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Minimum and maximum order of magnitude of the discrepancy of $(n\alpha)$

by

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Dedicated to Prof. Wolfgang Schmidt
on the occasion of his sixtieth birthday

It is a classical result of P. Bohl [5], W. Sierpiński [21, 22] and H. Weyl [25, 26] that the sequence $(n\alpha)_{n\geq 1}$ is uniformly distributed modulo 1 if and only if α is irrational. The discrepancies

$$D_N^*(\alpha) = \sup_{0\leq x\leq 1} \Big| \sum_{n=1}^{N} c_{[0,x)}(\{n\alpha\}) - Nx \Big|$$

and

$$D_N(\alpha) = \sup_{0\leq x<y\leq 1} \Big| \sum_{n=1}^{N} c_{[x,y)}(\{n\alpha\}) - N(y-x) \Big|$$

measure the deviation of this sequence from an ideal distribution. (Here $N \in \mathbb{N}$, c_M is the characteristic function of the set M and $\{x\} = x - [x]$ denotes the fractional part of x.) The speed of convergence in the limit relations

$$\lim_{N\to\infty} \frac{1}{N} D_N^*(\alpha) = 0 \quad \text{and} \quad \lim_{N\to\infty} \frac{1}{N} D_N(\alpha) = 0$$

is used as a measure for the quality of distribution and was studied by many authors. Initially the problem was tackled by H. Behnke [3, 4], A. Ostrowski [14], G. H. Hardy and J. E. Littlewood [10], and E. Hecke [11]. More recently, it was taken up by H. Niederreiter [13], J. Lesca [12], V. T. Sós [23, 24], Y. Dupain [7, 8], Y. Dupain and V. T. Sós [9], L. Ramshaw [15] and J. Schoißengeier [17, 18, 20].

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In [17] it was proved that $\liminf_{N\to\infty} D_N^*(\alpha) = 1$ for all irrational α. To determine the maximum order of $D_N^*(\alpha)$, the quantities

$$\omega_N^+(\alpha) = \sup_{0\le x\le 1} \Big(\sum_{n=1}^{N} c_{[0,x)}(\{n\alpha\}) - Nx\Big)$$

and

$$\omega_N^-(\alpha) = \sup_{0\le x\le 1} \Big(Nx - \sum_{n=1}^{N} c_{[0,x)}(\{n\alpha\})\Big)$$

were introduced by J. Schoißengeier [20] who determined

$$\max_{1\le N<q_{m+1}} \omega_N^+(\alpha) \quad \text{and} \quad \max_{1\le N<q_{m+1}} \omega_N^-(\alpha)$$

up to an absolute error in terms of the continued fraction expansion of α. Utilizing $D_N^*(\alpha) = \max(\omega_N^+(\alpha), \omega_N^-(\alpha))$ one arrives at the maximum order of $D_N^*(\alpha)$.

It is the purpose of this paper to prove analogous results for the minimum and maximum order of $D_N(\alpha)$. We calculate $\max_{1\le N<q_{m+1}} D_N(\alpha)$ in terms of the continued fraction expansion of α up to an absolute error (where q_m denotes the denominator of the mth convergent of α). Using this we describe the maximum order of the sequence $(D_N(\alpha))_{N\ge 1}$ and calculate $\limsup_{N\to\infty} D_N(\alpha)/\log N$ for all α for which $D_N(\alpha) = O(\log N)$ is satisfied. Finally, we determine the minimum order of $(D_N(\alpha))_{N\ge1}$ which turns out to be closely connected to the Lagrange spectrum.

1. The maximum order. We will use the following notations: α will always denote an irrational real number with regular continued fraction expansion $\alpha = [a_0, a_1, a_2, \ldots]$ ($a_0 \in \mathbb{Z}$ and $a_1, a_2, \ldots \in \mathbb{N}$) and convergents $(p_m/q_m)_{m\ge0}$. For all $i, j \ge 0$ let

$$s_{ij} = q_{\min(i,j)}(q_{\max(i,j)}\alpha - p_{\max(i,j)})$$

and

$$\varepsilon_i = \frac{1}{2}(1-(-1)^{a_{i+1}}) \prod_{\substack{0\le j\le i\\ j\equiv i \ (\mathrm{mod}\ 2)}} (-1)^{a_{j+1}}.$$

We are now prepared to state our first main result.

Theorem 1.1. *For $m \ge 0$ let $N_m = \frac{1}{2}\sum_{i=0}^{m}(a_{i+1} + (-1)^m\varepsilon_i)q_i$. Then as $m \to \infty$,*

$$4 \max_{1\le N<q_{m+1}} D_N(\alpha) = \sum_{i=0}^{m} a_{i+1} - \sum_{0\le i\le m} \sum_{\substack{0\le j\le m\\ j\equiv i \ (\mathrm{mod}\ 2)}} \varepsilon_i\varepsilon_j |s_{ij}| + O(1)$$

and

$$\max_{1\le N<q_{m+1}} D_N(\alpha)=\begin{cases} D_{N_m}(\alpha)+O(1) & \text{if } N_m<q_{m+1},\\ D_{N_{m-1}}(\alpha)+O(1) & \text{otherwise.}\end{cases}$$

The implicit constants are absolute.

Proof. We introduce

$$S_m=\frac14\sum_{i=0}^{m}a_{i+1}-\frac14\sum_{0\le i\le m}\ \sum_{\substack{0\le j\le m\\ j\equiv i \ (\mathrm{mod}\ 2)}}\varepsilon_i\varepsilon_j|s_{ij}|$$

as a convenient shorthand notation.

Employing $c_{[0,\{x-y\})}(\{x\})-\{x-y\}=\{y\}-\{x\}$ for all $x,y\in\mathbb{R}$ we have for $0\le k,l\le N<q_{m+1}$,

$$(*)\quad \Delta_N(k,l)$$
$$:=\sum_{n=1}^{N}c_{[0,\{k\alpha\})}(\{n\alpha\})-N\{k\alpha\}-\sum_{n=1}^{N}c_{[0,\{l\alpha\})}(\{n\alpha\})+N\{l\alpha\}$$
$$=\sum_{n=1}^{N}(\{(n-k)\alpha\}-\{n\alpha\})-\sum_{n=1}^{N}(\{(n-l)\alpha\}-\{n\alpha\})$$
$$=\sum_{n=1}^{k-1}\{-n\alpha\}+\sum_{n=1}^{N-k}\{n\alpha\}-\sum_{n=1}^{l-1}\{-n\alpha\}-\sum_{n=1}^{N-l}\{n\alpha\}$$
$$=k-1-\sum_{n=1}^{k-1}\{n\alpha\}-(l-1)+\sum_{n=1}^{l-1}\{n\alpha\}+\sum_{n=1}^{N-k}\{n\alpha\}-\sum_{n=1}^{N-l}\{n\alpha\}$$
$$=\sum_{n=1}^{l-1}(\{n\alpha\}-1/2)+\sum_{n=1}^{N-k}(\{n\alpha\}-1/2)-\sum_{n=1}^{k-1}(\{n\alpha\}-1/2)-\sum_{n=1}^{N-l}(\{n\alpha\}-1/2)$$
$$\le 2\max_{1\le M<q_{m+1}}\sum_{n=1}^{M}B_1(n\alpha)-2\min_{1\le M<q_{m+1}}\sum_{n=1}^{M}B_1(n\alpha)=S_m+O(1).$$

Here $B_1(x)=\{x\}-1/2$ denotes the first Bernoulli polynomial. The last step made use of Corollary 2 in §2 of [19]. Using $D_N(\alpha)=1+\max_{1\le k,l\le N}\Delta_N(k,l)$ we get $\max_{1\le N<q_{m+1}}D_N(\alpha)\le S_m+c$ with an absolute constant $c>0$. To obtain equality we set

$$k:=1+\frac12\sum_{\substack{0\le i\le m\\ i\equiv 0 \ (\mathrm{mod}\ 2)}}(a_{i+1}+(-1)^m\varepsilon_i)q_i,$$
$$l:=1+\frac12\sum_{\substack{0\le i\le m\\ i\equiv 1 \ (\mathrm{mod}\ 2)}}(a_{i+1}+(-1)^m\varepsilon_i)q_i$$

and $\widehat{N}_m := k+l-1 = N_m+1$. Obviously $l-1 = \widehat{N}_m - k$ and $k-1 = \widehat{N}_m - l$. According to Corollary 2 in §2 of [19] we have equality in $(*)$. Had we proved $\widehat{N}_m < q_{m+1}$ we would have completed the proof of the theorem. It is of no importance that $\widehat{N}_m = N_m + 1$ as $D_{N+1}(\alpha) = D_N(\alpha) + O(1)$ with an absolute implied constant. A trivial estimation yields

$$N_m \le \frac{1}{2}\sum_{i=0}^{m} 2a_{i+1}q_i = q_{m+1} + q_m - 1.$$

If $a_{m+1} \ge 3$ we even have

$$N_m < (q_m + q_{m-1}) + \frac{1}{2}(a_{m+1}+1)q_m \le q_{m+1}.$$

Thus, $N_m \ge q_{m+1}$ only if $a_{m+1} \le 2$. But in this case we may safely change to $N_{m-1} < q_m + q_{m-1} \le q_{m+1}$ as $S_m = S_{m-1} + O(a_{m+1})$ with an absolute implied constant.

We conclude the proof with a remark: Obviously $N_m \ge q_m$ if $a_{m+1} \ge 2$. If $a_{m+1} = a_m = 1$ it is possible that $a_{m+1} + (-1)^m\varepsilon_m = a_m + (-1)^{m-1}\varepsilon_{m-1} = 0$ but by the definition of the ε_i it is impossible to have also $a_{m-1} + (-1)^{m-2}\varepsilon_{m-2} = 0$. Therefore $N_m \ge q_{m-2}$.

Remark. Using Corollary 1 in §2 of [19] the Bernoulli polynomials can be replaced by Dedekind sums in the above estimate. This indicates a close connection between discrepancies and Dedekind sums which was first pointed out and explored by U. Dieter (oral communication).

COROLLARY 1.2. *Let α be an irrational number. For $N \in \mathbb{N}$ we define $m \in \mathbb{N}$ by the property $q_m \le N < q_{m+1}$. Then*

$$\limsup_{N\to\infty} D_N(\alpha) \Big/ \Big(\sum_{i=0}^{m} a_{i+1} - \sum_{0\le i\le m} \sum_{\substack{0\le j\le m\\ j\equiv i \ (\mathrm{mod}\ 2)}} \varepsilon_i\varepsilon_j|s_{ij}|\Big) = \frac{1}{4}.$$

Proof. This can be proved along the same lines as Corollary 1 in §2 of [20].

Remark. Another proof of Theorem 1.1 which uses a completely different method is to be found in [1]. It yields more precise information on where the maximum is attained at the cost of a much longer proof.

2. The maximum order for numbers of bounded density. By a well known theorem of W. M. Schmidt [16] for every α an infinity of positive integers N such that $D_N(\alpha) \ge (66\log 4)^{-1}\log N$ exist. On the other hand, it was first observed by H. Behnke [4] that $D_N(\alpha) = O(\log N)$ if and only if $\alpha = [a_0, a_1, a_2, \ldots]$ is of bounded density (i.e. $\sum_{i=0}^{m} a_{i+1} = O(m)$ as

$m \to \infty$). For these numbers we are now able to compute the infimum of all possible implied constants in the estimate $D_N(\alpha) = O(\log N)$.

THEOREM 2.1. *Let* α *be a number of bounded density. Then*

$$\nu(\alpha) := \limsup_{N\to\infty} \frac{D_N(\alpha)}{\log N} = \frac{1}{4} \limsup_{m\to\infty} \frac{1}{\log q_m} \Big(\sum_{i=0}^{m} a_{i+1} - \sum_{0\le i\le m} \sum_{\substack{0\le j\le m \\ j\equiv i \ (\mathrm{mod}\ 2)}} \varepsilon_i \varepsilon_j |s_{ij}| \Big).$$

Proof. This may be proved as Theorem 1 in §3 of [20].

Theorem 2.1 implies a property of the function ν which was first shown by L. Ramshaw [15]:

COROLLARY 2.2. *Let* α, β *be two numbers of bounded density. Assume that there exists a matrix* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z})$ *such that* $\beta = (a\alpha + b)/(c\alpha + d)$. *Then* $\nu(\beta) = \nu(\alpha)$.

Proof. This follows immediately from Theorem 2.1 and various parts of the proof of Theorem 2 in §3 of [20].

Remark. The analogous map $\nu^*(\alpha) = \limsup_{N\to\infty} D_N^*(\alpha)/\log N$ is studied in [1, 2]. The image of ν^* has the property $\nu^*(B) = [\nu^*(\sqrt{2}), \infty)$. (Here B denotes the set of all numbers of bounded density.) In the present case we are able to prove $[\nu(\sqrt{2}), \infty) \subseteq \nu(B)$ but $\nu((1+\sqrt{5})/2) < \nu(\sqrt{2})$.

In the case of quadratic irrationalities there is a formula which does not contain any limit processes:

THEOREM 2.3. *Let* $\alpha = [0, \overline{a_1, \ldots, a_e}]$ *where* $2 \mid e$ *and set*

$$\eta_t = \prod_{\substack{0\le\sigma<e \\ \sigma\equiv t \ (\mathrm{mod}\ 2)}} (-1)^{a_{\sigma+1}} \qquad \textit{for } t \in \{0, 1\}.$$

Then

$$\begin{aligned}\nu(\alpha) = \frac{1}{4\log(q_e + \alpha q_{e-1})} \Big(&\sum_{i=0}^{e-1} a_{i+1} \\ &+ \sum_{t=0}^{1} (2t-1) \frac{q_{e-1}}{2\eta_t - q_e - p_{e-1}} \mathcal{N}\Big(\sum_{\substack{0\le i<e \\ i\equiv t \ (\mathrm{mod}\ 2)}} \varepsilon_i (q_i\alpha - p_i) \Big) \\ &+ \sum_{t=0}^{1} (2t-1) \sum_{\substack{0\le i<e \\ i\equiv t \ (\mathrm{mod}\ 2)}} \sum_{\substack{0\le j<e \\ j\equiv t \ (\mathrm{mod}\ 2)}} \varepsilon_i \varepsilon_j q_i p_j \,\mathrm{sgn}\,(i-j) \Big),\end{aligned}$$

where $\mathcal{N}$ *denotes the norm of the quadratic field* $\mathbb{Q}(\alpha)$.

P r o o f. Here the same applies as to Theorem 1 of §4 in [20].

N o t e. In view of Corollary 2.2 the assumption on the shape of the continued fraction expansion of α does not exclude any quadratic irrationalities. Note also that the period e is not assumed to be of minimal length.

We finish the section with two special cases of Theorem 2.3.

COROLLARY 2.4. *Let* $\alpha = [0, \overline{a, b}]$ *with* $a, b \in \mathbb{N}$. *Then*

$$\nu(\alpha) = \frac{1}{4\log(1+b/\alpha)}\left(a+b-\frac{1}{2}\cdot\frac{1-(-1)^a}{ab+2(1-(-1)^a)}-\frac{1}{2}\cdot\frac{1-(-1)^b}{ab+2(1-(-1)^b)}\right).$$

COROLLARY 2.5. *Let* $\alpha = [0, \overline{a}]$ *with* $a \in \mathbb{N}$. *Then*

$$\nu(\alpha) = \frac{a}{4\log(1/\alpha)}\left(1-\frac{1}{2}\cdot\frac{1-(-1)^a}{a^2+4}\right).$$

R e m a r k. Corollary 2.5 was first proved by L. Ramshaw [15].

3. The maximum order for special Hurwitz continued fractions

THEOREM 3.1. *Let* $t \in \mathbb{N}$ *and* $\alpha_t = \coth(1/t) = [t, 3t, 5t, \ldots]$. *Then as* $m \to \infty$,

$$\max_{1\le N<q_{m+1}} D_N(\alpha_t) = \frac{1}{4}tm^2 + tm + \frac{1}{16t}((-1)^t - 1)\log m + O(1).$$

P r o o f. The proof runs analogously to that of Theorem 1 in §5 of [20].

THEOREM 3.2. *Let* $t \in \mathbb{N}$. *Then*

$$\limsup_{N\to\infty} D_N\left(\coth\frac{1}{t}\right)\left(\frac{\log\log N}{\log N}\right)^2 = \frac{t}{4}, \tag{1}$$

$$\limsup_{N\to\infty} D_N(\sqrt[t]{e})\left(\frac{\log\log N}{\log N}\right)^2 = \frac{t}{4}, \tag{2}$$

$$\limsup_{N\to\infty} D_N(\sqrt[2t+1]{e^2})\left(\frac{\log\log N}{\log N}\right)^2 = \frac{2t+1}{4}. \tag{3}$$

P r o o f. Using the well known continued fraction expansions of the numbers $\coth(1/t)$, $\sqrt[t]{e}$ and $\sqrt[2t+1]{e^2}$ we proceed according to the following scheme. First we calculate estimates

$$\sum_{i=0}^{m} a_{i+1} \sim C_1(t)m^2 \quad \text{and} \quad \sum_{i=1}^{m} \log a_i \sim C_2(t)m\log m$$

as $m \to \infty$. Since

$$\sum_{i=1}^{m}\log a_i \le \log q_m \le \sum_{i=1}^{m}\log(a_i+1) = \sum_{i=1}^{m}\log a_i + O(m)$$

we have

$$\log q_m \sim C_2(t) m \log m.$$

For each $N \in \mathbb{N}$ we define $m \in \mathbb{N}$ via the relation $q_m \le N < q_{m+1}$. Since $\log q_{m+1} \sim \log q_m$ as $m \to \infty$ we infer $\log N \sim C_2(t) m \log m$ as $N \to \infty$. This yields $\log\log N \sim \log m$ and $\log N \sim C_2(t) m \log m \sim C_2(t) m \log\log N$ as $N \to \infty$. Putting all together we find

$$S_m \sim \sum_{i=0}^{m} a_{i+1} \sim C_1(t) m^2 \sim \frac{C_1(t)}{C_2(t)^2} \left(\frac{\log N}{\log\log N} \right)^2,$$

where we made use of

$$\sum_{0 \le i \le m} \sum_{\substack{0 \le j \le m \\ j \equiv i \ (\mathrm{mod}\ 2)}} \varepsilon_i \varepsilon_j |s_{ij}| = O(m).$$

The result now follows from Corollary 1.2.

4. The minimum order. We continue by introducing a few more notations. Let $q_m \le N < q_{m+1}$. There is a unique expansion $N = \sum_{j=0}^{m} b_j q_j$ where $0 \le b_j \le a_{j+1}$ for all j, $b_0 < a_1$ and $b_j = a_{j+1} \Rightarrow b_{j-1} = 0$ for $j \ge 1$. For $j \ge -1$ we define $A_j = \sum_{\mu=0}^{m} b_\mu s_{\mu j}$. Let i_N be the smallest integer $j \ge 0$ such that $b_j \ne 0$. Set

$$s := \min\{ j \mid 2 \nmid j,\ 1 \le j \le m,\ A_j > 0,\ A_{j+2} > 0 \Rightarrow b_{j+1} < a_{j+2} \}$$

and

$$t := \min\{ j \mid 2 \nmid j,\ 1 \le j \le m,\ A_{j-1} < 0 < A_{j+1},\ A_{j+2} > 0 \Rightarrow b_{j+1} < a_{j+2} - 1 \},$$

where $\min \emptyset := \infty$. Finally, we define

$$u := \begin{cases} 0 & \text{if } 2 \mid i_N \text{ and } (b_0 < a_1 - 1 \text{ or } A_1 < 0), \\ \min\{s, t\} & \text{otherwise.} \end{cases}$$

THEOREM 4.1.

(1) $$\omega_N^+(\alpha) = \sum_{\substack{u \le j \le m \\ j \equiv 0 \ (\mathrm{mod}\ 2)}} b_j (1 - A_j) + \sum_{\substack{u \le j \le m \\ A_{j+1} < 0 < A_{j-1} \\ j \equiv 0 \ (\mathrm{mod}\ 2)}} A_j - \sum_{\substack{u \le j \le m \\ A_{j-1} \le 0 < A_{j+1} \\ j \equiv 0 \ (\mathrm{mod}\ 2)}} A_j - \sum_{\substack{u \le j \le m \\ A_j < 0 \\ j \equiv 0 \ (\mathrm{mod}\ 2)}} a_{j+1} A_j + (\delta_{u,0} - 1) A_u,$$

(2) $$\omega_N^-(\alpha) = \omega_N^+(\alpha) + A_0 - \sum_{j=0}^{m} b_j ((-1)^j - A_j)$$

and

$$D_N(\alpha) = 2\omega_N^+(\alpha) + A_0 - \sum_{j=0}^{m} b_j((-1)^j - A_j). \tag{3}$$

P r o o f. Though not explicitly stated, (1) and (2) are contained in Theorem 1 of §8 in [17]. (Note the slightly different definition of the A_j.) (3) follows immediately from $D_N(\alpha) = \omega_N^+(\alpha) + \omega_N^-(\alpha)$.

LEMMA 4.2. *Let $q_m \le N < q_{m+1}$. Then*

$$\omega_N^+(\alpha) = \max_{1\le k\le N}(\sigma^{-1}(k) - N\{k\alpha\}) \ge q_m|q_m\alpha - p_m|$$

and

$$\omega_N^-(\alpha) = 1 + \max_{1\le k\le N}(N\{k\alpha\} - \sigma^{-1}(k)) \ge q_m|q_m\alpha - p_m|,$$

where $\sigma : \{1,\ldots,N\} \to \{1,\ldots,N\}$ is the unique permutation which satisfies $\{\alpha\sigma(i)\} < \{\alpha\sigma(i+1)\}$ for $1 \le i < N$.

P r o o f. There is a k_0 ($1 \le k_0 \le N$) such that $\sigma^{-1}(k_0) = N$. We have

$$\sigma^{-1}(k_0) - N\{k_0\alpha\} = N|1 + [k_0\alpha] - k_0\alpha| \ge q_m|q_m\alpha - p_m|$$

as $|q\alpha - p| \ge |q_m\alpha - p_m| > |q_{m+1}\alpha - p_{m+1}|$ for all $(q,p) \ne (q_{m+1}, p_{m+1})$ with $0 \le q < q_{m+1}$. The second assumption is proved analogously.

THEOREM 4.3. *If α is an irrational number, then*

$$\liminf_{N\to\infty} D_N(\alpha) = 1 + \liminf_{m\to\infty} q_m|q_m\alpha - p_m|.$$

P r o o f. As in the proof of Corollary 1 in §9 of [17] we compute $D_{bq_m}(\alpha)$ (where $1 \le b \le a_{m+1}$) using Theorem 4.1 and arrive at

$$D_{bq_m}(\alpha) = b - (b-2)bq_m|q_m\alpha - p_m| - b|q_m\alpha - p_m|.$$

Putting $b = 1$ leads to

$$\liminf_{N\to\infty} D_N(\alpha) \le 1 + \liminf_{m\to\infty} q_m|q_m\alpha - p_m|.$$

To prove the reverse inequality let $\varepsilon > 0$ and N such that $D_N^*(\alpha) = \max(\omega_N^+(\alpha), \omega_N^-(\alpha)) > 1 - \varepsilon$. (The existence of such an N is guaranteed by Corollary 2 in §9 of [17].) If (without loss of generality) $\omega_N^+(\alpha) > 1 - \varepsilon$ then $D_N(\alpha) = \omega_N^+(\alpha) + \omega_N^-(\alpha) > 1 + q_m|q_m\alpha - p_m| - \varepsilon$ by Lemma 4.2.

R e m a r k. As proved in the above theorem, $D_N(\alpha)$ behaves like $D_N^*(\alpha)$ if the sequence $(a_j)_{j\ge1}$ of partial quotients is unbounded, otherwise it is closely related to the Lagrange spectrum. As this set has been studied thoroughly there is an abundance of information available on the set $\mathcal{S} := \{\liminf_{N\to\infty} D_N(\alpha) \mid \alpha \in \mathbb{R}\backslash\mathbb{Q}\}$ (see [6]). We restrict ourselves to state just a few of the known facts:

$\mathcal{S}$ is a closed subset of the interval $[1, 1 + 1/\sqrt{5}]$ with $\min \mathcal{S} = 1$ and $\max S = 1+1/\sqrt{5}$. Its subset $\mathcal{S} \cap (1+1/3, 1+1/\sqrt{5}]$ consists of the numbers $1 + m/\sqrt{9m^2 - 4}$ where m is a positive integer such that

$$m^2 + m_1^2 + m_2^2 = 3mm_1m_2$$

for some positive integers $m_1 \le m$ and $m_2 \le m$. The three largest numbers of $\mathcal{S}$ are $1 + 1/\sqrt{5}$, $1 + 1/\sqrt{8}$ and $1 + 5/\sqrt{221}$. Let

$$\mu_0 = \frac{253589820 + 283748\sqrt{462}}{491993569} = 4.527829\ldots$$

Then $[1, 1+1/\mu_0] \subseteq \mathcal{S}$ and there is no interval I such that $[1, 1+1/\mu_0] \subsetneqq I \subseteq \mathcal{S}$. On the other hand, there are gaps in $\mathcal{S}$ such as $J = (1+1/\sqrt{13}, 1+1/\sqrt{12})$, i.e. $J \cap \mathcal{S} = \emptyset$ but $1 + 1/\sqrt{12} \in \mathcal{S}$ and $1 + 1/\sqrt{13} \in \mathcal{S}$.

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