p^{-2})^{-1} = 1 + O\biggl(\frac{1}{\log y}\biggr),$$

of which the lemma is an immediate consequence. ∎

Lemma 4. *Let δ be a real number with $0 < \delta \le 1/2$, and let*

$$Q(y) = -(1+y)\log(1+y) = \sum_{n=2}^{\infty} \frac{(-1)^{n-1} y^n}{n(n-1)} \quad \textit{for } 0 \le y \le 1.$$

Then

(i) $\#\{n \le x : |\omega(n) - L_2 x| > \delta L_2 x\} \ll \dfrac{\delta^{-1} x}{\sqrt{L_2 x}} e^{Q(\delta) L_2 x}$.

The implied constant is absolute.

(ii) *For δ-sufficiently large n, we have $\omega(n) \le (1+\delta)(\log n)/L_2 n$.*

Proof. The first inequality follows from Theorems (3.18) and (3.20) of Karl Norton's paper [NO2]. In Theorem (3.18), we take $E = P$. In Theorem (3.20), we choose $E = P$, and $\beta = 1/2$. Then, we combine the estimates. For a very detailed history of this and related results, as well as a discussion of the importance of the techniques applied to them, see [NO2].

The second inequality is a result of the discussion in [HW], beginning with the last sentence on p. 354. ∎

3. Technical lemmas

LEMMA 5. *If k is an odd, squarefree positive integer, then $d(n)$ divides n for all integers n of the form $n = 2^{k-1}km$, with $\mu(m) \neq 0$, $(m, 2k) = 1$, and $\omega(m) \leq k - 1 - \omega(k)$.*

Proof. By the multiplicativity of $d(n)$ and the pairwise coprimality of 2, k, and m, we have $d(n) = kd(k)d(m)$. Since k and m are squarefree, we deduce that $d(n) = 2^{\omega(k)+\omega(m)}k$. Hence, if $\omega(m)$ does not exceed $k-1-\omega(k)$, then $d(n)$ divides $2^{\omega(k)+k-1-\omega(k)}k = 2^{k-1}k$, which completes the proof. ∎

We use the last lemma to derive a lower bound for $D(x)$, and the next lemma to derive an upper bound for this function. In the next lemma and thereafter, we will let

$$T = \{n : \text{if } p \mid n, \text{ then } p^2 \mid n\}$$

denote the set of powerful, or squarefull, positive integers.

LEMMA 6. *Let k be a given positive integer. If $\nu_2(n) = k - 1$, and $d(n)$ divides n, then n has the form*

$$n = 2^{k-1}(\operatorname{odd}(kl))mt,$$

where $m \in \mathcal{S}$, $t \in T$, m and t are odd, any prime divisor of l also divides k, and

$$\omega(m) \leq k - 1 - \nu_2(k) - \nu_2(d(\operatorname{odd}(kl))).$$

Clearly, we can take l to be odd, here.

Proof. Since $d(n)$ is multiplicative, and $2^{k-1} \parallel n$, we have $k \mid d(n)$, whence k is a divisor of n. Thus, $2^{k-1}\operatorname{odd}(k)$ divides n. Let

$$l = \prod_{p \mid \operatorname{odd}(k)} p^{\nu_p(n/k)},$$

so that $n/(2^{k-1}\operatorname{odd}(kl))$ is an integer coprime to $2k$. Since any positive integer can be written as the product of a squarefull and a squarefree integer by separating its canonical decomposition accordingly, we can write $n/(2^{k-1}\operatorname{odd}(kl)) = mt$, with $m \in \mathcal{S}$, $t \in T$, $(m, t) = (m, 2k) = (t, 2k) = 1$. Hence, $n = 2^{k-1}(\operatorname{odd}(kl))mt$. Since d is a multiplicative function, we find that $kd(\operatorname{odd}(kl))2^{\omega(m)}d(t) \mid n$. We deduce the lemma by evaluating the function ν_2 at both sides.

LEMMA 7. *Let*

$$\pi_j(x, k) = \#\{n \leq x : \omega(n) = j,\ \mu(n) \neq 0,\ (n, k) = 1\},$$

for all integers $j \geq 0$ *and* $k \geq 1$, *and real numbers* $x \geq 0$. *Further*, *define*

$$f(z) = \frac{1}{\Gamma(1+z)} \prod_{p} \left(1 - \frac{1}{p}\right)^{z} \left(1 + \frac{z}{p}\right),$$

and

$$f_n(z) = \prod_{p \mid n} \left(1 + \frac{z}{p}\right)^{-1},$$

for all positive real z. *Let* A *and* B *be fixed positive real numbers. Then*

$$\pi_j(x,k) = \frac{x}{\log x} \frac{(L_2 x)^{j-1}}{(j-1)!} \left(f\left(\frac{j-1}{L_2 x}\right) f_k\left(\frac{j-1}{L_2 x}\right) + O_{A,B}\left(\frac{j(L_3(16k))^3}{(L_2 x)^2}\right)\right),$$

uniformly in

$$1 \leq j \leq BL_2 x, \qquad k \leq \exp((\log x)^A), \qquad \prod_{p \mid k} p \leq (\log x)^A. \tag{6}$$

Moreover, *we have*

$$1 \geq f_k\left(\frac{j-1}{L_2 x}\right) \geq f_k(B) \gg_B (L_2(3k))^{-B} \tag{7}$$

uniformly in the range (6).

Proof. The first conclusion follows at once from Theorem 2 of [SP3]. For more information, see the first paragraph on p. 85 of [SP1]. We deduce the remaining conclusion from the first conclusion of Lemma 2 of [SP1], with $b = B$.

Lemma 8. *For any constant* B, *we have* $1 \ll_B f(z) \ll 1$ *uniformly in the interval* $0 \leq z \leq B$.

Proof. The function is nonvanishing on the compact interval $0 \leq z \leq B$, and can be shown to be continuous thereon (see [SE]). ∎

In view of Lemma 6, we need to control the size of the largest squarefull divisor of an integer n, with perhaps some exceptional integers. We do so with the following result.

Lemma 9. *For* $1 \leq y \leq x$, *we have*

$$\#\{n \leq x : \text{the largest squarefull divisor of } n \text{ exceeds } y\} \ll x/\sqrt{y}. \tag{8}$$

The implied constant is absolute.

Proof. If we partition the set of integers n on the left according to the value of the largest squarefull divisor t of n, we find that the quantity on the left is at most

$$\sum_{t>y,\, t\in T}\ \sum_{n\le x,\, t\mid n} 1.$$

Now the inner sum is trivially at most x/t. Thus, the left side of (8) does not exceed

$$x\sum_{t>y,\, t\in T} 1/t.$$

If we sum by parts, and apply the result that the number of squarefull integers not exceeding any real number $w \ge 1$ is $O(\sqrt{w})$, we obtain the lemma. The estimate $O(\sqrt{w})$ follows from an estimate of Erdős and Szekeres [ES]. ■

Also, in view of Lemma 6, we need a precise result analogous to Lemma 9 when we are concerned with powers of 2, and not with all squarefull numbers.

LEMMA 10. *For $1 \le w \le x$ we have*

$$\#\{n \le x : \nu_2(n) > w\} \ll x/2^w.$$

P r o o f. If we partition the set on the left according to the value of $\nu_2(n)$, we find that the quantity on the left-hand side is

$$\sum_{k>w}\ \sum_{n\le x,\, 2^k \| n} 1 \le \sum_{k>w}\ \sum_{n\le x,\, 2^k \mid n} 1.$$

The last sum on n is, trivially, at most $x/2^k$. So,

$$\#\{n \le x : \nu_2(n) > w\} \le \sum_{k>w} \frac{x}{2^k} \ll \frac{x}{2^w},$$

since the last sum is a geometric series. ■

Note that Lemma 10 does not follow directly from Lemma 9.

We also require a result which allows us to take the series $\sum_{j=0}^{\infty} y^j/j! = e^y$, perturb some of the subscripts j with $|j-y|$ sufficiently small, and still have a series whose value is approximable by e^y. An estimate of this type, sufficient for our purposes, is the next lemma.

LEMMA 11. *For $|j-y| \le y/\log y$, and for $1 \le m \le 10\log y$, we have*

$$\frac{y^{j-m}}{(j-m)!} = \frac{y^j}{j!}\left(1 + O\left(\frac{m}{\log y}\right)\right).$$

The implied constant is absolute.

P r o o f. We have

$$\frac{y^{j-m}}{(j-m)!} = \frac{y^j}{j!}\prod_{i=0}^{m-1}\frac{j-i}{y} = \frac{y^j}{j!}\prod_{i=0}^{m-1}\left(1 + \frac{j-i-y}{y}\right).$$

For y sufficiently large,

$$\frac{-2}{\log y} \le \frac{j-i-y}{y} \le \frac{2}{\log y} \quad \text{for all } i,j.$$

Thus,

$$\frac{y^j}{j!}\left(1-\frac{2}{\log y}\right)^m < \frac{y^{j-m}}{(j-m)!} < \frac{y^j}{j!}\left(1+\frac{2}{\log y}\right)^m.$$

The lemma is now an immediate consequence of the Binomial Theorem. ∎

Lemma 12.

$$\sum_{n,\,|n-x|\ge\sqrt{x\log x}} \frac{x^n}{n!} \ll \frac{e^x}{\sqrt{x\log x}}.$$

Proof. This is Lemma 5 of [SP1]. For much more precise estimates of this sort, we refer the reader to [NO1]. ∎

The next lemma allows us to bound $f_n(z)$ and related products, for some z.

Lemma 13. *Let B and C be fixed real numbers with $B > 0$ and with $0 < C < 2$, and let $0 \le b \le B$ and $0 \le c \le C$. We have*

(i) $\displaystyle\prod_{p|k}\left(1+\frac{b}{p}\right) \ll (L_2(3k))^b$ *uniformly in b and k;*

(ii) $\displaystyle\prod_{p|k}\left(1-\frac{c}{p}\right)^{-1} \ll (L_2(3k))^c$ *uniformly in c and k.*

Proof. This lemma follows from Lemma 2 of [SP1]. ∎

4. A lower bound for $D(x)$. If we partition the set of positive integers n such that $d(n)$ divides n according to the value $k-1$ of $\nu_2(n)$, we obtain

$$D(x) \ge \sum_{\frac14 L_2x<k\le\frac34 L_2x} \sum_{\substack{n\le x\\ 2^{k-1}\|n\\ d(n)|n}} 1 \ge \sum_{\substack{\frac14 L_2x<k\le\frac34 L_2x\\ k\in\mathcal{S}'}} \sum_{\substack{n\le x\\ 2^{k-1}\|n\\ d(n)|n}} 1.$$

In view of Lemma 5, we have

$$D(x) \ge \sum_{\substack{\frac14 L_2x<k\le\frac34 L_2x\\ k\in\mathcal{S}'}} \sum_{\substack{m\in\mathcal{S},\,2^{k-1}km\le x\\ (m,2k)=1,\,\omega(m)\le k-1-\omega(k)}} 1.$$

First, we solve the inequality $2^{k-1}km \le x$ for m. Then we partition the set of integers m contributing to the last sum, according to the value of $\omega(m)$.

Ergo,

$$D(x) \geq \sum_{\substack{\frac{1}{4}L_2x<k\leq\frac{3}{4}L_2x\\ k\in\mathcal{S}'}} \sum_{1\leq j\leq k-1-\omega(k)} \sum_{\substack{m\in\mathcal{S},\, m\leq 2^{1-k}x/k\\ (m,2k)=1,\omega(m)=j}} 1.$$

We deduce from Lemma 7, with $A = B = 1$, that the innermost sum is

$$\frac{v}{\log v}\frac{(L_2v)^{j-1}}{(j-1)!}\biggl(f\Bigl(\frac{j-1}{L_2v}\Bigr)f_k\Bigl(\frac{j-1}{L_2v}\Bigr)+O\Bigl(\frac{j(L_4x)^3}{(L_2x)^2}\Bigr)\biggr),$$

where $v = 2^{1-k}x/k$. Now

$$\log v = \log x + (1-k)\log 2 - \log k = (\log x)(1 + O((L_2x)/\log x)) \tag{9}$$

for $k \leq L_2x$. Hence,

$$L_2v = L_2x + \log(1 + O((L_2x)/\log x)) = L_2x + O((L_2x)/\log x).$$

From the Binomial Theorem, we can conclude that

$$\frac{(L_2v)^{j-1}}{(j-1)!} = \frac{(L_2x)^{j-1}}{(j-1)!}\biggl(1+O\Bigl(\frac{1}{\log x}\Bigr)\biggr)^{j-1} = \frac{(L_2x)^{j-1}}{(j-1)!}\biggl(1+O\Bigl(\frac{jL_2x}{\log x}\Bigr)\biggr)$$

uniformly in $k \leq L_2x$, $j \leq L_2x$. It now follows from (7) and Lemma 8 that the last sum on m is

$$\frac{x}{2^{k-1}k\log x}\frac{(L_2x)^{j-1}}{(j-1)!}\biggl(f\Bigl(\frac{j-1}{L_2v}\Bigr)f_k\Bigl(\frac{j-1}{L_2v}\Bigr)+O\Bigl(\frac{j(L_4x)^3}{(L_2x)^2}\Bigr)\biggr),$$

uniformly in $k \leq L_2x,\ j \leq L_2x$. Thus, we can deduce from Lemma 8 and from (7), with $B = 3/4$, that

$$D(x) \gg \frac{x(L_4x)^{-3/4}}{\log x} \sum_{\substack{\frac{1}{4}L_2x<k\leq\frac{3}{4}L_2x\\ k\in\mathcal{S}'}} \frac{1}{2^k k} \sum_{1\leq j\leq k-1-\omega(k)} \frac{(L_2x)^{j-1}}{(j-1)!}. \tag{10}$$

Next, we replace the inner sum by the last summand, to obtain

$$\begin{aligned} D(x) &\gg \frac{x(L_4x)^{-3/4}}{\log x} \sum_{\substack{\frac{1}{4}L_2x<k\leq\frac{3}{4}L_2x\\ k\in\mathcal{S}'}} \frac{1}{2^k k}\frac{(L_2x)^{k-2-\omega(k)}}{(k-2-\omega(k))!} \\ &\gg \frac{x(L_4x)^{-3/4}}{(\log x)L_2x} \sum_{\substack{k,|\frac{1}{2}L_2x-k|\leq\frac{1}{2}(L_2x)/L_3x\\ k\in\mathcal{S}'}} \frac{1}{2^{\omega(k)}}\frac{\bigl(\frac{1}{2}L_2x\bigr)^{k-2-\omega(k)}}{(k-2-\omega(k))!}. \end{aligned}$$

Here, the replacement of the range $\frac{1}{4}L_2x < k \leq \frac{3}{4}L_2x$ by the inequality $\bigl|\frac{1}{2}L_2x - k\bigr| \leq \frac{1}{2}(L_2x)/L_3x$ just makes the sum on k tinier. Now, we want to apply Lemma 11 to replace $k-2-\omega(k)$ by k twice, on the right-hand side. To do so, we first use Lemma 4(i) to deduce that $2+\omega(k) \leq 3(\log k)/L_2k$

$\le \frac{1}{2}L_2x$. Then we apply the aforementioned Lemma 11 to get

$$D(x) \gg \frac{x(L_4x)^{-3/4}}{(\log x)L_2x} \sum_{\substack{|k-\frac{1}{2}L_2x|\le\frac{1}{2}(L_2x)/L_3x \\ k\in\mathcal{S}'}} \frac{1}{2^{\omega(k)}} \frac{\left(\frac{1}{2}L_2x\right)^k}{k!}.$$

When we rewrite the sum on the right as

$$\sum_{\substack{k=1 \\ k\in\mathcal{S}'}}^{\infty} \frac{1}{2^{\omega(k)}} \frac{\left(\frac{1}{2}L_2x\right)^k}{k!} - \sum_{\substack{|k-\frac{1}{2}L_2x|>\frac{1}{2}(L_2x)/L_3x \\ k\in\mathcal{S}'}} \frac{1}{2^{\omega(k)}} \frac{\left(\frac{1}{2}L_2x\right)^k}{k!},$$

and observe that the last sum will increase if we omit the condition $k \in \mathcal{S}'$ from beneath it, we are able to deduce that

$$D(x) \gg \frac{x(L_4x)^{-3/4}}{(\log x)L_2x} \sum_{\substack{k=1 \\ k\in\mathcal{S}'}}^{\infty} \frac{1}{2^{\omega(k)}} \frac{\left(\frac{1}{2}L_2x\right)^k}{k!} + E(x), \tag{11}$$

where

$$E(x) = O\left(\frac{x(L_4x)^{-3/4}}{(\log x)L_2x} \sum_{|k-\frac{1}{2}L_2x|>\frac{1}{2}(L_2x)/L_3x} \frac{1}{2^{\omega(k)}} \frac{\left(\frac{1}{2}L_2x\right)^k}{k!}\right).$$

We majorize $1/2^{\omega(k)}$ by $1/2 \le 1$, in the last summation. Then we deduce from Lemma 12 that

$$E(x) \ll \frac{x(L_4x)^{-3/4}}{\sqrt{\log x}(L_2x)^{3/2}L_3x}. \tag{12}$$

To show that $E(x)$ is of smaller order than the main term on the right of (11), we restrict the sum on k in (11) to the range $\left|k-\frac{1}{2}L_2x\right| \le \sqrt{(L_2x)L_3x}$, and then apply Lemma 4(i) to bound $1/2^{\omega(k)}$ from below in that range. Thus,

$$\sum_{\substack{k=1 \\ k\in\mathcal{S}'}}^{\infty} \frac{1}{2^{\omega(k)}} \frac{\left(\frac{1}{2}L_2x\right)^k}{k!} > \sum_{\substack{|k-\frac{1}{2}L_2x|\le\sqrt{(L_2x)L_3x} \\ k\in\mathcal{S}'}} \frac{1}{2^{2(\log k)/L_2k}} \frac{\left(\frac{1}{2}L_2x\right)^k}{k!}.$$

Now for k contributing to the right-hand side, the exponent $2(\log k)/L_2k$ is less than $.1L_3x$. Ergo,

$$\frac{1}{2^{2(\log k)/L_2k}} > (L_2x)^{-.1\log 2} > (L_2x)^{-.07}.$$

So, upon using the identity

$$\sum_{|k-\frac{1}{2}L_2x|\le\sqrt{(L_2x)L_3x}} = \sum_{\text{all } k} - \sum_{|k-\frac{1}{2}L_2x|>\sqrt{(L_2x)L_3x}},$$

we find that

$$\sum_{\substack{k=1\\ k\in\mathcal{S}'}}^{\infty}\frac{1}{2^{\omega(k)}}\frac{\left(\frac12 L_2x\right)^k}{k!}$$
$$> (L_2x)^{-.07}\Bigg(\sum_{\substack{k=0\\ k\in\mathcal{S}'}}^{\infty}\frac{\left(\frac12 L_2x\right)^k}{k!}+O\Bigg(\sum_{|k-\frac12 L_2x|>\sqrt{(L_2x)L_3x}}\frac{\left(\frac12 L_2x\right)^k}{k!}\Bigg)\Bigg).$$

By Lemma 3, the penultimate sum is

$$\frac{2}{\pi^2}e^{\frac12 L_2x}\left(1+O\left(\frac{1}{L_3x}\right)\right)>\frac{1}{10}\sqrt{\log x}.$$

The ultimate sum is estimated with the aid of Lemma 12. Accordingly,

$$\sum_{\substack{k=1\\ k\in\mathcal{S}'}}^{\infty}\frac{1}{2^{\omega(k)}}\frac{\left(\frac12 L_2x\right)^k}{k!}>(L_2x)^{-.07}\left(\frac{1}{10}\sqrt{\log x}+O\left(\sqrt{\frac{\log x}{(L_2x)L_3x}}\right)\right)$$
$$\gg (L_2x)^{-.07}\sqrt{\log x}.$$

Consequently, (12) implies that

$$E(x)=o\left(\frac{x(L_4x)^{-3/4}}{(\log x)L_2x}\sum_{k=1}^{\infty}\frac{1}{2^{\omega(k)}}\frac{\left(\frac12 L_2x\right)^k}{k!}\right).$$

The first inequality of Theorem 1 now follows from (11), for a sufficiently small constant c_0.

Note. With more effort, we can replace the factor $(L_4x)^{-3/4}$ by $1/\sqrt{L_4x}$. The main idea is to restrict the sum on k to the range $\frac14 L_2x<k\le\frac12 L_2x+g(x)$, for an appropriate function $g(x)$, prior to the application of the lower bound in (7).

5. An upper bound for $D(x)$**.** In view of Lemma 6, we have

$$D(x)\le\sum_{n=2^{k-1}(\mathrm{odd}(kl))mt\le x}'' 1,$$

where the double dashes imply that $m\in\mathcal{S}$, $t\in T$, $\prod_{p\mid l}p\mid k$, and

$$\omega(m)\le k-1-\nu_2(k)-\nu_2(d(\mathrm{odd}(kl))). \tag{13}$$

Now, we apply Lemma 9 with $y=(\log x)^2$, to restrict the size of the divisor t of n. We simultaneously constrain the size of the largest squarefull divisor

$r(kl)$ of $\mathrm{odd}(kl)$. This application yields

$$D(x) \le \sideset{}{''}\sum_{\substack{n=2^{k-1}(\mathrm{odd}(kl))mt\le x\\ t\le(\log x)^2\\ r(kl)\le(\log x)^2}} 1 + O\left(\frac{x}{\log x}\right).$$

When we similarly apply Lemma 10 with $w = \frac{3}{4}L_2x$ to restrict the range of k, and note that $\prod_{p|l} p \mid k$ if and only if $l \mid k^l$, we find that

$$D(x) \ll \sum_{\substack{t\le(\log x)^2\\ t\in T}} \sum_{1\le k\le\frac{3}{4}L_2x} \sum_{\substack{l,\, l|k^l\\ r(kl)\le(\log x)^2}} \sum_{\substack{m\le\frac{x}{2^{k-1}(\mathrm{odd}(kl))t}\\ (13)\text{ holds}}} 1 + \frac{x}{(\log x)^{.501}}. \tag{14}$$

Assume that l contributes to the penultimate sum in (14). If p is a prime divisor of $\mathrm{odd}(kl)$, then p makes a positive contribution to $\omega(s(k)) - \omega(l)$ if and only if we have

$$p \parallel k \quad \text{and} \quad p \nmid l. \tag{15}$$

If (15) holds, then p contributes 1 to the additive function

$$\nu_2(d(\mathrm{odd}(kl))) \tag{16}$$

of kl. In any case, p contributes not less than 0 to (16). So,

$$\nu_2(d(\mathrm{odd}(kl))) \ge \omega(s(k)) - \omega(l). \tag{17}$$

Combining (17) with (13) yields

$$\omega(m) \le k - 1 - \nu_2(k) - \omega(s(k)) + \omega(l).$$

Furthermore, $\omega(m) \le k - 1 - \nu_2(k)$, by (13). It therefore follows that

$$\omega(m) \le \min\{k - 1 - \nu_2(k), k - 1 - \nu_2(k) - \omega(s(k)) + \omega(l)\} := h(k,l). \tag{18}$$

So, we can conclude from our last estimate for $D(x)$ that

$$D(x) \le \sum_{\substack{t\le(\log x)^2\\ t\in T}} \sum_{1\le k\le\frac{3}{4}L_2x} \sum_{\substack{l,\, l|k^l\\ r(kl)\le(\log x)^2}} \sum_{\substack{m\le\frac{x}{2^{k-1}(\mathrm{odd}(kl))t}\\ (18)\text{ holds}}} 1 + O\left(\frac{x}{(\log x)^{.501}}\right). \tag{19}$$

Upon partitioning the set of m contributing to the last sum according to the value of $\omega(m)$, and applying Lemmas 7 and 8, we find that this last sum is of order

$$1 + \sum_{1\le j\le h(k,l)} \frac{w}{\log w}\frac{(L_2w)^{j-1}}{(j-1)!} f_k\left(\frac{j-1}{L_2w}\right),$$

where $w = x/(2^{k-1}\,\mathrm{odd}(klt))$. Since $w < x$, we have $L_2 w < L_2 x$, and $(j-1)/L_2 w > (j-1)/L_2 x$. By definition, $f_k(z)$ is a decreasing function of z, so that $f_k((j-1)/L_2 w) \le f_k((j-1)/L_2 x)$. To bound $1/\log w$, we further observe that the constraint $r(kl) \le (\log x)^2$ forces l not to exceed $(\log x)^2$, inasmuch as any prime divisor of l must also divide $\mathrm{odd}(k)$. Hence, we can deduce from the inequalities $t \le (\log x)^2$ and $1 \le k \le \frac{3}{4} L_2 x$ that $\log w \gg \log x$. Consequently, the innermost sum in (19) is of order

$$1 + \sum_{1 \le j \le h(k,l)} \frac{x}{\log x} \frac{f_k\left(\frac{j-1}{L_2 x}\right)}{2^{k-1}\,\mathrm{odd}(klt)} \frac{(L_2 x)^{j-1}}{(j-1)!}. \tag{20}$$

Now

$$\frac{y^{j+1}}{(j+1)!} = \frac{y^j}{j!} \frac{y}{j+1}. \tag{21}$$

Since $1 \ll f_k((j-1)/L_2 x) \ll 1$ for $1 \le j \le k \le 4 L_2 x$, Equation (21) suggests that the function

$$f_k\left(\frac{j-1}{L_2 x}\right) \frac{(L_2 x)^{j-1}}{(j-1)!} \tag{22}$$

might be increasing geometrically in the range $j \le k \le \frac{3}{4} L_2 x$. To show that this is indeed the case, we use the definition of $f_k(z)$ to compute $f_k(j/L_2 x)/f_k((j-1)/L_2 x)$, and show that this quantity is not too small. We get

$$\begin{aligned}
\frac{f_k\left(\dfrac{j}{L_2 x}\right)}{f_k\left(\dfrac{j-1}{L_2 x}\right)} &= \prod_{p \mid k} \left(1 + \frac{j}{pL_2 x}\right)^{-1} \left(1 + \frac{j-1}{pL_2 x}\right) \\
&= \prod_{p \mid k} \left(1 - \frac{j}{pL_2 x} + \left(\frac{j}{pL_2 x}\right)^2 \left(1 - \frac{j}{pL_2 x}\right)^{-1}\right) \left(1 + \frac{j-1}{pL_2 x}\right) \\
&> \prod_{p \mid k} \left(1 - \frac{j}{pL_2 x}\right) \left(1 + \frac{j-1}{pL_2 x}\right) \\
&= \prod_{p \mid k} \left(1 - \frac{1}{pL_2 x} - \frac{j}{pL_2 x} \frac{j-1}{pL_2 x}\right).
\end{aligned}$$

But for $j \le k \le \frac{3}{4} L_2 x$, we have $(j-1)/L_2 x < j/L_2 x \le \frac{3}{4}$. Accordingly,

$$\frac{f_k\left(\dfrac{j}{L_2 x}\right)}{f_k\left(\dfrac{j-1}{L_2 x}\right)} \ge \prod_{p \mid k} \left(1 - \frac{1}{pL_2 x} - \frac{9/16}{p^2}\right).$$

If $p < .03L_2x$, we bound $-1/(pL_2x)$ from below by $.03/p^2$. Otherwise, we observe that since $p \mid k$, and $k \le \frac{3}{4}L_2x$, we have $p \le \frac{3}{4}L_2x$, so that

$$-\frac{1}{L_2x} \ge \frac{3/4}{p}.$$

Thus, $-1/(pL_2x) \ge (3/4)/p^2$ if $p \ge .03L_2x$. So, we have

$$\frac{f_k\left(\dfrac{j}{L_2x}\right)}{f_k\left(\dfrac{j-1}{L_2x}\right)} \ge \Bigg(\prod_{p\mid k,\, p<.04L_2x}\left(1-\frac{9/16}{p^2}\right)\Bigg)\Bigg(\prod_{p\mid k,\, p\ge .04L_2x}\left(1-\frac{18/16}{p^2}\right)\Bigg).$$

If $k \le \frac{3}{4}L_2x$, and x is sufficiently large, then there can be at most one prime $p \ge .03L_2x$ with $p \mid k$. So, the last product exceeds $1 - 1.3175/(.03L_2x)^2 > .9999$, for x large enough. Thus, if we omit the constraints on p from the penultimate product, we can conclude that

$$\frac{f_k\left(\dfrac{j}{L_2x}\right)}{f_k\left(\dfrac{j-1}{L_2x}\right)} \ge .9999\prod_p\left(1-\frac{9/16}{p^2}\right) > 0.76 \quad \text{for } j \le k \le \frac{3}{4}L_2x.$$

It therefore follows from (21) that the function (22) grows geometrically (or faster) for j, k in this range. In this event, the sum (20) has the same order of magnitude as the term of the sum with j as large as possible. Consequently,

$$\begin{aligned}(23)\qquad D(x) &\ll \frac{x}{\log x}\sum_{\substack{t\le(\log x)^2\\ t\in T}}\frac{1}{t}\sum_{1\le k\le\frac{3}{4}L_2x}\frac{1}{2^{k-1}\operatorname{odd}(k)}\\ &\quad\times\sum_{\substack{l,\, l\mid k^l\\ r(kl)\le(\log x)^2}}\frac{f_k((h(k,l)-1)/L_2x)}{\operatorname{odd}(l)}\frac{(L_2x)^{h(k,l)-1}}{(h(k,l)-1)!}+\frac{x}{(\log x)^{.501}}.\end{aligned}$$

Since $1/\operatorname{odd}(l) < 2/l$, we replace $\operatorname{odd}(l)$ by l. Then we interchange the order of summation to make the sum on t the innermost sum. (Note that the constraint $r(kl) \le (\log x)^2$ implies that our iterated sum is finite.) Then we apply the fact that the sum of the reciprocals of the squarefull integers converges (see Erdős and Szekeres' paper [ES]) to obtain the bound

$$\begin{aligned}D(x) &\ll \frac{x}{\log x}\sum_{1\le k\le\frac{3}{4}L_2x}\frac{1}{2^{k-1}\operatorname{odd}(k)}\\ &\quad\times\sum_{\substack{l,\, l\mid k^l\\ r(kl)\le(\log x)^2}}\frac{1}{l}f_k\left(\frac{h(k,l)-1}{L_2x}\right)\frac{(L_2x)^{h(k,l)-1}}{(h(k,l)-1)!}+\frac{x}{(\log x)^{.501}}.\end{aligned}$$

Upon multiplying and dividing by $2^{h(k,l)-1}$, we find that

$$(24)\qquad D(x) \ll \frac{x}{\log x} \sum_{1 \le k \le \frac{3}{4} L_2 x} \frac{2^{-\omega(s(k))}}{2^{\nu_2(k)}\,\mathrm{odd}(k)} \times \sum_{\substack{l,\, l \mid k^l \\ r(kl) \le (\log x)^2}} \frac{2^{\omega(l)}}{l} f_k\left(\frac{h(k,l)-1}{L_2 x}\right) \frac{\left(\frac{1}{2} L_2 x\right)^{h(k,l)-1}}{(h(k,l)-1)!} + \frac{x}{(\log x)^{.501}}.$$

Next, we substitute k for $2^{\nu_2(k)}\,\mathrm{odd}(k)$. Then, we partition the sum on k according to whether or not $\left|k - \frac{1}{2} L_2 x\right| \le \frac{1}{2}(L_2 x)/L_3 x$. So,

$$(25)\qquad D(x) \ll \frac{x}{\log x}\left(\sum\nolimits_1 + \sum\nolimits_2\right) + \frac{x}{(\log x)^{.501}},$$

where

$$\sum\nolimits_1 = \sum_{k,\, |k - \frac{1}{2} L_2 x| \le \frac{1}{2}(L_2 x)/L_3 x} \frac{2^{-\omega(s(k))}}{k} \sum_l ,$$

$$\sum\nolimits_2 = \sum_{\substack{k \le \frac{3}{4} L_2 x \\ k,\, |k - \frac{1}{2} L_2 x| > \frac{1}{2}(L_2 x)/L_3 x}} \frac{2^{-\omega(s(k))}}{k} \sum_l ,$$

and the sum over l is as in (24). Now by (18),

$$k - (h(k,l) - 1) \le 2 + \nu_2(k) + \omega(k),$$

and $k - (h(k,l) - 1) \ge 0$ for k sufficiently large. Thus, Lemma 4(ii) implies that

$$(26)\qquad 0 \le k - (h(k,l) - 1) \le 2 + \frac{\log k}{\log 2} + \frac{2 \log k}{L_2 k}$$

for k sufficiently large. Consequently, we can apply Lemma 11 with $m = k - (h(k,l) - 1)$, and $y = \frac{1}{2} L_2 x$, to obtain

$$(27)\qquad \sum\nolimits_1 \ll \sum_{k,\, |k - \frac{1}{2} L_2 x| \le \frac{1}{2}(L_2 x)/L_3 x} \frac{2^{-\omega(s(k))}}{k} \frac{\left(\frac{1}{2} L_2 x\right)^k}{k!} \sum_{l,\, l \mid k^l} \frac{2^{\omega(l)}}{l} f_k\left(\frac{h(k,l)-1}{L_2 x}\right).$$

Here, we have omitted the condition that $r(kl) \le (\log x)^2$, and thereby increased the right-hand side.

Utilize Lemma 7 to replace $f_k((h(k,l) - 1)/L_2 x)$ by 1 in the definition of $\sum_2$. Then let $J = J(x)$ be the smallest integer exceeding $2L_3 x$, so that $J > 2 + \nu_2(k) + \omega(s(k)) - \omega(l)$, provided that $k \le \frac{3}{4} L_2 x$, and that x is sufficiently large. Moreover, the function $\left(\frac{1}{2} L_2 x\right)^j / j!$ of j is increasing for $j < \frac{1}{2} L_2 x - 1$, and decreasing for $j > \frac{1}{2} L_2 x$. We partition the sum $\sum_2$

according to whether k exceeds $\frac{1}{2}L_2x$. If k does exceed $\frac{1}{2}L_2x$, we replace $h(k,l)$ by $k-J+1$, and otherwise we replace $h(k,l)$ by $k-1$. Accordingly,

$$\sum\nolimits_2 \le \sum\nolimits_3 + \sum\nolimits_4 \tag{28}$$

where

$$\sum\nolimits_3 = \sum_{1\le k\le \frac{1}{2}L_2x-\frac{1}{2}(L_2x)/L_3x} \frac{2^{-\omega(s(k))}}{k}\frac{\left(\frac{1}{2}L_2x\right)^k}{k!}\sum_{l,\, l|k^l}\frac{2^{\omega(l)}}{l}, \tag{29}$$

$$\sum\nolimits_4 = \sum_{\frac{1}{2}L_2x+\frac{1}{2}(L_2x)/L_3x<k\le \frac{3}{4}L_2x} \frac{2^{-\omega(s(k))}}{k}\frac{\left(\frac{1}{2}L_2x\right)^{k-J(x)}}{(k-J(x))!}\sum_{l,\, l|k^l}\frac{2^{\omega(l)}}{l}. \tag{30}$$

Again, we have deleted the constraint $r(kl)\le(\log x)^2$.

Since $\omega(n)$ is additive, and takes the value 1 at nontrivial prime powers, we have

$$\sum_{l,\, l|k^l}\frac{2^{\omega(l)}}{l} = \prod_{p|k}\left(1+\sum_{h=1}^{\infty}\frac{2}{p^h}\right).$$

Ergo, we can conclude from Lemma 13 with $c=2$ that

$$\sum_{l,\, l|k^l}\frac{2^{\omega(l)}}{l} < \prod_{p|k}\sum_{h=0}^{\infty}\left(\frac{2}{p}\right)^h = \prod_{p|k}\left(1-\frac{2}{p}\right)^{-1} \ll (L_2k)^2. \tag{31}$$

When we combine this estimate with the inequality $k<L_2x$, we find that

$$\sum_{l,\, l|k^l}\frac{2^{\omega(l)}}{l} \ll (L_4x)^2.$$

When we apply this bound to estimate the innermost sum in each of (29) and (30), we find that

$$\sum\nolimits_3 \ll (L_4x)^3 \sum_{1\le k\le \frac{1}{2}L_2x-\frac{1}{2}(L_2x)/L_3x} \frac{2^{-\omega(s(k))}}{k}\frac{\left(\frac{1}{2}L_2x\right)^k}{k!}, \tag{32}$$

$$\sum\nolimits_4 \ll (L_4x)^3 \sum_{\frac{1}{2}L_2x+\frac{1}{2}(L_2x)/L_3x<k\le \frac{3}{4}L_2x} \frac{2^{-\omega(s(k))}}{k}\frac{\left(\frac{1}{2}L_2x\right)^{k-J(x)}}{(k-J(x))!}. \tag{33}$$

Next, we bound $2^{-\omega(s(k))}$ by 1 in (32) and (33), and majorize $1/k$ by $1/L_2x$ in (33). In (32), we observe that

$$\frac{1}{k}\frac{\left(\frac{1}{2}L_2x\right)^k}{k!} = \frac{1}{L_2x}\frac{k+1}{k}\frac{\left(\frac{1}{2}L_2x\right)^{k+1}}{(k+1)!}.$$

So,

$$\sum\nolimits_3 \ll \frac{(L_4x)^3}{L_2x} \sum_{1\le k\le \frac12 L_2x-\frac12(L_2x)/L_3x} \frac{\left(\frac12 L_2x\right)^{k+1}}{(k+1)!},$$

$$\sum\nolimits_4 \ll \frac{(L_4x)^3}{L_2x} \sum_{\frac12 L_2x+\frac12(L_2x)/L_3x<k\le\frac34 L_2x} \frac{\left(\frac12 L_2x\right)^{k-J(x)}}{(k-J(x))!}.$$

Since $J(x) < 2L_3x+1$, the quantity $k+J(x)$ runs through only integers exceeding $\frac12 L_2x+\frac12(L_2x)/L_3x-2L_3x-1$. Thus,

$$\sum\nolimits_3 + \sum\nolimits_4 \le \frac{(L_4x)^3}{L_2x} \sum_{|k-\frac12 L_2x|\ge\sqrt{\frac12(L_2x)\log(\frac12 L_2x)}} \frac{\left(\frac12 L_2x\right)^{k}}{k!}.$$

By Lemma 13,

$$\sum\nolimits_3 + \sum\nolimits_4 \ll \frac{(L_4x)^3}{(L_2x)^{3/2}}\sqrt{\log x}.$$

Recalling (25) and (28) gives

$$D(x) \ll \frac{x}{\log x}\sum\nolimits_1 + \frac{x}{\sqrt{\log x}}\frac{(L_4x)^3}{(L_2x)^{3/2}}. \tag{34}$$

It therefore remains to estimate $\sum_1$. Toward that end, we note that (23) and (26) insure that

$$\frac{h(k,l)-1}{L_2x} = \frac12 + O\left(\frac{1}{L_3x}\right).$$

So, we can conclude from (27) that

$$\sum\nolimits_1 \ll \sum_{|k-\frac12 L_2x|\le\frac12(L_2x)/L_3x} f_k\left(\frac12+O\left(\frac{1}{L_3x}\right)\right) \times \frac{2^{-\omega(s(k))}}{k}\frac{\left(\frac12 L_2x\right)^{k}}{k!}\sum_{l,\, l\mid k^l}\frac{2^{\omega(s(l))}}{l}.$$

By (31),

$$\sum\nolimits_1 \ll \sum_{|k-\frac12 L_2x|\le\frac12(L_2x)/L_3x} f_k\left(\frac12+O\left(\frac{1}{L_3x}\right)\right) \times \prod_{p\mid k}\left(1-\frac{2}{p}\right)^{-1}\frac{2^{-\omega(s(k))}}{k}\frac{\left(\frac12 L_2x\right)^{k}}{k!}.$$

Now in view of our notation, we have

$$(35)\qquad f_k\left(\frac{1}{2}+O\left(\frac{1}{L_3x}\right)\right)\prod_{p|k}\left(1-\frac{2}{p}\right)^{-1}$$
$$=\prod_{p|k}\left(1+\frac{1/2}{p}+O\left(\frac{1}{pL_3x}\right)\right)^{-1}\left(1-\frac{2}{p}\right)^{-1}$$
$$=\prod_{p|k}\left(1-\frac{3/2}{p}\right)^{-1}\left(1+O\left(\frac{1}{pL_3x}\right)\right)^{-1}\left(1+O\left(\frac{1}{p^2}\right)\right).$$

Since $k\le 2L_2x$, it follows from Lemma 13(ii) that

$$(36)\qquad \prod_{p|k}\left(1+O\left(\frac{1}{pL_3x}\right)\right)^{-1}\ll (L_4x)^{O(1/L_3x)}\ll 1.$$

Furthermore,

$$(37)\qquad \prod_{p|s(k)}\left(1+\frac{O(1)}{p^2}\right)\ll\prod_{p}\left(1+\frac{O(1)}{p^2}\right),$$

and the last product converges. Finally, another application of Lemma 13(ii) yields

$$(38)\qquad \prod_{p|k}\left(1-\frac{3/2}{p}\right)^{-1}\ll (L_2k)^{3/2}\ll (L_2x)^{3/2}.$$

Combining (35)–(38), and substituting the result into our last bound for $\sum_1$, yields the inequality

$$\sum\nolimits_1\ll (L_4x)^{3/2}\sum_{|k-\frac{1}{2}L_2x|\le\frac{1}{2}(L_2x)/L_3x}\frac{2^{-\omega(s(k))}}{k}\frac{\left(\frac{1}{2}L_2x\right)^k}{k!}.$$

Clearly, we can replace the first k in the denominator by L_2x. When we substitute the result into (34), we arrive at

$$(39)\qquad D(x)\ll\frac{x}{\log x}\frac{(L_4x)^{3/2}}{L_2x}\sum_{|k-\frac{1}{2}L_2x|\le\frac{1}{2}(L_2x)/L_3x}2^{-\omega(s(k))}\frac{\left(\frac{1}{2}L_2x\right)^k}{k!}$$
$$+\frac{x}{\sqrt{\log x}}\left(\frac{L_4x}{L_2x}\right)^{3/2}.$$

Next, we show that the term $(x/\sqrt{\log x})(L_4x/L_2x)^{3/2}$ is of smaller order than the other part of the right-hand side. By Lemma 4(i), any k contributing to the sum satisfies

$$2^{-\omega(s(k))}\ge 2^{-\omega(k)}\ge 2^{-2(\log k)/L_2k}>2^{-3(L_3x)/L_4x}.$$

Therefore, the other part is of order at least

$$\frac{x}{\log x}\frac{1}{L_2x}2^{-3(L_3x)/L_4x}\sum_{|k-\frac{1}{2}L_2x|\le\frac{1}{2}(L_2x)/L_3x}\frac{\left(\frac{1}{2}L_2x\right)^k}{k!}.$$

In the same manner as we argued at the end of Section 4, we utilize Lemma 12 to obtain

$$\begin{aligned}\sum_{|k-\frac{1}{2}L_2x|\le\frac{1}{2}(L_2x)/L_3x}&\frac{\left(\frac{1}{2}L_2x\right)^k}{k!}\\ &=\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}L_2x\right)^k}{k!}+O\bigg(\sum_{|k-\frac{1}{2}L_2x|>\frac{1}{2}(L_2x)/L_3x}\frac{\left(\frac{1}{2}L_2x\right)^k}{k!}\bigg)\\ &=\sqrt{\log x}+O\bigg(\frac{\sqrt{\log x}}{\sqrt{(L_2x)L_3x}}\bigg).\end{aligned}$$

Consequently, the other part is of order at least

$$\frac{x}{\sqrt{\log x}}\frac{1}{L_2x}2^{-3(L_3x)/L_4x},$$

which is of greater order than $(x/\sqrt{\log x})(L_4x/L_2x)^{3/2}$. It follows that

$$D(x)\ll\frac{x}{\log x}\frac{(L_4x)^{3/2}}{L_2x}\sum_{|k-\frac{1}{2}L_2x|\le\frac{1}{2}(L_2x)/L_3x}2^{-\omega(s(k))}\frac{\left(\frac{1}{2}L_2x\right)^k}{k!}. \tag{40}$$

Upon removal of the inequality below the sum, the upper bound part of Theorem 1 follows, for a sufficiently large constant c_1.

Note. If we merely want to establish the weaker result obtained by replacing the extreme right-hand side of the inequality in Theorem 1 by its product with $\sqrt{L_4x}$, we can shorten the proof by bounding f_k by 1 in (20). Then we observe that $(L_2x)^{j-1}/(j-1)!$ is geometrically increasing.

6. The proof of Theorem 2. Concluding remarks. We state the generalization of the lower estimate in Theorem 1 to the κ-fold iterated divisor functions $d_\kappa(n)$, as the next theorem.

THEOREM 3.

$$D_\kappa(x)\gg_\kappa\frac{x}{(\log x)(L_2x)^{\kappa-1}}\sum_{\substack{i=1\\ i\equiv1\bmod(\kappa!)^2\\ \binom{i+\kappa-2}{\kappa-1}\ squarefree}}^{\infty}\frac{\left(\frac{1}{\kappa}L_2x\right)^i}{i!\kappa^{\omega(i)}}\quad for\ \kappa\ge2.$$

The proof requires the following generalization of Lemma 3(i).

Lemma 14. *If $\kappa \geq 2$, then there exists a constant $c_1(\kappa) > 0$ for which*

$$\sum_{\substack{n=0 \\ n \equiv 1 \bmod (\kappa!)^2 \\ \binom{n+\kappa-2}{\kappa-1} \text{ squarefree}}}^{\infty} \frac{y^n}{n!} = c_1(\kappa)e^y + O\left(\frac{e^y}{\sqrt{y}}\right) \quad \text{for } y \geq 1.$$

The proof of Lemma 14 is similar to the proof of Lemma 13(ii). We omit the details. We remark that the binomial coefficient $\binom{n+\kappa-2}{\kappa-1}$ can be replaced by any integer-valued polynomial $\psi(n)$ and the congruence $n \equiv 1 \bmod (\kappa!)^2$ can be replaced by any finite set of congruences in n. (Of course, the constant $c_1(\kappa)$ is replaced by a constant depending upon the system of congruences in n, and upon the polynomial.) Moreover, the constant will be positive if and only if $\psi(n)$ assumes at least one squarefree value. ∎

The details of the proof of Theorem 3 are analogous to the arguments of Section 4. First, we let $\mathcal{S}'' = \mathcal{S}''(\kappa)$ denote the set of positive integers k for which

$$k \equiv 1 \bmod (\kappa!)^2 \quad \text{and} \quad \binom{k+\kappa-2}{\kappa-1} \text{ is squarefree.}$$

Then we partition the set of positive integers n such that $d_\kappa(n)$ divides n according to the integer k such that $\kappa^k \mid n$, but $\kappa^{k+1} \nmid n$. Here, $\kappa = d_\kappa(p)$ for any prime p, so that κ plays the role of 2. Thus,

$$D_\kappa(x) \geq \sum_{\frac{1}{2\kappa} L_2 x < k \leq \frac{3}{4} L_2 x} \sum_{\substack{n \leq x \\ \kappa^k \mid n,\, \kappa^{k+1} \nmid n \\ d_\kappa(n) \mid n}} 1 \geq \sum_{\substack{\frac{1}{2\kappa} L_2 x < k \leq \frac{3}{4} L_2 x \\ k \in \mathcal{S}''}} \sum_{\substack{n \leq x \\ \kappa^k \mid n \\ (\kappa, n/\kappa^k) = 1 \\ d_\kappa(n) \mid n}} 1.$$

The rest of the argument is almost identical to that of Section 4, the main differences being the application of Lemma 14 in lieu of Lemma 3(i), and that in (10), the quantity $1/(2^k k)$ is replaced by

$$\frac{1}{\kappa^k \binom{k+\kappa-2}{\kappa-1}^{\omega(\kappa)}}.$$

Then we immediately note that

$$\frac{1}{\binom{k+\kappa-2}{\kappa-1}} = \frac{(\kappa-1)!}{(k+1)(k+2)\ldots(k+\kappa-2)} \gg \frac{1}{(L_2 x)^{\kappa-1}}.$$

We leave the remainder of the details to the reader. ∎

The bulk of the contribution to the sum in Theorem 3 occurs for k near

$\frac{1}{\kappa}L_2x$. Indeed, it follows from Theorem 3 that

$$D_\kappa(x) \gg \frac{x}{(\log x)(L_2x)^{\kappa-1}} \sum_{\substack{|k-\frac{1}{\kappa}L_2x|<\frac{1}{\kappa}(L_2x)/L_3x \\ k\in\mathcal{S}''}} \frac{1}{\kappa^{\omega(k)}} \frac{\left(\frac{1}{\kappa}L_2x\right)^k}{k!}.$$

Let $\varepsilon > 0$ be given. For integers k contributing to the sum, we can deduce from Lemma 4(ii) that

$$\frac{1}{\kappa^{\omega(k)}} \geq \frac{1}{\kappa^{(1+\varepsilon/2)(\log k)/L_2k}} > \frac{1}{\kappa^{(1+\varepsilon)(L_3x)/L_4x}}.$$

We deduce that

$$D_\kappa(x) \gg \frac{x}{(\log x)(L_2x)^{\kappa-1}} \kappa^{-(1+\varepsilon)(L_3x)/L_4x} \sum_{\substack{k\in\mathcal{S}'' \\ |k-\frac{1}{\kappa}L_2x|\leq\frac{1}{\kappa}(L_2x)/L_3x}} \frac{\left(\frac{1}{\kappa}L_2x\right)^k}{k!}. \tag{41}$$

Now we can conclude from Lemma 12 that

$$\begin{aligned}
&\sum_{\substack{k\in\mathcal{S}'' \\ |k-\frac{1}{\kappa}L_2x|\leq\frac{1}{\kappa}(L_2x)/L_3x}} \frac{\left(\frac{1}{\kappa}L_2x\right)^k}{k!} \\
&\quad = \sum_{\substack{k=0 \\ k\in\mathcal{S}''}}^{\infty} \frac{\left(\frac{1}{\kappa}L_2x\right)^k}{k!} + O\Bigg(\sum_{\substack{k=0 \\ |k-\frac{1}{\kappa}L_2x|>\frac{1}{\kappa}(L_2x)/L_3x}}^{\infty} \frac{\left(\frac{1}{\kappa}L_2x\right)^k}{k!}\Bigg) \\
&\quad = \sum_{\substack{k=0 \\ k\in\mathcal{S}''}}^{\infty} \frac{\left(\frac{1}{\kappa}L_2x\right)^k}{k!} + O\bigg(\frac{e^{\frac{1}{\kappa}L_2x}}{\sqrt{L_2x}\sqrt{L_3x}}\bigg).
\end{aligned}$$

Furthermore, by Lemma 14, this quantity is

$$c_1(\kappa)(\log x)^{1/\kappa} + O((\log x)^{1/\kappa}/\sqrt{(L_2x)(L_3x)}) \gg (\log x)^{1/\kappa}.$$

Combining this bound with (41) yields

$$D_\kappa(x) \gg_{\varepsilon,\kappa} \frac{x}{(\log x)^{1-1/\kappa}(L_2x)^{\kappa-1}} \kappa^{-(1+\varepsilon)(L_3x)/L_4x}. \tag{42}$$

When κ is a nontrivial prime power, Theorem 2 follows from (42) and the older upper bound for $D_\kappa(x)$ stated in the introduction. We remark that we can now prove that upper bound with the function $\xi_p(x)$ omitted. When κ is not a prime power, we can obtain Theorem 2 either by generalizing the older upper bound for $D_\kappa(x)$, or by generalizing the argument in Section 5. We take the latter approach.

Let the prime decomposition of κ be $\kappa = \prod_{i=1}^{u} p_i^{\alpha_i}$. If n is any positive integer such that $d_\kappa(n) \mid n$, then it follows from the relation

$$d_\kappa(p^j) = \binom{j+\kappa-1}{\kappa-2}, \tag{43}$$

and the multiplicativity of $d_\kappa(n)$ that

$$\prod_{i=1}^{u} \binom{\nu_{p_i}(n)+\kappa-1}{\kappa-2} = d\Big(\prod_{i=1}^{u} p_i^{\nu_{p_i}(n)}\Big) \Big| \, d(n),$$

whence

$$\prod_{i=1}^{u} \binom{\nu_{p_i}(n)+\kappa-1}{\kappa-2} \Big| \, n.$$

So, if $k_i = \nu_{p_i}(n)$ for $i = 1, \ldots, u$, then

$$H := \Big(\Big(\prod_{i=1}^{u} p_i^{k_i}\Big) \operatorname{odd}_\kappa(\beta(k_1, \ldots, k_u))\Big)^{-1} n$$

is an integer, where $\operatorname{odd}_\kappa(j)$ denotes the maximal divisor of j which is coprime to κ, and where

$$\beta(k_1, \ldots, k_u) = \beta_\kappa(k_1, \ldots, k_u) = \prod_{i=1}^{u} \binom{k_i+\kappa-2}{\kappa-1}.$$

Clearly, $(H, \kappa) = 1$. Let l be the largest divisor of H such that every prime divisor of l also divides $\beta(k_1, \ldots, k_u)$, let t be the largest squarefull divisor of H/l, and let $m = H/(lt)$. Then in view of our notation, we have

$$m \in \mathcal{S},\ t \in T,\ (t, \kappa\beta(k_1, \ldots, k_u)) = (t, m) = (m, \kappa\beta(k_1, \ldots, k_u)) = 1, \tag{44}$$

$$n = (\operatorname{odd}_\kappa(\beta(k_1, \ldots, k_u)))\Big(\prod_{i=1}^{u} p_i^{k_i}\Big) ltm, \tag{45}$$

$$(l, \kappa) = 1, \qquad \prod_{p \mid l} p \,\Big|\, \beta(k_1, \ldots, k_u). \tag{46}$$

Next, we compare the exact power of p_i dividing both n and $d(n)$, and utilize the relation $d_\kappa(n) \mid n$ to obtain an upper bound on the number of prime divisors of m. From (44)–(46), the multiplicativity of $d_\kappa(n)$, and the fact that

$$d_\kappa(p) = \kappa \tag{47}$$

for every prime p, we can conclude that

$$d_\kappa(n) = d_\kappa(l \operatorname{odd}_\kappa(\beta(k_1, \ldots, k_u))\beta(k_1, \ldots, k_u)) d_\kappa(t) \kappa^{\omega(m)}.$$

Now since $d_\kappa(n) \mid n$, we must have $\nu_{p_i}(d_\kappa(n)) \le k_i$ for all i. Ergo,

$$\nu_{p_i}(d_\kappa(l \operatorname{odd}_\kappa(\beta(k_1, \ldots, k_u)))) + \nu_{p_i}(\beta(k_1, \ldots, k_u)) + \nu_{p_i}(d_\kappa(t)) + \alpha_i\omega(m) \le k_i$$

for all i. Hence,

$$\omega(m) \le (k_i - \nu_{p_i}(d_\kappa(l\,\mathrm{odd}_\kappa(\beta(k_1,\dots,k_u)))) - \nu_{p_i}(\beta(k_1,\dots,k_u)))/\alpha_i \tag{48}$$

for all i. Denote the product of all primes exactly dividing $\beta(k_1,\dots,k_u)$, and not dividing κ, by $s_\kappa(k_1,\dots,k_u)$, and denote the largest squarefull divisor of $\mathrm{odd}_\kappa(\beta(k_1,\dots,k_u))$ by $r_\kappa(k_1,\dots,k_u)$. Then the number of primes exactly dividing $\mathrm{odd}_\kappa(l\beta(k_1,\dots,k_u))$ is at least $\omega(s_\kappa(k_1,\dots,k_u)) - \omega(s(l))$. Therefore, it follows from (47) that

$$\kappa^{\omega(s_\kappa(k_1,\dots,k_u))-s(l)} \mid d_\kappa(\mathrm{odd}_\kappa(l\beta(k_1,\dots,k_u))).$$

Accordingly,

$$\nu_{p_i}(d_\kappa(\mathrm{odd}_\kappa(l\beta(k_1,\dots,k_u)))) \ge (\omega(s_\kappa(k_1,\dots,k_u)) - \omega(s(l)))\alpha_i.$$

We deduce from (48) that

$$\omega(m) \le k_i/\alpha_i - \omega(s_\kappa(k_1,\dots,k_u)) + \omega(s(l)) - \nu_{p_i}(\beta(k_1,\dots,k_u))/\alpha_i \tag{49}$$

for all i. Since $\omega(s(l)) \le \omega(l)$, we deduce that any m satisfying (48) must also satisfy

$$\omega(m) \le k_i/\alpha_i - \omega(s_\kappa(k_1,\dots,k_u)) + \omega(l) - \nu_{p_i}(\beta(k_1,\dots,k_u))/\alpha_i \tag{50}$$

for all i.

For the next part of the proof, we simultaneously partition the set of $n \le x$ for which $d_\kappa(n) \mid n$ according to the value of each k_i. We then partition each of the resulting subsets according to the value of t. Then we subdivide each of the new subsets resulting from the last partition according to the value of l. Thus, we obtain

$$D_\kappa(x) \le \sum_{(k_1,\dots,k_u)\in(\mathbb{Z}_{\ge 0})^{(u)}} \sum_{t\in T} \sum_{\substack{l \\ p\mid l \Rightarrow p\mid\beta(k_1,\dots,k_u)}} \times \sum_{\substack{m\le x(\Pi_{i=1}^{u} p_i^{-k_i})(lt\,\mathrm{odd}_\kappa(\beta(k_1,\dots,k_u)))^{-1} \\ (m,\kappa\beta(k_1,\dots,k_u))=1 \\ (50)\text{ holds}}} 1. \tag{51}$$

As with $D(x) = D_2(x)$, we truncate the ranges of the variables k_i and the ranges of t and l to

$$t \le \log x, \qquad \frac{1}{2p_i} L_2 x < k_i \nu_{p_i}(x) \le \frac{3}{4} L_2 x, \tag{52}$$

$$r(l\,\mathrm{odd}_\kappa(\beta(k_1,\dots,k_u))) \le (\log x)^2. \tag{53}$$

The error made can be neglected—indeed, the bulk of the contribution to the sum will come from u-tuples $(k_1,\dots,k_u)$ with k_i near $p_i^{-\nu_{p_i}(k_i)} L_2 x$, $1 \le i \le u$ (see below).

First, we delete the condition

$$(m, \kappa\beta(k_1, \ldots, k_u)) = 1. \tag{54}$$

Then, we partition the sum on m according to the value j of $\omega(m)$. The sum on m then becomes

$$O(1) + \sum_{1 \le j \le \min_{1 \le i \le u}\left\{\frac{k_i}{\alpha_i} - \omega(s_\kappa(k_1, \ldots, k_u)) + \omega(l) - \frac{\nu_{p_i}(\beta(k_1, \ldots, k_u))}{\alpha_i}\right\}}, \tag{55}$$

where the jth summand is

$$\#\left\{m \le \frac{x \prod_{i=1}^{u} p_i^{-k_i}}{lt\, \mathrm{odd}_\kappa(\beta(k_1, \ldots, k_u))} : \omega(m) = j\right\}. \tag{56}$$

We briefly sketch the remainder of the argument, since much of the analysis is extremely similar to the arguments given in Section 5. Let k denote the greatest integer not exceeding the minimum in (55). So, our last upper bound for $D_\kappa(x)$ can be rewritten

$$D_\kappa(x) \le \sum_{k=1}^{\infty} \sideset{}{'}\sum_{(k_1, \ldots, k_u) \in (\mathbb{Z}_{\ge 0})^{(u)}} \sum_{t \in T} \sum_{l, p \mid l \Rightarrow p \mid \beta(k_1, \ldots, k_u)} \sum_{1 \le j \le k},$$

where the jth summand equals the jth summand in (55), and where the dash indicates that the sum on $(k_1, \ldots, k_u)$ is over u-tuples such that the greatest integer of the minimum in (55) has the value k. Then we argue that we can truncate the sums to the ranges

$$\begin{gathered} k \le \frac{3}{4} L_2 x; \qquad \frac{k_i}{\alpha_i} \le \frac{3}{4} L_2 x \quad \text{ for } 1 \le i \le u; \\ t \le (\log x)^2; \qquad r(kl) \le (\log x)^2. \end{gathered} \tag{57}$$

As in the argument given in Section 5, we show that the jth summand grows geometrically, so that the sum on j has the same order of magnitude as the jth summand with $j = k$. We obtain the upper bound

$$\begin{aligned} D_\kappa(x) \ll \frac{x}{\log x} \sum_{k} \sideset{}{'}\sum_{(k_1, \ldots, k_u)} & \frac{\prod_{i=1}^{u} p_i^{-k_i}}{\mathrm{odd}_\kappa(\beta(k_1, \ldots, k_u))} \\ & \times \sum_{\substack{t \in T \\ (57)\text{ holds}}} \frac{1}{t} \sum_{\substack{l \\ p \mid l \Rightarrow p \mid \beta(k_1, \ldots, k_u)}} \frac{1}{l} \frac{(L_2 x)^k}{k!} + \frac{x}{\sqrt{\log x}}. \end{aligned}$$

Again as in the argument given in Section 5, we show that the sum on t can be omitted; thus,

$$D_\kappa(x) \ll \frac{x}{\log x} \sum_{k \le \frac{3}{4} L_2 x} \sideset{}{'}\sum_{\substack{(k_1, \ldots, k_u) \\ \frac{k_i}{\alpha_i} \le \frac{3}{4} L_2 x}} \frac{\prod_{i=1}^{u} p_i^{-k_i}}{\mathrm{odd}_\kappa(\beta(k_1, \ldots, k_u))} \tag{58}$$

$$\times \sum_{\substack{l,p|l\Rightarrow p|\beta(k_1,\ldots,k_u)\\ r(kl)\le(\log x)^2}} \frac{1}{l}\frac{(L_2x)^k}{k!} + \frac{x}{\sqrt{\log x}}.$$

For any u-tuple, $k_1,\ldots,k_u$, there is a value of i for which the minimum referred to in (55) is attained. (There may be more than one such value.) We partition the sum on $k_1,\ldots,k_u$, according to which value of i (or values of i) give(s) that minimum. Thus,

$$D_\kappa(x) \ll \frac{x}{\log x}\sum_{h=1}^{u} \sum_{\substack{k_1,\ldots,k_u\\ \frac{k_i}{\alpha_i}\le\frac{3}{4}L_2x}}'' + \frac{x}{\sqrt{\log x}}$$

where the $(k_1,\ldots,k_u)$-summand equals the corresponding summand in (58), and where the double dash means that $(k_1,\ldots,k_u)$ contributes to the sum in (58) and that the minimum in (55) is attained for $i=h$. (Note that once k_h is chosen, the value of k is determined, so that we no longer need that sum.)

Again as in Section 5, we multiply and divide by a suitable power of k, to replace $(L_2x)^k$ by $((1/k)L_2x)^k$. Then we partition the sum on k according to whether

$$\left|\frac{k_h}{\alpha_h} - \frac{1}{\kappa}L_2x\right| \le \frac{1}{\kappa}(L_2x)/L_3x. \tag{59}$$

When (59) holds, we replace $(L_2x)^k/k!$ by $(L_2x)^{[k_h/\alpha_h]}/[k_h/\alpha_h]!$. We show that the part of the sum with (59) failing to hold can be neglected. In this way, we get

$$D_\kappa(x) \ll_\kappa \frac{x}{\log x}\sum_{h=1}^{u} \sum_{\substack{(k_1,\ldots,k_u)\\ \frac{k_i}{\alpha_i}\le\frac{3}{4}L_2x\\ (59)\text{ holds}}}'' \frac{I(k_1,\ldots,k_u;h)}{\kappa^{\omega(s_\kappa(k_1,\ldots,k_u))}} \times \frac{\left(\frac{1}{\kappa}L_2x\right)^{[k_h/\alpha_h]}}{[k_h/\alpha_h]!} \sum_{\substack{l\\ p|l\Rightarrow p|\beta(k_1,\ldots,k_u)}} \frac{\kappa^{\omega(l)}}{l} + E_1(x) \tag{60}$$

where

$$E_1(x) = o\left(\frac{x}{(\log x)^{1-1/\kappa}}\frac{1}{(\log\log x)^{1-\kappa+1/4}}\right), \tag{61}$$

and where

$$I(k_1,\ldots,k_u;h) = \frac{k^{[k_h/\alpha_h-\nu_{p_h}(\beta(k_1,\ldots,k_u))/\alpha_h]}}{(\prod_{i=1}^{u} p_i^{k_i})\,\mathrm{odd}_\kappa(\beta(k_1,\ldots,k_u))}.$$

It follows from the Fundamental Theorem of Arithmetic that

$$I(k_1,\ldots,k_u;h) \ll_\kappa \frac{1}{\beta(k_1,\ldots,k_u)} \prod_{\substack{i=1\\ i\neq h}}^{u} p_i^{k_h-k_i+\nu_{p_i}(\beta(k_1,\ldots,k_u))-\nu_{p_h}(\beta(k_1,\ldots,k_u))}. \tag{62}$$

By construction, $s\big(\binom{k_h+\kappa-2}{\kappa-1}\big)$ divides $s_\kappa(k_1,\ldots,k_u)$. Hence,

$$\omega\left(s\left(\binom{k_h+\kappa-2}{\kappa-1}\right)\right) \le \omega(s_\kappa(k_1,\ldots,k_u)).$$

Moreover, since

$$\binom{k_h+\kappa-2}{\kappa-1} = \frac{1}{(\kappa-1)!}\prod_{v=0}^{k_h+\kappa-2}(k_h+v),$$

we have

$$\omega\Big(s\Big(\prod_{v=0}^{k_h+\kappa-2}(k_h+v)\Big)\Big) - \omega((\kappa-1)!) \le \omega(s_\kappa(k_1,\ldots,k_u)).$$

Hence, $\kappa^{-\omega(s_\kappa(k_1,\ldots,k_u))} \le \kappa^{-\gamma(k_h,\kappa)}$, where

$$\gamma(n,\kappa) = \omega\Big(s\Big(\prod_{v=0}^{n-2}(n+v)\Big)\Big). \tag{63}$$

Next, we estimate the sum on l in a similar manner to the way we estimated the sums on l in Section 5. The result is

$$\sum_{\substack{l\\ p\mid l\Rightarrow p\mid\beta(k_1,\ldots,k_u)}} \frac{\kappa^{\omega(l)}}{l} \ll_\kappa (L_4x)^{1+\kappa}. \tag{64}$$

Combining (60) with (62)–(64) yields

$$D_\kappa(x) \ll \frac{x(L_4x)^{\kappa+1}}{\log x}\sum_{h=1}^{u}\frac{\kappa^{-\gamma(k_h,\kappa)}}{\beta(k_1,\ldots,k_u)} \times \sum_{\substack{(k_1,\ldots,k_u)\\ k_i\le\frac{3}{4}L_2x\\ \text{(59) holds}}}'' \frac{\big(\frac{1}{\kappa}L_2x\big)^{[k_h/\alpha_h]}}{[k_h/\alpha_h]!} I'(k_1,\ldots,k_u;h) + E_1(x) \tag{65}$$

where $I'(k_1,\ldots,k_u;h)$ is the expression on the right of (62). In view of the meaning of the double dash on the sum over $(k_1,\ldots,k_u)$, we have

$$k_i - \nu_{p_i}(\beta(k_1,\ldots,k_u)) \ge k_h - \nu_{p_i}(\beta(k_1,\ldots,k_u)) \tag{66}$$

for all i. So, $k_i \geq (L_2x)/(2\kappa)$. It follows that

$$\frac{1}{\beta(k_1,\ldots,k_u)} \ll (L_2x)^{(1-\kappa)\omega(\kappa)}.$$

Next, we rewrite the sum over $(k_1,\ldots,k_u)$ as a u-fold iterated sum, with k_h outside, and each of the other variables inside:

$$\sum_{(k_1,\ldots,k_u)}'' = \sum_{\substack{k_h \\ (59)\text{ holds}}} \sum_{k_1} \sum_{k_2} \underset{\substack{k_i \leq \frac{3}{4}L_2x \text{ for all } i \\ (66) \text{ holds for all } i}}{\cdots} \sum_{k_u}.$$

(Here, of course, the second sum on the right is over k_2 if $h = 1$.) Thus, we have

$$D_\kappa(x) \ll_\kappa \frac{x(L_4x)^{\kappa+1}}{(\log x)(L_2x)^{(\kappa-1)\omega(\kappa)}} \sum_{h=1}^{u} \sum_{\substack{k_h \\ (59)\text{ holds}}} \kappa^{-\gamma(k_h,\kappa)} \frac{\left(\frac{1}{\kappa}L_2x\right)^{[k_h/\alpha_h]}}{[k_h/\alpha_h]!}$$
$$\times \prod_{\substack{i=1 \\ i\neq h}}^{u} \sum_{\substack{k_i \leq \frac{3}{4}L_2x \\ (66)\text{ holds}}} R(k_h,k_i) + E_1(x)$$

where

$$R(m,n) = R(m,n,\kappa;k_1,\ldots,k_u) = p^{k_h-k_i+\nu_{p_i}(\beta(k_1,\ldots,k_u))-\nu_{p_h}(\beta(k_1,\ldots,k_u))}. \tag{67}$$

By (66), the exponent on p in (67) is a nonpositive integer. If we write $-g$ for that exponent, we discover that

$$R(m,n) = p^{-g}\#\{(h,i) : k_h - k_i + \nu_{p_i}(\beta(k_1,\ldots,k_u)) - \nu_{p_h}(\beta(k_1,\ldots,k_u)) = -g\}.$$

Estimating $p^gR(m,n)$ is, thus, related to the problem of estimating the number of positive integers $n \leq y$ for which $n - \nu_p(n)$ assumes a fixed value. Solving this related problem immediately yields an upper bound on $p^gR(m,n)$. But, in our earlier paper on the subject (see Lemma 4 of [SP1]), we showed that

$$\#\{n \leq y : n - \nu_2(n) = m\} \leq \xi((\log y)/\log 2), \tag{68}$$

where $\xi(x)$ is defined in the introduction to the present paper. A similar proof yields a comparable result for $\nu_p(n)$, where p is a fixed prime. The replacement of $\nu_p(n)$ by an expression of the form $\nu_p(n+j)$, where j is fixed, poses no real additional difficulty. To estimate $p^gR(m,n)$, we must replace $\nu_p(n+j)$ by a finite product of, say, t linear factors $n+j_l$ in n. Then, we argue that for $\nu_p(\prod_{l=1}^{t}(n+j_l))$ sufficiently large (as a function of the j_l), we must have

$$\nu_p\Big(\prod_l (n+j_l)\Big) = \nu_p(n+j_L) + M,$$

for some subscript L, where the integer M is bounded in magnitude by t. The result we get is comparable to (68), but with $\le$ replaced by $\ll$, and with ν_2 replaced by ν_p. In particular, we have $p^g R(m,n) \ll L_v x$ for any integer $v \ge 2$. So, by (66), the sum on k_i in (67) is of order

$$\sum_{g=0}^{\infty} \frac{L_v\big(\frac{3}{4}L_2 x\big)}{p^g} \ll L_v x.$$

Therefore,

$$D_\kappa(x) \ll_{\kappa,v} \frac{x(L_4 x)^{\kappa+1}(L_v x)^{\omega(\kappa)}}{(\log x)(L_2 x)^{(\kappa-1)\omega(\kappa)}} \sum_{h=1}^{u} \sum_{k_h} \kappa^{-\gamma(k_h,\kappa)} \frac{\big(\frac{1}{\kappa}L_2 x\big)^{[k_h/\alpha_h]}}{[k_h/\alpha_h]!}. \tag{69}$$

This is our upper bound. The next theorem is an immediate consequence.

Theorem 4. *For any integers* κ, v *exceeding* 1, *we have*

$$D_\kappa(x) \ll_{\kappa,v} \frac{x(L_4 x)^{\kappa+1}(L_v x)}{(\log x)(L_2 x)^{(\kappa-1)\omega(\kappa)}} \sum_{h=1}^{\omega(k)} \sum_{k_j=1}^{\infty} \kappa^{-\gamma(k_h,\kappa)} \frac{\big(\frac{1}{\kappa}L_2 x\big)^{[k_h/\alpha_h]}}{[k_h/\alpha_h]!},$$

where $\gamma(k_h,\kappa)$ *is defined by* (63), *and*

$$\kappa = \prod_{i=1}^{\omega(k)} p_i^{\alpha_i}.$$

Finally, we note that in the inner sum, the expression $[k_h/\alpha_h]$ has the same value whenever k_h satisfies $\alpha_h n \le k_h \le \alpha_n n + \alpha_h - 1$. So, if we replace $-\gamma(k_h,\kappa)$ by 0, and then group together those k_h-terms with $[k_h/\alpha_h]$ the same, we get the bound

$$O\bigg(\sum_{a=1}^{\infty} \frac{\big(\frac{1}{\kappa}L_2 x\big)^a}{a!}\bigg) = O\big(e^{\frac{1}{\kappa}L_2 x}\big)$$

for the inner sum. The upper bound implied by Theorem 2 follows, upon choosing, say, $v = 3$.

References

[ER] P. Erdős, *Über die kleinste quadratfreie Zahl einer arithmetischen Reihe*, Monatsh. Math. 64 (1960), 314–316.

[ES] P. Erdős und G. Szekeres, *Über die Anzahl der Abelschen Gruppen gegebener Ordnung und über ein verwandtes zahlentheoretisches Problem*, Acta Univ. Szeged. 7 (1934-35), 95–102.

[HW] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 4th ed., reprinted with corrections, Clarendon Press, Oxford, 1968.

[NO1] K. K. Norton, *Estimates for partial sums of the exponential series*, J. Math. Anal. Appl. 63 (1978), 265–296.

[NO2] —, *On the number of restricted prime factors of an integer. II*, Acta Math. 143 (1979), 9–38.

[SE] A. Selberg, *Note on a problem of L. G. Sathe*, J. Indian Math. Soc. (N.S.) (1954), 83–87.

[SP1] C. Spiro, *How often is the number of divisors of n a divisor of n?*, J. Number Theory 21 (1985), 81–100.

[SP2] —, *An iteration problem involving the divisor function*, Acta Arith. 46 (1986), 215–225.

[SP3] —, *Extensions of some formula of A. Selberg*, Internat. J. Math. Math. Sci. 8 (1985), 283–302.

POB 7445
NASHUA, NEW HAMPSHIRE 03060-7445
U.S.A.

Received on 30.4.1993
and in revised form on 27.12.1993 (2424)