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The Hausdorff dimension of sets arising in metric Diophantine approximation

by

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A real number x which satisfies $|qx-p| < q^{-\tau}$ for infinitely many $p, q \in \mathbb{Z}$ for any $\tau > 1$ is called *well-approximable*. The Hausdorff dimension of the set of such numbers was found in 1929 by Jarník [5]. Throughout this paper the modulus will be the sup norm, i.e.,

$$|\mathbf{q}| = \max_i \{|q_i|\}.$$

Let $W(m, n; \tau)$ be the set of matrices $X \in M_{m\times n}(\mathbb{R})$ which satisfy

$$|\mathbf{q}X - \mathbf{p}| < |\mathbf{q}|^{-\tau}$$

for infinitely many $\mathbf{q} \in \mathbb{Z}^m, \mathbf{p} \in \mathbb{Z}^n$. This set is a generalisation of the set of well-approximable numbers corresponding to $m = n = 1$. The Hausdorff dimension of $W(m, n; \tau)$ was obtained in [1]. The Hausdorff dimension of the subset

$$W_0 = W_0(m, n; \tau) = \{X \in \mathbb{R}^{mn} : |\mathbf{q}X| < |\mathbf{q}|^{-\tau} \text{ for infinitely many } \mathbf{q} \in \mathbb{Z}^m\}$$

of $W(m, n; \tau)$ was obtained in [2]. In W_0, all the vectors $\mathbf{p}$ which can be regarded as a general kind of numerator, are set to zero. There now arises the question of what happens for a more general selection of the vectors $\mathbf{p}$. To this end define $W_A(m, n; \tau)$ to be the set of matrices $X \in M_{m\times n}(\mathbb{R})$ such that

$$|\mathbf{q}X - \mathbf{p}| < |\mathbf{q}|^{-\tau}$$

for infinitely many $\mathbf{q} \in \mathbb{Z}^m$, $\mathbf{p} \in A$. In these sets $\widehat{W}$ will be used to denote the set of X restricted to the unit cube I. In this paper the Hausdorff dimension of $W_A(m, n; \tau)$ is studied when $A \subseteq \mathbb{Z}^n$ is a subgroup of $\mathbb{Z}^n$. Because A is a subgroup the result holds for $m + r > n$ where r is the dimension of the lattice A. Following the proof an application will be shown. This will give the Hausdorff dimension of the set of $X \in M_{m\times n}(I)$, where $I = (-1/2, 1/2]$, such that the system of inequalities

$$\max\{|\mathbf{q}\cdot\mathbf{x}^{(1)} - p_1|, \ldots, |\mathbf{q}\cdot\mathbf{x}^{(r)} - p_r|, |\mathbf{q}\cdot\mathbf{x}^{(r+1)}|, \ldots, |\mathbf{q}\cdot\mathbf{x}^{(n)}|\} < |\mathbf{q}|^{-\tau} \tag{1}$$

holds for infinitely many integer vectors $\mathbf{q} \in \mathbb{Z}^m$ and $\mathbf{p} \in A$. Here $\mathbf{x}^{(1)}, \ldots \ldots, \mathbf{x}^{(n)}$ are the column vectors of X. This is a set of mixed Diophantine inequalities and $A = \mathbb{Z}^r \times \{0\}^{n-r}$. Evidently the result for this set contains the result for W_0 in which $r = 0$. The Hausdorff dimension of $\widehat{W}_{\mathbb{Z}^r \times \{0\}^{n-r}}(m, n; \tau)$ in the case $m > 1, r = 1, n = 2$ was obtained in [3]. The case $r = 0, m = 1$ will be omitted; in this case $\widehat{W}(1, n; \tau) = \{0\}$ and so the Hausdorff dimension is 0.

As some of the proofs for the present paper are very similar to those of [2] they will be omitted. The upper bound is straightforward as is usual and is found using a covering and counting argument. The lower bound will be found in two parts as in [2]. For $m + r > n$ the more general result discussed above will be used. The second part will use this result to find the Hausdorff dimension of a cartesian product of two sets one of which is diffeomorphic to $\widehat{W}_{\mathbb{Z}^r \times \{0\}^{n-r}}(m, n; \tau)$ when $m + r \leq n$. This method is taken from [2] and the details will not be included here. Hereafter $\mathbb{R}^{mn}$ is identified with the set of $m \times n$ real matrices $M_{m \times n}(\mathbb{R})$. Also $M_{m \times n}(I)$ will be denoted by I^{mn}.

To obtain the Hausdorff dimension of W_A the idea of ubiquity [3] will be used.

Define the set $\Pi(\mathbf{p}, \mathbf{q})$ to be the set of matrices $X \in I^{mn}$ such that

$$|\mathbf{q}X - \mathbf{p}| = 0.$$

Also define

$$\mathcal{N}_A(N) = \mathrm{card}\{\mathbf{p} \in A : |\mathbf{p}| \leq N\} \quad \text{and} \quad \wp_A(p) = \mathrm{card}\{\mathbf{p} \in A : |\mathbf{p}| = p\}.$$

The symbol κ will be used to denote upper order at infinity. That is, for an increasing function f,

$$\kappa(f) = \limsup_{N \to \infty} \frac{\log f(N)}{\log N}.$$

Since in this paper A will always be a subgroup of the lattice it will have a dimension r and this will usually be equivalent to $\kappa(\mathcal{N})$. For example, consider those vectors with only even entries. As there may exist a subgroup for which this is not so the upper order notation will be maintained but in the example given $\kappa(\mathcal{N}) = r$.

Theorem. *When $m, n > 0$ are integers, for $m + \kappa(\mathcal{N}) > n$,*

$$\dim W_A(m, n; \tau) = (m-1)n + \frac{m + \kappa(\mathcal{N})}{\tau + 1} \quad \textit{when } \tau \geq \frac{m + \kappa(\mathcal{N})}{n} - 1.$$

First the upper bound result will be stated with a brief proof. Full details of this type of proof can be found in [1] and [2].

LEMMA 1. *For all integers* m, n,

$$\dim W_A(m,n;\tau) \le (m-1)n + \frac{m+\kappa(\mathcal{N})}{\tau+1} \quad \text{when } \tau \ge \frac{m+\kappa(\mathcal{N})}{n} - 1.$$

Proof. The number of mn-dimensional hypercubes $C \in C(\mathbf{p},\mathbf{q})$ of width $4|\mathbf{q}|^{-(\tau+1)}$ with centres on the $(m-1)n$-dimensional hyperplane $\Pi(\mathbf{p},\mathbf{q})$ which cover $\{X \in I^{mn} : |\mathbf{q}X - \mathbf{p}| < |\mathbf{q}|^{-\tau}\}$ is $\ll |\mathbf{q}|^{(\tau+1)(m-1)n}$. The collection of such hypercubes with $|\mathbf{q}| > N$ covers W_A for each N. The "t-volume" of this cover of $W_A(m,n;\tau)$ can be estimated by

$$\sum_{\substack{\mathbf{q}\in\mathbb{Z}^m\\ |\mathbf{q}|>N}} \sum_{\substack{\mathbf{p}\in A\\ |\mathbf{p}|<|\mathbf{q}|}} \sum_{C\in C(\mathbf{p},\mathbf{q})} 4^t|\mathbf{q}|^{-(\tau+1)t}.$$

Now take $t > (m-1)n + (m+\kappa(\mathcal{N}))/(\tau+1)$, i.e., take t to be $(m-1)n + (m+\kappa(\mathcal{N}))/(\tau+1) + \varepsilon$ for some $\varepsilon > 0$. The triple sum is then

$$\ll \sum_{q>N} q^{-(\tau+1)(m-1)n-m-\kappa(\mathcal{N})-\varepsilon(\tau+1)+(\tau+1)(m-1)n+\kappa(\mathcal{N})+(m-1)}$$

where $|\mathbf{q}| = q$. This becomes

$$\ll \sum_{q>N} q^{-1-\varepsilon},$$

which tends to 0 as $N \to \infty$ and proves the lemma.

For the lower bound consider the related lim sup set Λ_A, where

$$\Lambda_A = \{X \in I^{mn} : \operatorname{dist}(X,\Pi(\mathbf{p},\mathbf{q})) < \psi(|\mathbf{q}|) \text{ for infinitely many } \mathbf{q} \in \mathbb{Z}^m, \mathbf{p} \in A\},$$

and $\psi(|\mathbf{q}|) = m^{-1}|\mathbf{q}|^{-(\tau+1)}$. Here $\operatorname{dist}(X,\Pi(\mathbf{p},\mathbf{q}))$ represents the distance of X from the resonant set $\Pi(\mathbf{p},\mathbf{q})$ taken with the sup norm. Evidently $W_A \supseteq \Lambda_A$. Thus it suffices to find a lower bound for the Hausdorff dimension for Λ_A. In [3] the method of ubiquity was developed in order to find the Hausdorff dimensions of general lim sup sets. There follows a modified version (with appropriate notation for Λ_A) of ubiquity which can be shown to be equivalent to the full definition in [3] when the affine case is considered.

Ubiquity. Let Ω be a bounded open region in $\mathbb{R}^{mn}$ and let

$$\mathcal{R}_A = \{\Pi(\mathbf{p},\mathbf{q}) : \mathbf{q} \in \mathbb{Z}^m\backslash\{0\},\ \mathbf{p} \in A\}.$$

Also for each δ write

$$B(\Pi(\mathbf{p},\mathbf{q});\delta) = \{X \in \Omega : \operatorname{dist}(X,\Pi(\mathbf{p},\mathbf{q})) < \delta\}.$$

Then, if

$$\lim_{N\to\infty}\Big|\Omega\setminus\bigcup_{\substack{\mathbf{q}\\|\mathbf{q}|\le N\\ \mathbf{p}\in A}} B(\Pi(\mathbf{p},\mathbf{q});\varrho(N))\Big| = 0$$

and $\lim_{N\to\infty}\varrho(N)=0$, where ϱ is a decreasing function, the family $\mathcal{R}_A$ is called a *ubiquitous system* relative to ϱ. (For the details see [3].)

LEMMA 2. *$\mathcal{R}_A$ is ubiquitous with respect to the function*

$$\varrho(N)=(m+2)N^{-m/n}(\mathcal{N}_A(N/2))^{-1/n}\log N.$$

To prove this lemma we need the following.

LEMMA 3. *For $N>N_0(n,m,A)$ and every $X\in I^{mn}$ there exist integer vectors $\mathbf{q}\in\mathbb{Z}^m$, $\mathbf{p}\in A$ with $|\mathbf{q}|,|\mathbf{p}|\le N$ for N_0 large enough such that*

$$|\mathbf{q}X-\mathbf{p}|<(m+2)N^{1-m/n}(\mathcal{N}_A(N/2))^{-1/n}.$$

Proof. Consider those $\mathbf{q}$ with non-negative components and those $\mathbf{p}$ such that $|\mathbf{p}|\le N/2$. There are $(N+1)^m\mathcal{N}_A(N/2)$ vectors $\mathbf{q}X-\mathbf{p}$ and

$$-\frac{m+2}{2}N\le \mathbf{q}X-\mathbf{p}\le\frac{m+2}{2}N.$$

Divide the cube in $\mathbb{R}^n$ with centre 0, sidelength $(m+2)N$ and volume $(m+2)^nN^n$ into $N^m\mathcal{N}_A(N/2)$ smaller cubes of volume $(m+2)^nN^{n-m}\times(\mathcal{N}_A(N/2))^{-1}$ and sidelength $(m+2)N^{1-m/n}(\mathcal{N}_A(N/2))^{-1/n}$. As $(N+1)^m>N^m$ there must be two vectors $\mathbf{q}_1X-\mathbf{p}_1$, $\mathbf{q}_2X-\mathbf{p}_2$, say, in one small cube. Therefore

$$|(\mathbf{q}_1-\mathbf{q}_2)X-(\mathbf{p}_1-\mathbf{p}_2)|<(m+2)N^{1-m/n}(\mathcal{N}_A(N/2))^{-1/n}.$$

Evidently $\mathbf{q}_1-\mathbf{q}_2\in\mathbb{Z}^m$ and $|\mathbf{q}_1-\mathbf{q}_2|\le N$. Also $\mathbf{p}_1-\mathbf{p}_2\in A$ since A is closed under subtraction and $|\mathbf{p}_1-\mathbf{p}_2|\le N$ by choice of $\mathbf{p}_1$ and $\mathbf{p}_2$. Thus the lemma is proved.

In Lemma 3 the $\mathbf{p}$ which has been shown to exist will always be such that $|\mathbf{p}|<\frac{1}{2}|\mathbf{q}|+1$ as otherwise $|\mathbf{q}X-\mathbf{p}|>1$, which would be unacceptable.

Proof of Lemma 2. The $\mathbf{p},\mathbf{q}$ in the following proof are those which have been shown to exist for any X in the previous lemma. Let

$$E(N)=\{X\in I^{mn}:|\mathbf{q}|<N/\log N\}$$

and

$$D(N)=\{X\in I^{mn}:|X-\partial I^{mn}|\ge N^{-1}\}\setminus E(N).$$

Then

$$E(N) \subseteq \bigcup_{q=1}^{N/\log N} \bigcup_{p=1}^{k|\mathbf{q}|} \bigcup_{|\mathbf{q}|=q} \bigcup_{\substack{\mathbf{p}\in A\\ |\mathbf{p}|=p}} \{X \in I^{mn} : \\ |\mathbf{q}X - \mathbf{p}| < (m+2)N^{1-m/n}(\mathcal{N}_A(N/2))^{-1/n}\}.$$

Therefore

$$|E(N)| \le \sum_{q=1}^{N/\log N} \sum_{p=1}^{k|\mathbf{q}|} \sum_{|\mathbf{q}|=q} \sum_{\substack{\mathbf{p}\in A\\ |\mathbf{p}|=p}} \frac{2^n(m+2)^n N^{n-m}(\mathcal{N}_A(N/2))^{-1}}{|\mathbf{q}|^n}$$

$$\ll N^{n-m}(\mathcal{N}_A(N/2))^{-1} \sum_{q=1}^{N/\log N} \mathcal{N}_A(kq)q^{m-1-n}.$$

For $m + \kappa(\mathcal{N}) > n$ this is

$$\ll N^{n-m}(\mathcal{N}_A(N/2))^{-1}\left(\frac{N}{\log N}\right)^{m-n-1}\left(\frac{N}{\log N}\right)\mathcal{N}_A\left(\frac{N}{\log N}\right). \tag{2}$$

This is $\ll (\log N)^{n-m-\kappa(\mathcal{N})+\varepsilon}$ for all $\varepsilon > 0$ since it can be shown that

$$\frac{\mathcal{N}_A(N/\log N)}{\mathcal{N}_A(N/2)} \ll (\log N)^{-\kappa(\mathcal{N})+\varepsilon}.$$

Thus $|E(N)| \to 0$ as $N \to \infty$ when $m + \kappa(\mathcal{N}) > n$.

Thus $\lim_{N\to\infty} |E(N)| = 0$ and $\lim_{N\to\infty} |I^{mn}\backslash D(N)| = 0$. Now let $\widetilde{X} \in D(N)$ and choose a $\widetilde{\mathbf{q}}$ such that

$$|\widetilde{\mathbf{q}}\widetilde{X} - \widetilde{\mathbf{p}}| \le (m+2)N^{1-m/n}(\mathcal{N}_A(N/2))^{-1/n}, \\ N/\log N \le |\widetilde{\mathbf{q}}| \le N$$

as is possible by Lemma 3.

Now $|\widetilde{\mathbf{q}}| = |\widetilde{q}_i|$ for some i. Let $\delta_j = (\widetilde{p}_j - \widetilde{\mathbf{q}}\widetilde{\mathbf{x}}^{(j)})/|\widetilde{q}_i|$, for $1 \le j \le n$. Then $\widetilde{\mathbf{q}}(\widetilde{\mathbf{x}}^{(j)} + \delta_j\mathbf{e}^{(i)}) - \widetilde{p}_j = 0$ for $j = 1, \ldots, n$ where $\mathbf{e}^{(i)}$ denotes the ith basis vector. Also

$$|\delta_j| \le \frac{(m+2)N^{1-m/n}(\mathcal{N}_A(N/2))^{-1/n}}{|\widetilde{\mathbf{q}}|} \\ \le (m+2)N^{-m/n}\log N(\mathcal{N}_A(N/2))^{-1/n},$$

for $1 \le j \le n$. Therefore $X = (\widetilde{\mathbf{x}}^{(1)} + \delta_1\mathbf{e}^{(i)}, \ldots, \widetilde{\mathbf{x}}^{(n)} + \delta_n\mathbf{e}^{(i)})$ is a point in the resonant set and

$$\mathrm{dist}(\widetilde{X}, X) \le (m+2)N^{-m/n}(\mathcal{N}_A(N/2))^{-1/n}\log N.$$

Now, let

$$\varrho(N) = (m+2)N^{-m/n}(\mathcal{N}_A(N/2))^{-1/n}\log N,$$

so that

$$D(N) \subseteq A(N) = \bigcup_{\substack{\mathbf{q}\in\mathbb{Z}^m,\mathbf{p}\in A\\ |\mathbf{q}|\le N}} B(\Pi(\mathbf{p},\mathbf{q}),\varrho(N))$$

giving $\lim_{N\to\infty} |\Omega\backslash A(N)| = 0$. Thus $\mathcal{R}_A$ is ubiquitous with respect to $\varrho(N)$ for $m+\kappa(\mathcal{N}) > n$.

From Theorem 1 in [3],

$$\dim \Lambda_A \ge \dim \mathcal{R}_A + \gamma \operatorname{codim} \mathcal{R}_A$$

where

$$\gamma = \limsup_{N\to\infty} \frac{\log \varrho(N)}{\log \psi(N)},$$

and $\mathcal{R}_A$ is the set of resonant sets $\Pi(\mathbf{p},\mathbf{q})$. Now, $\dim \mathcal{R}_A = (m-1)n$, $\operatorname{codim} \mathcal{R}_A = n$, $\varrho(N) = (m+2)N^{-m/n}(\mathcal{N}_A(N/2))^{-1/n}\log N$ and $\psi(N) = m^{-1}N^{-(\tau+1)}$. Hence

$$\begin{aligned}\gamma &= \limsup_{N\to\infty} \frac{\log \varrho(N)}{\log N}\frac{\log N}{\log \psi(N)}\\ &= \limsup_{N\to\infty} \left[\left(-\frac{m}{n} - \frac{1}{n}\frac{\log \mathcal{N}_A(N/2)}{\log N} + \frac{\log\log N}{\log N}\right)\left(\frac{-1}{\tau+1}\right)\right]\\ &= \frac{m+\kappa(\mathcal{N})}{n(\tau+1)}.\end{aligned}$$

Thus, since $W_A(m,n;\tau) \supseteq \Lambda_A$,

$$\dim W_A(m,n;\tau) \ge \dim \Lambda_A \ge (m-1)n + \frac{m+\kappa(\mathcal{N})}{\tau+1}.$$

So the theorem is proved for $m+\kappa(\mathcal{N}) > n$.

Now we apply this result to $W_{\mathbb{Z}^r\times\{0\}^{n-r}}(m,n;\tau)$. If $W_{\mathbb{Z}^r\times\{0\}^{n-r}}(m,n;\tau)$ then $\mathcal{N}_A(N) = N^r$. Thus

$$\kappa(\mathcal{N}) = \lim_{N\to\infty} \frac{\log N^r}{\log N} = r.$$

Hence for $m+r>n$,

$$\dim W_{\mathbb{Z}^r\times\{0\}^{n-r}}(m,n;\tau) \ge (m-1)n + \frac{m+r}{\tau+1}.$$

The upper bound result, Lemma 1, also holds giving

$$\dim \widehat{W}_A(m,n;\tau) = (m-1)n + \frac{m+r}{\tau+1}$$

for $m+r>n$ where $A = \mathbb{Z}^r \times \{0\}^{n-r}$.

To obtain the Hausdorff dimension for $W_{\mathbb{Z}^r\times\{0\}^{n-r}}(m,n;\tau)$ when $m+r \le n$ the result obtained above will be used. In the following method a subset $\widetilde{W}_A(m,n;\tau)$ will be needed. This is the set of matrices in $\widehat{W}_A(m,n;\tau)$ such that the column vectors $\mathbf{x}^{(r+1)}, \ldots, \mathbf{x}^{(n)}$ are linearly independent. It can be readily verified that this set has the same dimension as $\widehat{W}_A(m,n;\tau)$. In fact the set of X for which those vectors are linearly dependent is of lower dimension and so plays no major part.

First two lemmas will be stated, the proof of the first can be found in [1], [2] and [6] and the proof of the second can be found in [4].

LEMMA 4. *For any real interval*, (a,b), *and set* $X \subseteq \mathbb{R}^k$, *the Hausdorff dimension of the set* $(X \times (a,b)^p)$, *where* p *is a positive integer*, *is*

$$\dim(X \times (a,b)^p) = \dim X + p.$$

LEMMA 5. *If there exists an onto function* $f : X \to Y$ *such that* f *is one-one and obeys a bi-Lipschitz condition then* $\dim Y = \dim X$.

For simplicity only the case $n = m + r$ will be obtained; the result then extends easily to $n > m + r$. Let G denote the set of vectors

$$\Big(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m+r-1)}, \sum_{k=1}^{m-1} a_k \mathbf{x}^{(r+k)}\Big)$$

such that

$$(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m+r-1)}) \in \widetilde{W}_A(m, m+r-1;\tau)$$

and

$$a_k \in \left(\frac{-1}{m-1}, \frac{1}{m-1}\right).$$

As in [2], $G \subseteq \widehat{W}_A(m, m+r;\tau)$. Define the function

$$f : \widetilde{W}_A(m, m+r-1;\tau) \times \left(\frac{-1}{m-1}, \frac{1}{m-1}\right)^{m-1} \to G$$

by

$$(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m+r-1)}, a_1, \ldots, a_{m-1}) \mapsto \Big(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m+r-1)}, \sum_{k=1}^{m-1} a_k \mathbf{x}^{(r+k)}\Big).$$

Exactly as in [2] this function can be shown to be locally bi-Lipschitz. Therefore, from the two lemmas above

$$\begin{aligned}\dim \widehat{W}_A(m, m+r;\tau) &\geq \dim G \\ &= \dim \widetilde{W}_A(m, m+r-1;\tau) \times \left(\frac{-1}{m-1}, \frac{1}{m-1}\right)^{m-1},\end{aligned}$$

which gives

$$\dim \widehat{W}_A(m,n;\tau) \geq (m-1)(m+r) + \frac{m+r}{\tau+1},$$

for $m+r=n$. For $m+r>n$ extend the function f as in [2]. In this case the cartesian product is between $\widetilde{W}_A(m,m+r-1;\tau)$ and a "cube" in $(n-m-r+1)(m-1)$ dimensions. This gives the result that

$$\dim W_{\mathbb{Z}^r\times\{0\}^{n-r}}(m,n;\tau)\ge(m-1)n+\frac{m+r}{\tau+1},$$

and finally from Lemma 1 that

$$\dim W_{\mathbb{Z}^r\times\{0\}^{n-r}}(m,n;\tau)=(m-1)n+\frac{m+r}{\tau+1}.$$

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References

[1] J. D. Bovey and M. M. Dodson, *The Hausdorff dimension of systems of linear forms*, Acta Arith. 45 (1986), 337–358.

[2] H. Dickinson, *The Hausdorff dimension of systems of simultaneously small linear forms*, Mathematika 40 (1993), 367–374.

[3] M. M. Dodson, B. P. Rynne and J. A. G. Vickers, *Diophantine approximation and a lower bound for Hausdorff dimension*, ibid. 37 (1990), 59–73.

[4] K. J. Falconer, *Fractal Geometry—Mathematical Foundations and Applications*, Wiley, Chichester, 1990.

[5] V. Jarník, *Diophantische Approximationen und Hausdorffsches Mass,* Mat. Sb. 36 (1929), 371–382.

[6] D. G. Larman, *On Hausdorff measure in finite dimensional compact metric spaces*, Proc. London Math. Soc. (3) 17 (1967), 193–206.

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