ACTA ARITHMETICA
LXVIII.2 (1994)

Values of polynomials with integer coefficients and distance to their common zeros

by

FRANCESCO AMOROSO (Pisa)

1. Introduction. Let $f_1, \ldots, f_m \in \mathbb{Z}[x_1, \ldots, x_n]$ be polynomials of maximum degree D and height (= maximum absolute value of the coefficients) $\le H$ defining an affine variety $\mathbb{V} \subset \mathbb{C}^n$ of codimension k. Denote by dist the distance in $\mathbb{C}^n$ with respect to the norm $|\omega| = \max_i |\omega_i|$. In [B] W. D. Brownawell proved the following inequality of Łojasiewicz type:

For any $\omega \in \mathbb{C}^n$ we have

$$\min\{\mathrm{dist}(\omega, \mathbb{V}), 1\}^{(n+1)^2} \le C_1^D (H \max\{1, |\omega|\}^2)^{C_2} \max_i |f_i(\omega)|^{D^{-n}}$$

where $C_1 = \exp\{11(n+1)^5\}$ and $C_2 = (n+1)^2$.

This result is essentially the best possible, except perhaps for the values of the constants and for the exponent $(n+1)^2$ in the left hand side. S. Ji, J. Kollár and B. Shiffman [J-K-S] have recently proved a similar result for polynomials over a field of arbitrary characteristic without this exponent but with an ineffective dependence on the coefficients. In spite of that, we can look for other relations between the values of the f_i's and the distance to their common zeros in $\mathbb{C}^n$. For a polynomial $f \in \mathbb{Z}[x_1, \ldots, x_n]$ we denote its size (= degree + logarithmic height) by $t(f)$; for $\alpha \in \mathbb{C}^n$ we also denote by $t(\alpha)$ the minimum size of a non-zero polynomial $f \in \mathbb{Z}[x_1, \ldots, x_n]$ for which $f(\alpha) = 0$ (if there are no such polynomials we put $t(\alpha) = \infty$). In this paper we deal with the following problem:

Let ω be in the unit ball of $\mathbb{C}^n$ and suppose that

$$(1) \qquad \max_i |f_i(\omega)| < \exp\{-C\{\max_i t(f_i)\}^\tau\}$$

for some C greater than a constant $A = A(n)$ and for some $\tau \ge n+1$. Find the best value $\eta = \eta(\tau, n, k)$ for which there exist constants $e = e(n,k)$ and

1991 *Mathematics Subject Classification*: Primary 11J25; Secondary 11J82.

$B = B(n)$ such that

$$\min_{\substack{\alpha\in\mathbb{C}^n\\ f_i(\alpha)=0}} |\alpha-\omega| < \exp\{-B^{-1}C^e t(\alpha)^\eta\}. \tag{2}$$

Roughly speaking, we are looking for an upper bound for transcendence measures in terms of approximation measures (for definitions see [P2]). If $n = 1$, this problem is completely solved: we can take $\eta = \tau$. In the general case, only partial results are known. For example, using a theorem of P. Philippon, it is easy to see that we can choose $\eta = \tau - n$ (here $-n$ corresponds to D^{-n} in Brownawell's inequality), and we conjecture that this exponent can be replaced by τ. In the present paper we prove this in three special cases: if $\tau = n + 1$, if $\mathbb{V}$ is discrete, or if $n = 2$. Our first result is the following theorem:

THEOREM 1. *For any integer $n \ge 1$ there exist two constants $A, B > 0$ having the following property. Let $f_1, \ldots, f_m$ and ω be as before and assume that* (1) *holds for some $\tau \ge n + 1$ and some $C > A$. Then, if the affine variety $\mathbb{V}$ defined by the f_i's has codimension k, we can find $\alpha \in \mathbb{V}$ such that* (2) *holds with*

$$\eta = \max\left\{n+1+\frac{\tau-(n+1)}{n+1-k}, \tau-n\right\} \tag{3}$$

and

$$e = \begin{cases} 1 & \text{if } \eta = \tau - n, \\ 2^{-n+k} & \text{otherwise.}\end{cases}$$

Notice that $\eta = \tau$ if $\tau = n + 1$ or if $k = n$ (i.e. if $\mathbb{V}$ is discrete).

The case $m = 1$ is of particular interest. First of all, Theorem 1 allows us to give a positive answer to the following conjecture of G. V. Chudnovsky (see [C], Problem 1.3, p. 178):

For any integer $n \ge 1$ there exists a positive constant C such that for almost all ω in the unit ball of $\mathbb{C}^n$ (*in the sense of the Lebesgue measure in $\mathbb{R}^{2n}$*) *the inequality $\log|f(\omega)| \le -Ct(f)^{n+1}$ has only finitely many solutions $f \in \mathbb{Z}[x_1, \ldots, x_n]$.*

Indeed, it is easy to see that for any $n \in \mathbb{N}$ there exists a positive constant C such that the set of ω's in the unit ball of $\mathbb{C}^n$ for which the inequality

$$|\alpha-\omega| < \exp\{-Ct(\alpha)^{n+1}\}$$

has infinitely many solutions $\alpha \in \mathbb{C}^n$ is negligible for the Lebesgue measure (see the proof of [A], Proposition 5). Using Theorem 1, we immediately obtain Chudnovsky's conjecture.

Moreover, for $m = 1$ and $n \ge 2$, (3) can be easily improved to

$$\eta = \max\left\{n+\frac{\tau-2}{n-1}, \tau-1\right\}$$

(see Theorem 2 in §3), which implies the full conjecture $\eta = \tau$ for $n = 2$. On the other hand, in [A] we proved (in a slightly weaker form) that we can choose for η the maximum between $\tau - 2 + \tau/n$ and the positive root of $x^2 + (1-\tau)x + n - 1 - \tau = 0$. This result approaches our conjecture for $\tau \to \infty$, but, unfortunately, the proof given in [A] contains some minor errors. In the appendix we shall give a proof of the slightly weaker result

$$\eta > 0, \quad \eta^2 + (1-\tau)\eta + n - \tau = 0$$

(which also approaches our conjecture) and corrections of other mistakes which occur in [A] (1).

2. Technical results. For the proofs, we use the theory of Chow forms, as developed by Yu. V. Nesterenko (see [N1], [N2] and [N3]) and by P. Philippon (see [P1] and [P2]). We briefly summarize the notations employed by Nesterenko. Given a homogeneous unmixed ideal I of rank $n+1-r$ in the ring $\mathbb{Z}[x_0, \ldots, x_n]$ having Chow form $F = F(u^1, \ldots, u^r) \in \mathbb{Z}[u_0^1, \ldots, u_n^r]$, we denote by $H(I)$ the maximum absolute value of the coefficients of F, by $N(I)$ the degree of F with respect to $u_0^1, \ldots, u_n^1$, and by $t(I)$ the number $N(I) + \log H(I)$. Given ω' in the projective space $\mathbb{P}^n$ over $\mathbb{C}$, we define $|I|_{\omega'}$ as

$$|I|_{\omega'} = H(\kappa(F))/|\omega'|^{rN(I)},$$

where $H(\kappa(F))$ is the maximum absolute value of the coefficients of the polynomial

$$\kappa(F) \in \mathbb{C}[s_{j,k}^i]_{\substack{i=1,\ldots,r \\ 0 \le j < k \le n}}$$

obtained by replacing in F the vectors u^i by $S^i\omega'$, with S^i ($i = 1, \ldots, r$) being skew-symmetric matrices in the new variables $s_{j,k}^i$ ($0 \le j < k \le n$). For more details, see [N3] (Nesterenko uses the notation $|I(\omega')|$ instead of $|I|_{\omega'}$). Given a homogeneous polynomial $Q \in \mathbb{Z}[x_0, \ldots, x_n]$ and $\omega' \in \mathbb{P}^n$ we let

$$|Q|_{\omega'} = |Q(\omega')|/|\omega'|^{\deg Q}.$$

We start with an easy consequence of the box principle.

LEMMA 1. *Let $n \ge 1$ be an integer and let $\omega' \in \mathbb{P}^n$. Then there exist two positive constants c_1 and c_2 depending only on n such that for any real number $N > c_1$ there exists a non-zero homogeneous polynomial $Q \in \mathbb{Z}[x_0, \ldots, x_n]$ with size $\le N$ satisfying*

$$|Q|_{\omega'} \le \exp\{-c_2 N^{n+1}\}.$$

Proof. Let H and d be two positive integers and let Λ be the set of homogeneous polynomials $Q \in \mathbb{Z}[x_0, \ldots, x_n]$ of degree d with non-negative

(1) I am grateful to Yuriĭ Nesterenko who drew my attention to these mistakes.

coefficients bounded by H. This set has cardinality $(H+1)^D$, $D=\binom{d+n}{n}$, and for any $Q\in\varLambda$ we have $|Q|_{\omega'}\le DH$. Let

$$\delta=\min_{Q_1,Q_2\in\varLambda,\,Q_1\neq Q_2}|Q_1-Q_2|_{\omega'}.$$

The ball in $\mathbb{C}$ with centre at the origin and radius $DH+\delta/2$ contains the disjoint union of the open balls of centre $Q(\omega')|\omega'|^{-d}$ ($Q\in\varLambda$) and radius $\delta/2$. This gives

$$\delta\le\frac{2DH}{(H+1)^D-1}\le 2DH^{1-D}$$

and so there exist two polynomials $Q_1,Q_2\in\varLambda$, $Q_1\neq Q_2$, such that

$$|Q_1-Q_2|_{\omega'}\le 2DH^{1-D}.$$

The polynomial $Q=Q_1-Q_2$ has degree d, height (= maximum absolute value of the coefficients) $\le H$ and satisfies $|Q|_{\omega'}\le 2DH^{1-D}$. The lemma follows upon taking $d=[N/2]$ and $H=[\exp\{N/2\}]$. ∎

Given ω', α' in the complex projective space $\mathbb{P}^n$, we put

$$d(\alpha',\omega')=\frac{\max_{0\le i<j\le n}|\omega_i'\alpha_j'-\omega_j'\alpha_i'|}{\max_{0\le i\le n}|\alpha_i'|\max_{0\le i\le n}|\omega_i'|}.$$

R e m a r k. Let $\omega'=(1,\omega)$ where ω is in the unit ball of $\mathbb{C}^n$ and assume $d(\alpha',\omega')<1$. Then $\alpha_0'\neq 0$ and the vector $\alpha\in\mathbb{C}^n$ defined by $\alpha_i=\alpha_i'/\alpha_0'$ ($i=1,\ldots,n$) satisfies $|\alpha-\omega|\le\max\{1,|\alpha|\}d(\alpha',\omega')$. This gives $\max\{1,|\alpha|\}\le(1-d(\alpha',\omega'))^{-1}$ and so

$$|\alpha-\omega|\le\frac{d(\alpha',\omega')}{1-d(\alpha',\omega')}.$$

In particular, if $d(\alpha',\omega')\le 1/2$, we have $|\alpha-\omega|\le 2d(\alpha',\omega')$.

LEMMA 2. *For any integer $n\ge 1$ there exists a constant $A>0$ having the following property. Let $k\le n$ be a positive integer, let $\tau\ge k+1$, $\eta\in[n+1,\tau+n-k]$ and $\theta>1$ be real numbers and let $\omega'\in\mathbb{P}^n$. Assume that there exists a homogeneous prime ideal $\wp\subset\mathbb{Z}[x_0,\ldots,x_n]$ of rank k such that $\wp\cap\mathbb{Z}=\{0\}$ and*

$$|\wp|_{\omega'}<\exp\{-Ct(\wp)^{\tau/k}\}$$

for some $C\ge A\theta$. Then either there exists $\alpha'\in\mathbb{V}_{\mathbb{P}}(\wp)$, the projective variety defined by $\wp$, such that

$$d(\alpha',\omega')<\exp\{-A^{-1}\theta t(\alpha')^{\eta}\},$$

or there exists a homogeneous prime ideal $\wp'\subset\mathbb{Z}[x_0,\ldots,x_n]$ of rank $k+1$ such that $\wp'\cap\mathbb{Z}=\{0\}$, $\wp'\supset\wp$ and

$$|\wp'|_{\omega'}<\exp\{-A^{-1}\theta^{-1}Ct(\wp')^{(n+1-\eta+\tau)/(k+1)}\}.$$

Moreover, if $k=n$ or if $\eta\le\tau-k$, the first case occurs.

Proof. Denote by $c_3, \ldots, c_{10}$ positive constants depending only on k, n, τ and η. If $\omega' \in \mathbb{V}_{\mathbb{P}}(\wp)$ we put $\alpha' = \omega$; otherwise let $\alpha' \in \mathbb{V}_{\mathbb{P}}(\wp)$ be such that $\delta = d(\omega', \alpha') > 0$ is minimal. Using Lemma 6 of [N3], we see that

$$-\delta > Ct(\wp)^{(\tau-k)/k} - c_3. \tag{4}$$

Moreover, Corollary 3 of [N1] gives

$$t(\alpha') \le c_4 t(\wp)^{1/k}. \tag{5}$$

Hence

$$-\delta > (Cc_4^{-\tau+k} - c_3)t(\alpha')^{\tau-k} \ge A^{-1}\theta t(\alpha')^{\eta} \tag{6}$$

provided that $\eta \le \tau - k$ and A is sufficiently large. Now assume $\eta > \tau - k$ and put

$$N = \theta^{-y} t(\wp)^{-x}(-\delta)^{y}$$

where

$$x = \frac{\eta - (n+1) + \tau/k}{\eta + (n+1)k - \tau} > 0 \quad \text{and} \quad y = \frac{k+1}{\eta + (n+1)k - \tau} \ge 1/n.$$

From (4) and from $\eta \le \tau + n - k$ we obtain

$$\begin{aligned} N &\ge \theta^{-y} t(\wp)^{-x}(Ct(\wp)^{\tau/k-1} - c_1)^{y} \\ &\ge \theta^{-y}(C - c_1)^{y} t(\wp)^{(\tau+n-k-\eta)/(\eta+(n+1)k-\tau)} \ge c_1 \end{aligned}$$

provided that A is sufficiently large. Therefore, Lemma 1 gives a non-zero homogeneous polynomial $Q \in \mathbb{Z}[x_0, \ldots, x_n]$ which satisfies

$$t(Q) \le N, \tag{7}$$

$$|Q|_{\omega'} \le \exp\{-c_2 N^{n+1}\}. \tag{8}$$

We distinguish three cases:

• First case: $Q \notin \wp$ and $\mu := c_2 N^{n+1}(-\delta)^{-1} < 1$. By (8) we have $|Q|_{\omega'} \le \exp\{\mu\delta\}$. If $k < n$, Lemma 4 of [N3] gives a homogeneous ideal $I \subset \mathbb{Z}[x_0, \ldots, x_n]$ of pure rank $k+1$ whose zeros coincide with the zeros of the ideal $(\wp, Q)$ and such that

$$t(I) \le c_5 t(Q) t(\wp), \tag{9}$$

$$\log |I|_{\omega'} \le \mu \log |\wp|_{\omega'} + c_6 t(\wp) t(Q). \tag{10}$$

Taking into account (10), (7), $\eta \le \tau + n - k$ and (9), we get

$$\begin{aligned} \log |I|_{\omega'} &\le -c_2 C N^{n+1}(-\delta)^{-1} t(\wp)^{\tau/k} + c_6 t(\wp) N \\ &= -c_2\theta^{-1} C (t(\wp)N)^{(n+1-\eta+\tau)/(k+1)} + c_6 t(\wp) N \\ &\le -(c_2\theta^{-1}C - c_6)(c_5^{-1} t(I))^{(n+1-\eta+\tau)/(k+1)}. \end{aligned}$$

Proposition 2 of [N2] gives a homogeneous prime ideal $\wp' \in \mathbb{Z}$ of rank $k+1$ whose zeros are zeros of I such that $\wp' \cap \mathbb{Z} = \{0\}$ and

$$(11)\qquad \log|\wp'|_{\omega'} < -c_7(\theta^{-1}C - c_8)t(\wp')^{(n+1-\eta+\tau)/(k+1)} \le -c_9\theta^{-1}Ct(\wp')^{(n+1-\eta+\tau)/(k+1)}$$

provided that A is sufficiently large.

If $k = n$, the same Lemma 4 of [N3] gives $\mu\log|\wp|_{\omega'} + c_6 t(\wp)t(Q) \ge 0$, which cannot occur if A is sufficiently large.

• Second case: $Q \notin \wp$ and $\mu \ge 1$. Taking into account (5) we obtain

$$(12)\qquad -\delta \ge c_{10}\theta^{(n+1)y/((n+1)y-1)}t(\alpha')^{k(n+1)x/((n+1)y-1)} \ge A^{-1}\theta t(\alpha')^{\eta}$$

since

$$\frac{k(n+1)x}{(n+1)y-1} - \eta = \frac{(\eta-n-1)(\eta-\tau+k(n+1))}{\tau+(n+1)-\eta} \ge 0.$$

• Third case: $Q \in \wp$. Using (7) and (5), we obtain

$$t(\alpha') \le t(Q) \le \theta^{-y}t(\wp)^{-x}(-\delta)^y \le c_4^{kx}\theta^{-y}t(\alpha')^{-kx}(-\delta)^y$$

and

$$(13)\qquad -\delta \ge A^{-1}\theta t(\alpha')^{\eta}.$$

Our assertion comes from (6), (11), (12) and (13). ∎

By induction we deduce the following

PROPOSITION 1. *For any integer $n \ge 1$ there exists a positive constant B having the following property. Let $k \le n$ be a positive integer and let $\omega' \in \mathbb{P}^n$. Assume that there exists a homogeneous prime ideal $\wp \subset \mathbb{Z}[x_0,\ldots,x_n]$ of rank k such that $\wp \cap \mathbb{Z} = \{0\}$ and*

$$|\wp|_{\omega'} < \exp\{-Ct(\wp)^{\tau/k}\}$$

for some $C \ge B$ and some $\tau \ge n+1$. Then there exists $\alpha' \in \mathbb{V}_{\mathbb{P}}(\wp)$ such that

$$d(\alpha',\omega') < \{-B^{-1}C^e t(\alpha')^{\eta}\}$$

where

$$\eta = \max\left\{n+1+\frac{\tau-(n+1)}{n+1-k}, \tau-k\right\}$$

and

$$e = \begin{cases} 1 & \text{if } \eta = \tau - k,\\ 2^{-n+k} & \text{otherwise.}\end{cases}$$

Proof. If $\eta = \tau - k$, Lemma 2 gives our claim. Assume

$$\eta = n+1+\frac{\tau-(n+1)}{n+1-k}.$$

From $\tau \geq n+1$ we obtain $\eta \geq n+1$. We shall prove the proposition by induction on k.

• $k=n$. Lemma 2, with $\theta = A^{-1}C$, gives $\alpha' \in \mathbb{V}_{\mathbb{P}}(\wp)$ such that

$$d(\alpha', \omega') < \exp\{-A^{-2}Ct(\alpha')^{\eta}\}.$$

• $k<n$. We apply Lemma 2 with $\theta = C^{1/2}$. If there exists $\alpha' \in \wp$ such that

$$d(\alpha', \omega') < \exp\{-A^{-1}C^{1/2}t(\alpha')^{\eta}\}$$

our assertion follows. Otherwise, there exists a homogeneous prime ideal $\wp' \supset \wp$ of rank $k+1$ such that $\wp' \cap \mathbb{Z} = \{0\}$ and

$$|\wp'|_{\omega'} < \exp\{-A^{-1}C^{1/2}t(\wp')^{\tau'/(k+1)}\},$$

with $\tau' = n+1-\eta+\tau$. By inductive hypothesis, we can find $\alpha' \in \wp$ with

$$d(\alpha', \omega') < \exp\{-B^{-1}C^{2^{-n+k}}t(\alpha')^{\eta'}\}$$

where

$$\eta' = n+1+\frac{\tau'-(n+1)}{n-k} = \eta. \ \blacksquare$$

Using Theorem 2 of [P2] (with $I_{N,1} = \ldots = I_{N,k+1} = (Q_N)$ and the polynomial Q_N of size $\leq N$ given by Lemma 1 as in the proof of Lemma 2) we find a result similar to the previous one but with a worse exponent:

For any integer n there exist constants $A, B > 0$ having the following property. Let $k \leq n$ be an integer, $\tau \geq n+1$ a real number and let $\omega' \in \mathbb{P}^n$. Assume that there exists a homogeneous prime ideal $\wp \subset \mathbb{Z}[x_0, \ldots, x_n]$ of rank k such that $\wp \cap \mathbb{Z} = \{0\}$ and

$$|\wp|_{\omega'} < \exp\{-At(\wp)^{\tau/k}\}.$$

Then we can find $(1, \alpha) \in \mathbb{C}^n$ such that

$$d(\alpha', \omega') < \exp\{-B^{-1}t(\alpha')^{\eta}\}$$

where

$$\eta = n+1+k\frac{\tau-(n+1)}{(n+1-k)\tau}.$$

3. Proof of the main results. We have a relation between the value of a homogeneous prime ideal $\wp$ at $\omega' \in \mathbb{P}^n$ and its projective distance from the variety defined by $\wp$. Our next task is to put it in terms of polynomials.

LEMMA 3. *Let $P_1, \ldots, P_m \in \mathbb{Z}[x_0, \ldots, x_n]$ be non-zero homogeneous polynomials of size $\leq T$ and let $\omega' \in \mathbb{P}^n$. Let $\varepsilon = \max_i |P_i|_{\omega'}$ and assume $\varepsilon < \exp\{-AT^{n+1}\}$ where $A > 0$ depends only on n. Then there exists* *an unmixed homogeneous ideal* $J\subset\mathbb{Z}[x_0,\dots,x_n]$ *of rank* $k\le n$ *such that* $\sqrt{J\mathbb{Q}[x_0,\dots,x_n]}\cap\mathbb{Z}[x_0,\dots,x_n]\supset I=(P_1,\dots,P_m)$ (2) *and*

$$t(J)\le B_1T^k,\qquad |J|_{\omega'}\le\varepsilon^{B_2^{-1}}$$

where A, B_1 *and* B_2 *are positive constants depending only on* n.

Proof. Denote by $c_{h,11},\dots,c_{h,16}$ $(h=1,\dots,n+1)$ positive constants depending only on n. We will show by induction that for $h=1,\dots,n+1$ there exist unmixed homogeneous ideals $J_h\subset\mathbb{Z}[x_0,\dots,x_n]$ of rank h such that $J_h\cap\mathbb{Z}=\{0\}$ (for $h\le n$) and

$$t(J_h)\le c_{h,11}^hT^h,\qquad |J_h|_{\omega'}\le\varepsilon^{c_{h,12}}. \tag{14_h}$$

Since the last inequalities fail for $h=n+1$, our assertion will be proved.

• $h=1$. We take $J_1=(P_1)$ and we apply Proposition 1 of [N3].

• $h\Rightarrow h+1$. Assume (14_h) satisfied for some $h\le n$ and for some ideal J_h. We denote by $J_{h,1}$ the intersection of the primary components of J_h whose radical contains I and by $J_{h,2}$ the intersection of the other components. Using [N2], Proposition 2, and Gelfond's inequality [G], Lemma II, p. 135, it is easy to see that

$$\begin{gathered} t(J_{h,1})\le c_{h,13}T^h,\qquad t(J_{h,2})\le c_{h,13}T^h,\\ |J_{h,1}|_{\omega'}|J_{h,2}|_{\omega'}<\varepsilon^{c_{h,12}}\exp\{c_{h,14}T^h\}\le\varepsilon^{c_{h,12}-c_{h,14}/A}. \end{gathered}\tag{15}$$

Since we are assuming that our claim is wrong, we must have $|J_{h,1}|_{\omega'}\ge\varepsilon^{B_2^{-1}}$; therefore

$$|J_{h,2}|_{\omega'}<\varepsilon^{c_{h,12}-c_{h,14}/A-1/B_2}. \tag{16}$$

A classical trick (see for instance [P1], Lemma 1.9) allows us to find homogeneous polynomials $a_1,\dots,a_m\in\mathbb{Z}[x_0,\dots,x_n]$ with $\deg a_j=\max(\deg P_i)-\deg P_j$ $(j=1,\dots,m)$ such that $P=a_1P_1+\dots+a_mP_m$ is not a zero-divisor on $\mathbb{Z}[x_0,\dots,x_n]/J_{h,2}$. Moreover, we can choose the a_i's in such a way that their heights are bounded by the number of irreducible components of $J_{h,2}$ and so, a fortiori, by $c_{h,13}T^h$. From this, we obtain

$$t(P)\le c_{h,15}T,\qquad |Q|_{\omega'}\le\varepsilon^{c_{h,16}}.$$

Using (15), (16) and the last inequalities, Proposition 3 of [N2] gives an unmixed ideal $J_{h+1}\subset\mathbb{Z}[x_0,\dots,x_n]$ of rank $h+1$ such that inequalities (14_{h+1}) hold. ∎

Using Proposition 2 of [N2], we easily deduce

PROPOSITION 2. *For any integer* $n\ge1$ *there exist two constants* $A,B>0$ *having the following property. Let* $\tau\ge n+1$ *be a real number and let* $\omega'\in\mathbb{P}^n$.

(2) $\operatorname{rank}(J)$ may be greater than $\operatorname{rank}(I)$.

Assume that there exist non-zero homogeneous polynomials $P_1, \ldots, P_m \in \mathbb{Z}[x_0, \ldots, x_n]$ *of size* $\leq T$ *such that* $\max_i |P_i|_{\omega'} < \exp\{-CT^\tau\}$ *for some* $C \geq A$. *Then there exists a homogeneous prime ideal* $\wp \subset \mathbb{Z}[x_0, \ldots, x_n]$ *of rank* $k \leq n$ *such that* $\wp \cap \mathbb{Z} = \{0\}$, $\wp \supset (P_1, \ldots, P_m)$ *and*

$$|\wp|_{\omega'} < \exp\{-B^{-1}Ct(\wp)^{\tau/k}\}.$$

Proof of Theorem 1. Let $f_1, \ldots, f_m$ be as in Theorem 1, let $P_i = {}^h f_i$ be the homogenization of f_i $(i = 1, \ldots, m)$ and let $\omega' = (1, \omega)$. Applying Proposition 1 to the homogeneous prime ideal $\wp$ given by Proposition 2 (which has rank $\geq k$ since $x_0 \notin \wp$) and using the remark before Lemma 2, we obtain our claim. ■

To improve the previous theorem when $m = 1$, we need the following lemma of Chudnovsky (see [C], Lemma 1.1, p. 424).

LEMMA 4. *Let* $f \in \mathbb{C}[x_1, \ldots, x_n]$ *of degree* $\leq d$ *and let* $\omega \in \mathbb{C}^n$. *Then for any* $\lambda \in \mathbb{N}^n$ *there exists a zero* $\alpha \in \mathbb{C}^n$ *of* f *such that*

$$\frac{1}{|\lambda|!}\left|\frac{\partial^\lambda f(\omega)}{\partial x^\lambda}\right||\alpha - \omega|^{|\lambda|} \leq 2^d |f(\omega)|$$

(*here* $|\lambda| = \lambda_1 + \ldots + \lambda_n$).

THEOREM 2. *For any integer* $n \geq 2$ *there exists a constant* $B > 0$ *having the following property. Let* $f \in \mathbb{Z}[x_1, \ldots, x_n]$ *of size* $\leq T$ *and let* ω *be in the unit ball of* $\mathbb{C}^n$ *such that*

$$|f(\omega)| < \exp\{-CT^\tau\}$$

for some $C \geq B$ *and some* $\tau \geq n + 1$. *Then there exists* $\alpha \in \mathbb{C}^n$ *on the hypersurface* $\{f = 0\}$ *such that*

$$|\alpha - \omega| < \exp\{-B^{-1}C^e t(\alpha)^\eta\}, \tag{17}$$

where

$$\eta = \max\left\{n + \frac{\tau - 2}{n - 1}, \tau - 1\right\}$$

and

$$e = \begin{cases} 1 & \text{if } \eta = \tau - 1, \\ 2^{-n+2} & \text{otherwise.} \end{cases}$$

Proof. We can assume f irreducible and $D_{x_1} f = \partial f / \partial x_1 \not\equiv 0$. Inequality (17) with $\eta = \tau - 1$ and $e = 1$ is easily proved applying Proposition 1 to the principal prime ideal $\wp = (f)$. Moreover, if

$$|D_{x_1} f(\omega)| \geq \exp\left\{-\frac{C}{2}t(f)^\tau\right\},$$

Lemma 4 gives $\alpha \in \mathbb{C}^n$ such that $f(\alpha) = 0$ and

$$\log|\alpha - \omega| < -\frac{C}{4} t(f)^{\tau}.$$

In this case, (17) is proved with $\eta = \tau$ and $e = 1$. Otherwise, using Proposition 2 with $P_1 = {}^h f$ and $P_2 = {}^h D_{x_1} f$, we can find a homogeneous prime ideal $\wp \subset \mathbb{Z}[x_1, \ldots, x_n]$ of rank ≥ 2 (actually $= 2$), containing the ideal $({}^h f, {}^h D_{x_1} f)$, such that $|\wp|_{\omega'} < \exp\{-c_{17} C t(\wp)^{\tau/2}\}$. Proposition 1 and the remark before Lemma 2 give (17) with

$$\eta = n + 1 + \frac{\tau - (n+1)}{n-1} = n + \frac{\tau - 2}{n-1}$$

and $e = 2^{-n+2}$. ∎

Appendix: Corrections to "Polynomials with high multiplicity" (Acta Arith. 56 (1990), 345–364). In this section we refer to lemmas, propositions, theorems, numbers of equations and lines of the paper [A] using italic type.

The inequalities (*5*) on *p. 354* are not true. More precisely, define for $k = 1, \ldots, k_0$ and $j = 1, \ldots, s_k$,

$$\Lambda_{jk} = \mathbb{V}_{\mathbb{P}}(\wp_{j,h}) \Big\backslash \bigcup_{h=1}^{k-1} \bigcup_{j=1}^{s_h} \mathbb{V}_{\mathbb{P}}(\wp_{j,h}),$$

where the symbols have the same meaning as in [A]. *Lemma 4* on *p. 354* gives

$$i_\omega(J_k) \geq \prod_{h=0}^{k-1} (t_k M - t_h M) \qquad \text{for any } \omega \in \Lambda_{jk}.$$

If Λ_{jk} is not empty, it is a non-empty Zariski open set in $\mathbb{V}_{\mathbb{P}}(\wp_{j,h})$, and so $e_{jk} \geq \prod_{h=0}^{k-1} (t_k M - t_h M)$ as claimed on *p. 355*, *l. 9*. So, inequalities (*5*) hold if $\Lambda_{jk} \neq \emptyset$. On the other hand, from (*4*) and the definition of these sets, it is easy to see that

$$\mathbb{V}_M \subset \bigcup_{k=1}^{k_0} \bigcup_{\substack{j=1,\ldots,s_k \\ \Lambda_{jk} \neq \emptyset}} \mathbb{V}_{\mathbb{P}}(\wp_{j,k}). \tag{18}$$

Now, the same arguments used on *p. 354*, *l. 8–11* give a polynomial

$$g_k \in \bigcap_{\substack{j=1,\ldots,s_k \\ \Lambda_{jk} \neq \emptyset}} \wp_{j,k}$$

of size $\leq c_6 T/M$. As in *l. 12* we put $g = \prod_{k=1}^{k_0} g_k$. Then (18) ensures that g is zero over $\mathbb{V}_M$ and we have $t(g) \leq c_7 T/M$.

Unfortunately, a problem now arises in the inequality in *l. −8/−7, p. 362* in the proof of *Theorem 2*, since (*5*) is available only if $\Lambda_{jk} \neq \emptyset$. This additional complication does not occur if $n = 2$ ($s_1 = 0$ since f is irreducible), so our result

$$\tau \le \eta + \max\left(0, \frac{4-\eta}{3}\right), \qquad n = 2,$$

is still true (but it is now sharpened by Theorem 2). In the general case, however, we can easily deduce from Proposition 2 and from *Theorem 1* a weak form of *Theorem 2*:

$$\tau \le \eta + \frac{n}{\eta + 1}.$$

A more precise formulation of this result is the following theorem, announced in the introduction:

THEOREM 3. *For any integer $n \ge 1$ there exist constants $A, B > 0$ having the following property. Let $f \in \mathbb{Z}[x_1, \ldots, x_n]$ and let ω be in the unit ball of $\mathbb{C}^n$. Suppose that $|f(\omega)| < \exp\{-CT^\tau\}$ for $C > A$ and $\tau \ge n+1$. Then we can find $\alpha \in \mathbb{C}^n$ on the hypersurface $\{f = 0\}$ such that*

$$|\alpha - \omega| < \exp\{-B^{-1}Ct(\alpha)^\eta\}$$

where η is the positive root of $\eta^2 + (1-\tau)\eta + n - \tau = 0$.

Proof. We define $M \ge 1$ as the first integer for which there exists $\lambda \in \mathbb{N}^n$ with $|\lambda| = M$ such that

$$\frac{1}{M!}\left|\frac{\partial^\lambda f(\omega)}{\partial x^\lambda}\right| > -\frac{C}{2}t(f)^\tau.$$

Let

$$u = \frac{\log M}{\log t(f)} \in [0,1].$$

Lemma 4 gives $\alpha \in \mathbb{C}^n$ with $f(\alpha) = 0$ and

$$|\alpha - \omega| < \left\{-\frac{C}{4}t(f)^{\tau-u}\right\}. \tag{19}$$

On the other hand, Proposition 2 with

$$\{P_1, \ldots, P_m\} = \left\{\frac{1}{\lambda!}\frac{\partial^\lambda f}{\partial x^\lambda},\ |\lambda| \le M-1\right\}$$

and Lemma 6 of [N3] give a point α of multiplicity $\ge M$ on the hypersurface $\{f = 0\}$ such that

$$|\alpha - \omega| < \exp\{-c_{18}Ct(f)^{\tau-n}\}.$$

By *Theorem 1*, $t(\alpha) \le c_{19}t(f)/M$, hence

$$|\alpha - \omega| < \exp\{-c_{20}Ct(\alpha)^{(\tau-n)/(1-u)}\}.$$

Combining the last inequality with inequality (19), we find $\omega \in \mathbb{C}^n$ on the hypersurface $\{f = 0\}$ which satisfies

$$|\alpha - \omega| < \exp\{-c_{21} C t(\alpha)^{\min\{(\tau-n)/(1-u), \tau-u\}}\}.$$

Since

$$\min_{0 \le u \le 1} \min\left\{\frac{\tau - n}{1 - u}, \tau - u\right\} = \eta,$$

our assertion follows. ∎

References

[A] F. Amoroso, *Polynomials with high multiplicity*, Acta Arith. 56 (1990), 345–364.

[B] W. D. Brownawell, *Local diophantine Nullstellen inequalities*, J. Amer. Math. Soc. 1 (1988), 311–322.

[C] G. V. Chudnovsky, *Contribution to the theory of transcendental numbers*, Math. Surveys Monographs 19, Amer. Math. Soc., 1984.

[G] A. O. Gelfond, *Transcendental and Algebraic Numbers*, Dover, New York, 1960.

[J-K-S] S. Ji, J. Kollár and B. Shiffman, *A global Łojasiewicz inequality for algebraic varieties*, Trans. Amer. Math. Soc. 329 (1992), 813–818.

[N1] Yu. V. Nesterenko, *Estimates for the characteristic function of a prime ideal*, Mat. Sb. 123 (1984), 11–34 = Math. USSR-Sb. 51 (1985), 9–32.

[N2] —, *On algebraic independence of algebraic powers of algebraic numbers*, Mat. Sb. 123 (1984), 435–459 = Math. USSR-Sb. 51 (1985), 429–453.

[N3] —, *On a measure of the algebraic independence of the values of certain functions*, Math. USSR-Sb. 56 (1987), 545–567.

[P1] P. Philippon, *Critères pour l'indépendance algébrique*, Inst. Hautes Études Sci. Publ. Math. 64 (1986), 5–52.

[P2] —, *Sur les mesures d'indépendance algébrique*, in: Séminaire de Théorie des Nombres, Paris 1983-84, Birkhäuser, 1985, 219–233.

[W] M. Waldschmidt, *Nombres transcendants*, Springer, Berlin, 1974.

DIPARTIMENTO DI MATEMATICA
VIA F. BUONARROTI 2
56127 PISA, ITALY

Received on 7.9.1992
and in revised form on 28.4.1994 (2301)