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On some divisor problems

by

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1. Introduction. We investigate the distribution of the divisor functions $d(1,1,2;n)$ and $d(1,1,2,2;n)$, which are defined as

$$d(1,1,2;n) = \#\{(n_1,n_2,n_3) \mid n_1,n_2,n_3 \in \mathbb{N},\ n_1n_2n_3^2 = n\},$$
$$d(1,1,2,2;n) = \#\{(n_1,n_2,n_3,n_4) \mid n_1,n_2,n_3,n_4 \in \mathbb{N},\ n_1n_2n_3^2n_4^2 = n\},$$

where $\mathbb{N}$ is the set of all natural numbers. Our results are:

Theorem 1.
$$\sum_{n\le x} d(1,1,2;n) = \textit{main terms} + O(x^{77/208+\varepsilon}).$$

Theorem 2.
$$\sum_{n\le x} d(1,1,2,2;n) = \textit{main terms} + O(x^{0.4+\varepsilon}).$$

Here ε is an arbitrarily small given positive number, and x is a large positive number. The exponent $77/208 = 0.3701\ldots$ of Theorem 1 improves the corresponding exponent $3/8 = 0.375$ of Schmidt [10], and the exponent 0.4 of Theorem 2 improves the exponent $45/109 = 0.412\ldots$ of Menzer and Seibold [9]. The connection of these divisor problems with the distribution of certain quantities of finite Abelian groups was first established in Krätzel [2]. Let $\tau(G)$ be the number of direct factors of a finite Abelian group G, and $t(G)$ be the number of unitary factors of G, and

$$T(x) = \sum \tau(G), \qquad T^*(x) = \sum t(G),$$

where the summations are over all Abelian groups of order not exceeding x. Then from the arguments of [2] we get

Corollary 1. $T(x) = \textit{main terms} + O(x^{0.4+2\varepsilon})$.

Corollary 2. $T^*(x) = \textit{main terms} + O(x^{77/208+2\varepsilon})$.

After certain reductions our problems are connected with multiple exponential sums, which can be estimated as accurately as possible by means of the method given in the author's preceding papers [3]–[8] on similar divisor problems. A sharper estimate of Huxley [1] will also be appealed to in proving Theorem 1.

The author wishes to thank his colleagues, M. N. Huxley of Cardiff and E. Krätzel of Jena, for sending reprints of [1] and [2], and for their encouragement.

2. Proof of Theorem 1. Let

$$S(a,b,c;x)=\sum_{n^am^{b+c}\le x, n\le m}\psi\bigg(\Big(\frac{x}{n^am^b}\Big)^{1/c}\bigg),\qquad \psi(t)=t-[t]-1/2.$$

We have

Lemma 1.

$$\sum_{n\le x} d(1,1,2;n)=\textit{main terms}+\Delta(1,1,2;x),$$

where

$$\Delta(1,1,2;x)=-2S(1,1,2;x)-2S(1,2,1;x)-2S(2,1,1;x)+O(x^{1/4}).$$

Proof. This is Lemma 5 of [2]. The expression for $\Delta(1,1,2;x)$ comes from a paper of Vogts (cf. Lemma 3 of Krätzel [2]).

For any permutation (a,b,c) of $(1,1,2)$, it suffices for us to consider $S(M,N;x)$, where M and N are integers with $2M\ge N$, $M^{b+c}N^a\le x$,

$$S(M,N;x):=S_{a,b,c}(M,N;x)=\sum_{(m,n)\in D}\psi\bigg(\Big(\frac{x}{n^am^b}\Big)^{1/c}\bigg),\qquad MN>x^{0.35},$$

and $D:=D(M,N)=\{(m,n)\mid m\sim M,\ n\sim N,\ m^{b+c}n^a\le x,\ n\le m\}$. Throughout this paper, we use $r\sim R$ and $r\cong R$ to mean $1\le r/R<2$ and $C_1\le r/R\le C_2$, respectively; C_i $(i=1,2,3,\ldots)$ will be some absolute constants. In order to introduce exponential sums we apply the familiar Fourier expansion treatment of the function $\psi(t)$; thus for a parameter $K\ge 100$, we get, as on p. 266 of [3], the following estimate:

$$(\ln x)^{-1}S(M,N;x) \ll MNK^{-1}+\sum_{1\le h\le K^2}\min(1/h,K/h^2)\Big|\sum_{(m,n)\in D}e(f(h,m,n))\Big|,$$

where

$$f(h,m,n)=h\Big(\frac{x}{n^am^b}\Big)^{1/c}.$$

Thus for some $H \in [1, K^2]$ we have

$$x^{-\varepsilon} S(M, N; x) \ll MNK^{-1} + \min(1, K/H)\Phi(H, M, N), \tag{1}$$

where

$$\begin{aligned} \Phi(H, M, N) &:= \Phi_{a,b,c}(H, M, N) \\ &= H^{-1} \sum_{h \sim H} \Big| \sum_{(m,n) \in D} e(f(h, m, n)) \Big|. \end{aligned} \tag{2}$$

Similarly to (1) and (2) of [7] we get (we have omitted the routine details for simplicity)

$$\begin{aligned} &\Phi(H, M, N) \\ &\quad \ll H^{-1}(M^2(HF)^{-1})^{1/2} \sum_{h \sim H} \Big| \sum_{(u,n) \in D_1} P(u)Q(n)e(g_1(h, u, n)) \Big| \\ &\qquad + (HF)^{1/2} + x^{1/3} \end{aligned} \tag{3}$$

and

$$\begin{aligned} \Phi(H, M, N) \ll MN(H^2F)^{-1} \sum_{h \sim H} \Big| \sum_{(u,v) \in D_2} R(u)S(v)e(g_2(h, u, v)) \Big| \\ + (HF)^{1/2} + x^{1/3}, \end{aligned} \tag{4}$$

where $F = (xM^{-b}N^{-a})^{1/c}$, D_1 and D_2 are subsets of $\{(u, n) \mid u/U \in [C_1, C_2], n \in [N, 2N)\}$ and $\{(u, v) \mid u/U \in [C_3, C_4], v/V \in [C_5, C_6]\}$, respectively, both are embraced by $O(1)$ algebraic curves, $P(\cdot)$, $Q(\cdot)$, $R(\cdot)$, $S(\cdot)$ are monomials of the form At^α, with A being the number independent of variables, and α being a rational, and

$$\begin{gathered} |P(\cdot)|, |Q(\cdot)|, |R(\cdot)|, |S(\cdot)| \le 1; \\ g_1(h, u, n) = C_7(xh^c u^b n^{-a})^{1/(c+b)}, \quad g_2(h, u, v) = C_8(xh^c u^b v^a)^{1/4}; \\ U = HFM^{-1}, \quad V = HFN^{-1}. \end{gathered}$$

We can apply Theorem 3 of [4] to estimate the triple exponential sum in (3), with the choice $(h, x, y) = (h, u, n)$; this yields

$$\begin{aligned} x^{-\varepsilon}\Phi(H, M, N) \ll\ & \sqrt[22]{H^8F^{11}M^3N^{13}} + (HF)^{1/2}N^{5/8} + \sqrt[16]{H^4F^4N^{17}} \\ & + \sqrt[32]{H^8F^{11}M^3N^{28}} + \sqrt[32]{H^{13}F^{16}M^3N^{18}} \\ & + \sqrt[4]{FMN^4} + \sqrt[4]{HF^2MN^2} + x^{1/3}. \end{aligned} \tag{5}$$

By putting the estimate (5) into (1) and choosing $K \in [0, x]$ optimally via Lemma 2 of [3], we get

LEMMA 2.

$$x^{-2\varepsilon}S(M,N;x) \ll \sqrt[30]{F^{11}M^{11}N^{21}} + \sqrt[24]{F^8M^8N^{18}} + \sqrt[20]{F^4M^4N^{21}} + \sqrt[40]{F^{11}M^{11}N^{36}} + \sqrt[45]{F^{16}M^{16}N^{31}} + \sqrt[5]{F^2M^2N^3} + (FMN^4)^{1/4} + x^{1/3}.$$

Since (a,b,c) is a permutation of $(1,1,2)$, $M \gg N$ and $M^{b+c}N^a \le x$, we have $F \ll x(MN^2)^{-1}$ and $N \ll x^{1/4}$, and thus by Lemma 2 we get

$$x^{-2\varepsilon}S(M,N;x) \ll \sqrt[5]{x^2N^{-1}} + (xN^2)^{1/4} + x^{0.36}. \tag{6}$$

We now use Huxley's results, which are better than those which can be deduced from [5]. By Theorem 4 of [1], for $(a,b,c)=(1,1,2)$ we have

$$x^{-\varepsilon}S(M,N;x) \ll N\left(\frac{Mx}{N}\right)^{23/146} \ll (x^{46}N^{123})^{1/219} \ll x^{0.36}; \tag{7}$$

for $(a,b,c)=(2,1,1)$ or $(1,2,1)$ we have

$$x^{-\varepsilon}S(M,N;x) \ll N(xN^{-2})^{23/73} = (x^{23}N^{27})^{1/73}. \tag{8}$$

From (6)–(8) we get

$$\begin{aligned} x^{-2\varepsilon}S(M,N;x) &\ll (xN^2)^{1/4} + \min((x^{23}N^{27})^{1/73}, \sqrt[5]{x^2N^{-1}}) + x^{0.36} \\ &\ll (xN^2)^{1/4} + x^{77/208}. \end{aligned} \tag{9}$$

To remove the term $(xN^2)^{1/4}$ we use Kolesnik's method.

LEMMA 3. *Let $f(x,y)$ be an algebraic function in the rectangle $D_0 = \{(x,y) \mid x\sim X, y\sim Y\}$ with $f(x,y) \sim_\Delta Ax^\alpha y^\beta$ throughout D_0, and let D be a subdomain of D_0 bounded by $O(1)$ algebraic curves. Suppose that $X \gg Y$, $N = XY$, $A>0$, $F = AX^\alpha Y^\beta$, $\alpha\beta(\alpha+\beta-1)(\alpha+\beta-2)\ne 0$, $0<\Delta<\varepsilon_0$, where ε_0 is a small number depending at most on α and β. Then*

$$\sum_{(x,y)\in D} e(f(x,y)) \ll_{\varepsilon,\alpha,\beta} \Big(\sqrt[6]{F^2N^3} + N^{5/6} + \sqrt[10]{\Delta^4Y^4F^2N^5} + \sqrt[8]{F^{-1}X^{-1}N^8} + NF^{-1/4} + \sqrt[4]{\Delta X^{-1}N^4} + NY^{-1/2}\Big)(NF)^{\varepsilon/2}.$$

Proof. See Lemma 1.5 of [6]. This result is due to Kolesnik.

By Cauchy's inequality and Weyl's inequality (cf. Lemma 3 of [3]), after a partial summation removing the smooth coefficient $S(v)$ together with an appeal to Lemma 1 of [3] relaxing the range of v, we get for the double summation over (u,v) in (4) the following estimate:

$$x^{-\varepsilon}\Big|\sum_{(u,v)\in D_2} R(u)S(v)e(g_2(u,v))\Big|^2 \ll \frac{(UV)^2}{Q} + \frac{UV}{Q}\sum_{1\le q\le Q}\Big|\sum_{(u,v)\in D(q)} e(g_3)\Big|,$$

where

$$D(q)=\{(u,v)\mid u\in[C_3U,C_4U],\ v\in[C_5V,C_6V],\ (v+q)\in[C_5V,C_6V]\},$$
$$g_3=g_3(h,u,v,q)=g_2(h,u,v+q)-g_2(h,u,v),$$
$$Q=\min(V(\ln x)^{-1},\sqrt[8]{(HF)^{-2}U^3V^5}).$$

If $Q\ll 1$ the above inequality holds obviously. Assume that $Q\gg 1$. We apply Lemma 3 to the inner double exponential sum over (u,v), with the choice $X\cong V$, $Y\cong U$, $\Delta=q/V$, $F\cong HFq/V$, to obtain

$$x^{-\varepsilon}\sum_{(u,v)\in D(q)} e(g_3)\ll\sqrt[6]{(HF)^2q^2U^3V}+(UV)^{5/6}+\sqrt[10]{(HF)^2q^6U^9V^{-1}}$$
$$+\sqrt[8]{(HF)^{-1}q^{-1}U^8V^8}+(HFq)^{-1/4}UV^{5/4}$$
$$+\sqrt[4]{qU^4V^2}+VU^{1/2},$$

and so

$$x^{-2\varepsilon}\sum_{(u,v)\in D_2} R(u)S(v)e(g_2(u,v)) \tag{10}$$
$$\ll (UV)^{11/12}+VU^{3/4}+\sqrt[16]{(HF)^2U^{13}V^{11}}+\sqrt[80]{(HF)^2U^{85}V^{51}}$$
$$+\sqrt[64]{(HF)^{-2}U^{67}V^{53}}+\sqrt[16]{(HF)^{-1}U^{16}V^{15}}$$
$$+\sqrt[128]{(HF)^{-6}U^{125}V^{123}}+UV(HF)^{-1/8}+\sqrt[64]{(HF)^{-6}U^{61}V^{67}}.$$

By substituting (10) in (4) we get

$$x^{-2\varepsilon}\Phi(H,M,N)\ll\sqrt[12]{(HF)^{10}MN}+\sqrt[16]{(HF)^{10}M^3N^5} \tag{11}$$
$$+\sqrt[4]{(HF)^3M}+\sqrt[80]{(HF)^{58}M^{-5}N^{29}}$$
$$+\sqrt[64]{(HF)^{54}M^{-3}N^{11}}+\sqrt[16]{(HF)^{14}N}$$
$$+\sqrt[128]{(HF)^{114}M^3N^5}+(HF)^{7/8}$$
$$+\sqrt[64]{(HF)^{58}M^3N^{-3}}+x^{1/3}.$$

We put the estimate of (11) in (1) and choose $K\in[0,x]$ optimally via Lemma 2 of [3] to get

$$x^{-3\varepsilon}S(M,N;x)\ll\sqrt[22]{F^{10}(MN)^{11}}+\sqrt[26]{F^{10}M^{13}N^{15}} \tag{12}$$
$$+\sqrt[7]{F^3M^4N^3}+\sqrt[138]{F^{58}M^{53}N^{87}}$$
$$+\sqrt[118]{F^{54}M^{51}N^{65}}+\sqrt[30]{F^{14}M^{14}N^{15}}$$
$$+\sqrt[242]{F^{114}M^{117}N^{119}}+\sqrt[15]{(FMN)^7}$$
$$+\sqrt[122]{F^{58}M^{61}N^{55}}+x^{1/3}$$

$$\begin{aligned}&\ll \sqrt[22]{x^{10}MN^{-9}} + \sqrt[26]{x^{10}M^3N^{-5}}\\&\quad + \sqrt[7]{x^3MN^{-3}} + \sqrt[138]{x^{58}M^{-5}N^{-29}} + x^{1/3}\\&\quad + \sqrt[118]{x^{54}M^{-3}N^{-43}} + \sqrt[30]{x^{14}N^{-13}}\\&\quad + \sqrt[242]{x^{114}M^3N^{-109}} + \sqrt[15]{x^7N^{-7}}\\&\quad + \sqrt[122]{x^{58}M^3N^{-61}},\end{aligned}$$

by using the fact that $F \ll xM^{-1}N^{-2}$. From (9) and (12) we deduce that

$$x^{-3\varepsilon}S(M,N;x) \ll \sum_{1\le i\le 9} E_i + x^{77/208},$$

where (note that $MN \ll x^{1/2}$ always holds)

$$\begin{aligned}E_1 &= \min((xN^2)^{1/4},\ \sqrt[22]{x^{10}MN^{-9}}) \le (x^{15}MN)^{1/42} \ll x^{31/84} < x^{0.37},\\E_2 &= \min((xN^2)^{1/4},\ \sqrt[26]{x^{10}M^3N^{-5}}) \le (x^{14}(MN)^3)^{1/42}\\&\ll x^{31/84} < x^{0.37},\\E_3 &= \min((xN^2)^{1/4}, \sqrt[7]{x^3MN^{-3}}) \le (x^5MN)^{1/15} \ll x^{11/30} < x^{0.37},\\E_4 &= \min((xN^2)^{1/4},\ \sqrt[138]{x^{58}N^{-34}}) \le x^{75/206} < x^{0.37},\\E_5 &= \min((xN^2)^{1/4},\ \sqrt[118]{x^{54}N^{-46}}) \le x^{77/210},\\E_6 &= \min((xN^2)^{1/4},\ \sqrt[30]{x^{14}N^{-13}}) \le x^{41/112} < x^{0.37},\\E_7 &= \min((xN^2)^{1/4},\ \sqrt[242]{x^{114}M^3N^{-109}}) \le (x^{170}(MN)^3)^{1/466}\\&\ll x^{343/932} < x^{0.37},\\E_8 &= \min((xN^2)^{1/4},\ \sqrt[15]{x^7N^{-7}}) \le x^{21/58} < x^{0.37},\\E_9 &= \min((xN^2)^{1/4},\ \sqrt[122]{x^{58}M^3N^{-61}}) \le (x^{90}(MN)^3)^{1/250}\\&\ll x^{91.5/250} < x^{0.37},\end{aligned}$$

whence the required estimate follows.

3. Proof of Theorem 2. Let

$$S(a,b,c,d;x) = \sum_{n_1^a n_2^b n_3^{c+d} \le x,\ 1\le n_1(\le)n_2\le n_3} \psi\biggl(\Bigl(\frac{x}{n_1^a n_2^b n_3^c}\Bigr)^{1/d}\biggr),$$

where $n_1(\le)n_2$ means that $n_1 \le n_2$ for $(a,b) = (a_i,a_j)$ with $i<j$, and $n_1 < n_2$ otherwise; here we have set $(a_1,a_2,a_3,a_4) = (1,1,2,2)$. Then we have

LEMMA 4.
$$\sum_{n\le x} d(1,1,2,2;n) = \textit{main terms} + \Delta(1,1,2,2;x),$$
where
$$\Delta(1,1,2,2;x) = -\sum_{(a,b,c,d)} S(a,b,c,d;x) + O(x^{1/3}),$$
and (a,b,c,d) *runs through all the ordered permutations of* $(1,1,2,2)$.

Proof. The expression for the remainder $\Delta(1,1,2,2;x)$ is due to Vogts, see [2].

In what follows we use the method presented in [8] for 4-dimensional exponential sums, but the details are much simpler here, and we omit many routine procedures. The reader is invited to consult [8]. It suffices for us to achieve an estimate of the type $S(a,b,c,d;\boldsymbol{N}) \ll x^{0.4+4\varepsilon}$, where $\boldsymbol{N} = (N_1,N_2,N_3)$, N_1, N_2 and N_3 are arbitrary positive integers with

$$(13)\qquad N_1 \ll N_2 \ll N_3, \quad N_1^a N_2^b N_3^{c+d} \le x, \quad N_1N_2N_3 > x^{0.37},$$

(a,b,c,d) is any permutation of $(1,1,2,2)$, and
$$S(a,b,c,d;\boldsymbol{N}) = \sum\nolimits^* \psi\bigg(\Big(\frac{x}{n_1^a n_2^b n_3^c}\Big)^{1/d}\bigg),$$
where $\sum^*$ denotes summation over lattice points (n_1,n_2,n_3) with
$$n_1^a n_2^b n_3^{c+d} \le x, \quad 1 \le n_1 (\le) n_2 \le n_3, \quad N_v \le n_v < 2N_v \quad (v=1,2,3).$$
Let $G = (xN_1^{-a}N_2^{-b}N_3^{-c})^{1/d}$. As in (12) of [8], we can deduce

LEMMA 5.
$$\begin{aligned} x^{-2\varepsilon} S(a,b,c,d;\boldsymbol{N}) \ll\ & \sqrt[30]{G^{11}N_1^{30}N_2^{21}N_3^{11}} + \sqrt[24]{(GN_3)^8 N_2^{18} N_1^{24}} \\ &+ \sqrt[20]{(GN_3)^4 N_2^{21} N_1^{20}} + \sqrt[40]{(GN_3)^{11} N_2^{36} N_1^{40}} \\ &+ \sqrt[45]{(GN_3)^{16} N_2^{31} N_1^{45}} + \sqrt[5]{(GN_3)^2 N_2^3 N_1^5} \\ &+ \sqrt[4]{GN_3 N_2^4 N_1^4} + x^{13/36}. \end{aligned}$$

Similarly to (17) of [8], we also have

LEMMA 6. $x^{-3\varepsilon} S(a,b,c,d;\boldsymbol{N}) \ll (GN_1N_2N_3)^{1/2} + x^{13/36}$.

Note that the term $x^{13/36}$ comes from an application of Lemma 1 of [8] (see also Lemma 1.4 of [6]) to the variable n_3 together with an estimate for the resulting "extra" term $R(h,n_1,n_2)$ (involving the use of the exponent pair $(1/6,4/6)$). From (13) it is seen that $G \ll x(N_1^2N_2^2N_3)^{-1}$ and $N_1N_2N_3 \ll x^{1/2}$. Thus Lemmas 5 and 6 give respectively (with $J = N_1N_2$)

$$(14)\qquad \begin{aligned} x^{-4\varepsilon} S(a,b,c,d;\boldsymbol{N}) \ll\ & \sqrt[30]{x^{11}J^{3.5}} + \sqrt[24]{x^8J^5} + \sqrt[20]{x^4J^{13}} \\ &+ \sqrt[40]{x^{11}J^{16}} + \sqrt[4]{xJ^2} + x^{0.4}, \end{aligned}$$

and

$$x^{-4\varepsilon} S(a,b,c,d;\boldsymbol{N}) \ll (xJ^{-1})^{1/2} + x^{13/36}. \qquad (15)$$

Now if $J \geq x^{0.2}$ then the required estimate follows from (15), and otherwise it follows from (14). Thus Theorem 2 has been verified.

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