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Mean square value of exponential sums related to representation of integers as sum of two squares

by

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1. Introduction. The problem we address here arises in the study of the error function in the shifted circle problem (see [BCDL]). Let $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ be a fixed point in a plane. Define

$$N(R;\alpha) = \#\{m \in \mathbb{Z}^2 : |m-\alpha| \le R\}$$

and

$$F(R;\alpha) = \frac{N(R;\alpha) - \pi R^2}{R^{1/2}}.$$

A long-standing famous conjecture of Hardy (see [H]) is to prove that when $R \to \infty$,

$$F(R;\alpha) = O(R^{\varepsilon}), \quad \forall \varepsilon > 0$$

(Hardy considered $\alpha = 0$). In [BCDL] and [B] it was proved that the mean square limit

$$D(\alpha) = \lim_{T\to\infty} \frac{1}{T} \int_1^T |F(R;\alpha)|^2 \, dR$$

exists and is equal to

$$D(\alpha) = (2\pi^2)^{-1} \sum_{n=1}^{\infty} n^{-3/2} |r_\alpha(n)|^2,$$

where

$$r_\alpha(n) = \sum_{k^2+l^2=n} e(\alpha_1 k + \alpha_2 l), \qquad e(t) = \exp(2\pi i t)$$

(for $\alpha = 0$ this reduces to a classical result of Cramér [C]). The existence of a limit distribution $p_\alpha(t)dt$ of $F(R;\alpha)$ was shown in [BCDL] as well:

$$\lim_{t\to\infty} \frac{1}{T} \int_{\{R:1\le R\le T, a\le F(R;\alpha)\le b\}} dR = \int_a^b p_\alpha(t)\, dt$$

for every $a < b$ (for $\alpha = 0$ this result is due to Heath-Brown, see [H-B]). The density $p_\alpha(t)$ was proved to be an analytic function in t which decays at infinity, roughly speaking, as $C\exp(-\lambda t^4)$.

In [BCDL] one of the key points in the proof was to evaluate the asymptotics of the series

$$S_\alpha(b) = \sum_{n=1}^{\infty} |r_\alpha(n)|^2 \exp(-n/b) \tag{1.1}$$

when $b \to \infty$. This gives a mean square value of $|r_\alpha(n)|$ as $n \to \infty$. In the present work we show that $S_\alpha(b)$ has an unexpected wild behavior. Namely, $S_\alpha(b)$, as a function of α, has a "bumpy" shape when $b \to \infty$, with a big bump at every rational point α. This behavior of $S_\alpha(b)$ is closely related to the fact, discovered in [BD], that the mean square limit $D(\alpha)$ has a sharp local maximum at every rational point. We prove here the following theorems:

THEOREM 1.1. *For any fixed* α,

$$\liminf_{b\to\infty}(b^{-1}S_\alpha(b)) \geq \pi. \tag{1.2}$$

THEOREM 1.2. *Except for an exceptional set of* α *of measure zero in* $\mathbb{R}^2$,

$$S_\alpha(b) = \pi b + O(b^{3/4+\varepsilon}) \quad \text{as } b \to \infty. \tag{1.3}$$

Remark. We prove in Theorem 1.5 that all rational α and all α sufficiently rapidly approximable by rationals belong to the exceptional set. Theorem 1.3 implies the weaker statement that

$$\lim_{b\to\infty}(b^{-1}S_\alpha(b)) = \pi \tag{1.4}$$

for almost all α. The power $3/4$ in Theorem 1.2 is best possible by virtue of (3.13) below.

THEOREM 1.3. *Assume that* $\alpha = (\alpha_1, \alpha_2)$ *is Diophantine*, *i.e.*,

$$|\alpha_1 k + \alpha_2 l - n| > C(k^2 + l^2)^{-D} \tag{1.5}$$

with some $C, D > 0$ *for all integers* k, l, n *with* $k^2 + l^2 \neq 0$. *Then* (1.4) *holds.*

THEOREM 1.4. *Assume that*

$$\alpha_1 p + \alpha_2 q - r = 0 \tag{1.6}$$

for some integer p, q, r *with coprime* $p, q \neq 0$. *Assume also that* α_1 *is Diophantine*, *i.e.*,

$$|k\alpha_1 - l| > C(k^2)^{-D}, \tag{1.7}$$

with some $C, D > 0$ *for all integers* k, l *with* $k \neq 0$. *Then*

$$\lim_{b\to\infty}(b^{-1}S_\alpha(b)) = \pi(1 + \varepsilon(pq)(p^2 + q^2)^{-1}), \tag{1.8}$$

where

$$\varepsilon(n) = \begin{cases} 1 & \text{if } n \text{ is even}, \\ 2 & \text{if } n \text{ is odd}. \end{cases}$$

Remark. Our proof shows that (1.6) without the assumption that either α_1 or α_2 is Diophantine implies

$$\liminf_{b\to\infty}(b^{-1}S_\alpha(b)) \ge \pi(1+\varepsilon(pq)(p^2+q^2)^{-1}). \tag{1.9}$$

THEOREM 1.5. *Suppose the vector α is rational, i.e., there exists an integer Q such that*

$$2Q\alpha_1 = n_1, \quad 2Q\alpha_2 = n_2 \tag{1.10}$$

are integers and $\gcd(Q, n_1, n_2) = 1$. *Then*

$$(b\log b)^{-1}S_\alpha(b) = C(Qr(Q))^{-1} + O(\log^{-1} b) \quad \text{as } b\to\infty, \tag{1.11}$$

where

$$r(Q) = \prod_{p\mid Q}(1+p^{-1}), \tag{1.12}$$

with the product taken over primes p dividing Q, and

$$\begin{aligned} &C = 3\ (Q \text{ even}), \quad C = 4\ (Q \text{ odd},\ n_1+n_2 \text{ even}),\\ &C = 2\ (Q \text{ odd},\ n_1+n_2 \text{ odd}). \end{aligned} \tag{1.13}$$

Remark. We have $Q < Qr(Q) \le \sigma(Q)$, where $\sigma(Q)$ is the sum of the divisors of Q. According to Theorem 323 in [HW],

$$\sigma(n) = O(n\log\log n).$$

COROLLARY 1. *For fixed rational α, the mean-square value of $r_\alpha(n)$ for n of the order of magnitude N is at least* $2(\sigma(Q))^{-1}\log(N/Q)$.

COROLLARY 2. *If α is an almost-rational vector, i.e., if an infinite sequence $\{Q_1, Q_2, \ldots\}$ of integers exists such that*

$$2Q_j\alpha = P_j + \varepsilon_j, \tag{1.14}$$

with P_j integral vectors and

$$(\sigma(Q_j))^{-1}\log(|\varepsilon_j|^{-1}) \to \infty \quad \text{as } j\to\infty, \tag{1.15}$$

then

$$\limsup_{b\to\infty}(b^{-1}S_\alpha(b)) = \infty. \tag{1.16}$$

The Tauberian theorem of Hardy and Littlewood (see [HL]) enables us to derive from Theorems 1.2–1.5 the asymptotics of

$$\sigma_\alpha(n) = n^{-1}\sum_{k=1}^{n}|r_\alpha(k)|^2$$

as $n\to\infty$. The theorem of Hardy and Littlewood is the following:

Theorem HL. *If $f(x) = \sum a_n x^n$ is a power series with positive coefficients, and*

$$f(x) \sim A(1-x)^{-1}|\log(1-x)|^{\alpha} \quad (x \to 1),$$

where $A > 0$ and $\alpha \geq 0$, then

$$a_1 + \ldots + a_n \sim An\log^{\alpha} n.$$

Define $a_n = |r_\alpha(n)|^2$, $1 - x = \exp(-b)$. Then we see from Theorems 1.2–1.5 and HL that for all Diophantine α,

$$\lim_{n\to\infty} \sigma_\alpha(n) = \pi; \tag{1.17}$$

for all α satisfying (1.6), (1.7),

$$\lim_{n\to\infty} \sigma_\alpha(n) = \pi(1 + \varepsilon(pq)(p^2+q^2)^{-1}); \tag{1.18}$$

and finally for all rational α,

$$\lim_{n\to\infty} (\log n)^{-1}\sigma_\alpha(n) = C(Qr(Q))^{-1}, \tag{1.19}$$

with $r(Q)$ and C defined in (1.12) and (1.13), respectively.

Theorems 1.1 and 1.5 were proved in [BCDL]. Here we prove Theorems 1.2–1.4 and Corollary 2 of Theorem 1.5.

2. Preliminaries from [BCDL]. Here we recall some results from [BCDL]. The sum (1.1) may be written

$$S_\alpha(b) = \sum_{m,m'} e(\alpha(m-m'))\exp(-m^2/b), \tag{2.1}$$

summed over integer vectors $m, m' \in \mathbb{Z}^2 \setminus \{0\}$ with $m^2 = m'^2$. As was shown in [BCDL], the sum (2.1) can be converted into an unrestricted sum,

$$S_\alpha(b) = \frac{1}{2}\sum_{k,l,j,h} e(h(l\alpha_1 - k\alpha_2))\exp(-(k^2+l^2)(j^2+h^2)/(4b)), \tag{2.2}$$

summed over all $(j,k,l,h) \in \mathbb{Z}^4$ satisfying

$$h^2 + j^2 \neq 0, \quad k^2 + l^2 \neq 0, \tag{2.3}$$

$$\text{either } j \equiv h \equiv 0, \text{ or } j \equiv h \equiv k \equiv l \equiv 1 \pmod 2, \tag{2.4}$$

and

$$k, l \text{ are relatively prime}, \tag{2.5}$$

which means that either $|k| + |l| = 1$, or $\gcd(|k|,|l|) = 1$.

According to the two possibilities in (2.4) we divide $S_\alpha(b)$ into even and odd parts,

$$S_\alpha(b) = S_{\rm e} + S_{\rm o}, \tag{2.6}$$

where the terms with j and h even are

$$S_{\mathrm{e}} = \frac{1}{2}\sum_{k,l}[F(w)F(0) - 1], \tag{2.7}$$

summed over integers (k,l) satisfying (2.5), and

$$S_{\mathrm{o}} = \frac{1}{2}\sum_{k,l} G(w)G(0), \tag{2.8}$$

summed over odd integers k and l satisfying (2.5). The functions (F,G) are defined by

$$\sum_{x} \exp(-x^2/a)e(xt) = F(t) \text{ or } G(t), \tag{2.9}$$

where the sum is over integer x for F and over half-odd-integer x for G. In (2.7)–(2.9) we have used the abbreviations $w = 2(l\alpha_1 - k\alpha_2)$, $a = b(k^2 + l^2)^{-1}$. By the Poisson summation formula, (2.9) gives

$$F(t) = (\pi a)^{1/2}\sum_{p} \exp(-\pi^2 a(p+t)^2), \tag{2.10}$$

$$G(t) = (\pi a)^{1/2}\sum_{p} (-1)^p \exp(-\pi^2 a(p+t)^2). \tag{2.11}$$

According to (2.9), the functions F and G are periodic with periods 1 and 2 respectively,

$$F(w+1) = F(w), \qquad G(w+1) = -G(w). \tag{2.12}$$

For $a \le 1$, (2.9) gives

$$F(w) = 1 + O(\exp(-a^{-1})), \qquad G(w) = O(\exp(-(4a)^{-1})). \tag{2.13}$$

For $a \ge 1$, (2.10) gives

$$F(w) = (\pi a)^{1/2}[\exp(-\pi^2 a\widehat{w}^2) + O(\exp(-(\pi/2)^2 a))], \tag{2.14}$$

$$G(w) = (-1)^w(\pi a)^{1/2}[\exp(-\pi^2 a\widehat{w}^2) + O(\exp(-(\pi/2)^2 a))], \tag{2.15}$$

where $\widehat{w}$ is the distance of w from the nearest integer.

3. Proof of Theorem 1.2. For Theorem 1.2 we divide the sum (2.1) into two parts

$$S_\alpha(b) = I(b) + R_\alpha(b), \tag{3.1}$$

where $I(b)$ consists of the terms with

$$m = m', \tag{3.2}$$

which are equal to the terms in (2.2) with $h = 0$. By (2.9) and (2.10),

$$I(b) = \Big(\sum_x \exp(-x^2/b)\Big)^2 = \pi b + O(b\exp(-\pi^2 b)). \tag{3.3}$$

By (3.1) and (3.3), Theorem 1.2 states that

$$R_\alpha(b) = O(b^{3/4+\varepsilon}) \tag{3.4}$$

except for a set of α of measure zero.

Consider the integral

$$J(b) = \int |R_\alpha(b)|^2\, d\alpha, \tag{3.5}$$

integrated over the square

$$0 < \alpha_1 < 1, \quad 0 < \alpha_2 < 1. \tag{3.6}$$

We represent $R_\alpha(b)$ by the sum (2.2) with the condition ($h \neq 0$) replacing (2.3). It is convenient to restrict the sum to positive h and drop the factor 1/2. When (2.2) is inserted into (3.5), the result is an eight-fold sum over the integers $(k, l, j, h, k', l', j', h')$. The integration over (3.6) eliminates all terms except those with

$$hl = h'l', \quad hk = h'k'. \tag{3.7}$$

Since h and h' are positive and the fractions k/l and k'/l' are reduced to their lowest terms by (2.5), (3.7) implies

$$h = h', \quad k = k', \quad l = l'. \tag{3.8}$$

The eight-fold sum collapses to a five-fold sum

$$J(b) = \sum_{k,l,h,j,j'} \exp[-(k^2 + l^2)(2h^2 + j^2 + j'^2)/(4b)], \tag{3.9}$$

with summations restricted only by

$$(k, l) = 1, \quad h > 0, \quad \text{either } (j, j', h) \text{ all even or } (j, j', h, k, l) \text{ all odd.} \tag{3.10}$$

When b is large, each of the sums over j and j' gives

$$[\pi b/(k^2 + l^2)]^{1/2} + O(1), \tag{3.11}$$

and the sum over h gives the same result multiplied by $2^{-3/2}$. Therefore (3.9) becomes

$$J(b) = 2^{-3/2} \sum_{k,l} (c_k + c_l)[\pi b/(k^2 + l^2)]^{3/2} + O(b), \tag{3.12}$$

where $c_k = 0$ for k even and $c_k = 1$ for k odd. The sum over (k, l) is convergent, so that

$$J(b) = Bb^{3/2} + O(b), \tag{3.13}$$

where B is a calculable constant, namely

$$B = (2\pi)^{3/2}((3+\sqrt{2})/7)\zeta(3/2)L(3/2)/\zeta(3), \tag{3.14}$$

where ζ and L are the Riemann and Dirichlet functions,

$$\zeta(s) = \sum_n n^{-s}, \qquad L(s) = \sum_n (-1)^{n-1}(2n-1)^{-s}. \tag{3.15}$$

We need to prove from (3.5) and (3.13) that (3.4) holds except for a set of α of measure zero. But (3.4) does not follow from (3.13) alone. We need in addition the fact that $R_\alpha(b)$ is a smoothly-varying function of b, so that it cannot become large at isolated peaks without violating (3.13). To prove (3.4) we require bounds on all the derivatives of $R_\alpha(b)$. It is convenient to use the notations

$$D = b^{-2}(d/db), \tag{3.16}$$

$$J_p(b) = \int |D^p R_\alpha(b)|^2\, d\alpha, \tag{3.17}$$

integrated over (3.6). The same analysis that led to (3.9) now gives

$$J_p(b) = 4^{-2p} \sum_{k,l,h,j,j'} \exp[-(k^2+l^2)(2h^2+j^2+j'^2)/(4b)] \times (h^2+j^2)^p(h^2+j'^2)^p(k^2+l^2)^{2p}. \tag{3.18}$$

The sums over (h,j,j') give

$$A_p(b/(k^2+l^2))^{2p+3/2}, \tag{3.19}$$

plus terms of lower order in b, with a numerical constant A_p. Inserting (3.19) into (3.18) gives

$$J_p(b) = A_p b^{2p+3/2} \sum_{k,l}(k^2+l^2)^{-3/2} = B_p b^{2p+3/2} + O(b^{2p+1}). \tag{3.20}$$

Thus $D^p R_\alpha(b)$ has the root-mean-square order of magnitude

$$b^{p+3/4}. \tag{3.21}$$

We have to prove that this same order of magnitude estimate holds pointwise, for almost all α, as $b \to \infty$ for fixed α.

We use an induction on p, working downward from $p+1$ to p. Our inductive hypothesis says that

$$|D^p R_\alpha(b)| < A b^{p+3/4+f(p)}, \tag{3.22}$$

with some positive $f(p)$ depending only on p, with A depending on p and α but not on b, except for a set of α of measure zero. We assume that (3.22) holds for $p+1$ and find for which $f(p)$ it will hold for p. Let $(b_1, b_2, \ldots)$ be a sequence of numbers tending to infinity, for example

$$b_j = j^m, \tag{3.23}$$

with an exponent m to be chosen later, such that

$$|b_{j+1} - b_j| < Ab_j^{1-1/m}. \tag{3.24}$$

The inductive hypothesis together with (3.24) implies that for every b in the range

$$b_j \le b < b_{j+1}, \tag{3.25}$$

we have

$$|D^p R_\alpha(b) - D^p R_\alpha(b_j)| < Ab^{p+3/4+f(p+1)-1/m}. \tag{3.26}$$

Comparing (3.26) with (3.22), we see that if

$$f(p+1) < f(p) + 1/m, \tag{3.27}$$

then (3.22) holds for all b if and only if

$$|D^p R_\alpha(b_j)| < Ab_j^{p+3/4+f(p)} \tag{3.28}$$

holds for all j and some A depending on α, with the usual exception of a set of α of measure zero. Therefore, to complete the induction it is only necessary to prove (3.28).

Let $m_{jp}(A)$ be the measure of the set of α for which (3.28) is false for a particular j. Comparing (3.28) with (3.17) and (3.20), we see that

$$m_{jp}(A) < B_p A^{-2} b_j^{-2f(p)} (1 + O(b_j^{-1/2})). \tag{3.29}$$

Therefore

$$\sum_j m_{jp}(A) < C_p A^{-2}, \tag{3.30}$$

where C_p is the sum of the coefficients on the right of (3.29). The series (3.30) converges and the sum is finite by (3.23) if

$$1/m < 2f(p). \tag{3.31}$$

The left side of (3.30) is an upper bound to the measure of the set of α for which (3.28) is false for a given A and at least one j. The set of α for which (3.28) is false for every A and some j has measure less than (3.30) for every A, i.e. has measure zero. So we have proved that (3.28) holds for almost all α if (3.31) holds. We proved before that (3.22) follows from (3.28) if (3.27) holds. Thus the induction of the hypothesis (3.22) from $p+1$ to p succeeds, provided that we can satisfy both (3.27) and (3.31) with the same m. This will be possible if and only if

$$f(p+1) < 3f(p). \tag{3.32}$$

To start the induction we use the estimate

$$|D^p R_\alpha(b)| < \sum_n n^p |r_0(n)|^2 \exp(-n/b) = O(b^{p+1+\varepsilon}), \tag{3.33}$$

which follows from

$$|r_\alpha(n)| \le r_0(n) = O(n^\varepsilon). \tag{3.34}$$

Choose any integer P. The inductive hypothesis (3.22) holds for $p = P$ by (3.33) if

$$f(P) > 1/4. \tag{3.35}$$

The induction requires only that (3.32) hold for $p < P$, which is true if we take

$$f(p) = K^{p-P}, \tag{3.36}$$

with any constant $K < 3$. So the induction is complete and proves (3.22) with $f(p)$ given by (3.36), for any value of P. But the choice of P is arbitrary. We can let $P \to \infty$ in (3.36) and deduce that (3.22) holds for any p provided that

$$f(p) > 0. \tag{3.37}$$

In particular, when $p = 0$, (3.22) with (3.37) implies (3.4), and Theorem 1.2 is proved.

4. Proof of Theorems 1.3 and 1.4

Proof of Theorem 1.3. By (2.6), $S_\alpha(b) = S_{\mathrm{e}} + S_{\mathrm{o}}$. Following [BCDL] we divide S_{e} into two parts, $S_{\mathrm{e}1} + S_{\mathrm{e}2}$, with $\widehat{w} > \delta$ and with $\widehat{w} \le \delta$, respectively, where $\delta > 0$ is an arbitrary small number. Similar division is defined for S_{o}. (B.75), (B.77) in [BCDL] prove

$$\lim_{\delta\to 0} \limsup_{b\to\infty} b^{-1}|S_1 - \pi| = 0, \tag{4.1}$$

where $S_1 = S_{\mathrm{e}1} + S_{\mathrm{o}1}$. Therefore Theorem 1.3 will be proved if we prove for $S_2 = S_{\mathrm{e}2} + S_{\mathrm{o}2}$ the following result:

LEMMA 4.1. *Assume that* $\alpha = (\alpha_1, \alpha_2)$ *is Diophantine*, *i.e.*, (1.5) *holds. Then*

$$\lim_{\delta\to 0} \limsup_{b\to\infty} b^{-1}|S_2| = 0. \tag{4.2}$$

Proof. We shall estimate $S_{\mathrm{e}2}$; $S_{\mathrm{o}2}$ can be estimated in the same way. We start with the definition of $S_{\mathrm{e}2}$:

$$S_{\mathrm{e}2} = \frac{1}{2}\sum_{k,l}[F(w)F(0) - 1]$$

with the summation over k, l with $(k,l) = 1$ and $\widehat{w} \le \delta$. Let us divide $S_{\mathrm{e}2}$ into four parts, $S_{\mathrm{e}2} = S_3 + S_4 + S_5 + S_6$, where

$$S_j = \frac{1}{2}\sum_{M_j}[F(w)F(0) - 1]$$

with

$$M_3 = \{k, l : (k, l) = 1; \widehat{w} \le \delta; a \le |\log \delta|^{-2}\};$$
$$M_4 = \{k, l : (k, l) = 1; \widehat{w} \le \delta; |\log \delta|^{-2} < a \le \delta^{-1/3}\};$$
$$M_5 = \{k, l : (k, l) = 1; \widehat{w} \le \delta; \delta^{-1/3} < a; \exp(-\pi^2 a \widehat{w}^2) \le a^{-1}\};$$
$$M_6 = \{k, l : (k, l) = 1; \widehat{w} \le \delta; \delta^{-1/3} < a; \exp(-\pi^2 a \widehat{w}^2) > a^{-1}\}.$$

Now we shall estimate in turn $S_3, \dots, S_6$. Without loss of generality we may assume that the summation in k, l goes over the region $|l| \ge |k|$, because the sum over the complementary set $|l| < |k|$ can be estimated in the same way.

In M_3, a is small, so by (2.13),

$$|F(w)F(0) - 1| \le C \exp(-a^{-1}) = C \exp(-(k^2 + l^2)/b),$$

hence

$$|S_3| \le C \sum_{k^2+l^2 \ge b|\log \delta|^2} \exp(-(k^2 + l^2)/b) \le C_0 b \exp(-|\log \delta|^2),$$

which satisfies (4.2).

From (2.13), (2.14), $|F(w)F(0) - 1| \le Ca$, hence

$$|S_4| \le C\delta^{-1/3} \sum_{M_4} 1.$$

By (B.48), (B.49) in [BCDL], for every fixed k the fraction of l with $\widehat{w} < \delta$ does not exceed $2\delta + 4/N$, hence

$$\sum_{M_4} 1 \le (2\delta + 4/N) \sum_{a \le \delta^{-1/3}} 1 = (2\delta + 4/N) \sum_{k^2+l^2 \le b\delta^{-1/3}} 1$$
$$\le C(2\delta + 4/N) b\delta^{-1/3}.$$

Hence

$$|S_4| \le C_0 b(2\delta + 4/N)\delta^{-2/3}.$$

Since we can take $N \to \infty$ as $b \to \infty$, S_4 also satisfies (4.2).

In M_5, by (2.14), $|F(w)| \le Ca^{1/2} \exp(-\pi^2 a \widehat{w}^2) \le C_0 a^{-1/2}$, hence

$$|F(w)F(0) - 1| \le C_1,$$

and

$$|S_5| \le C_1 \sum_{a \ge \delta^{-1/3}} 1 = C_1 \sum_{k^2+l^2 \le b\delta^{1/3}} 1 \le C_2 b\delta^{1/3}.$$

Thus S_5 satisfies (4.2).

In M_6, $\exp(-\pi^2 a\widehat{w}^2) > a^{-1}$, hence $\pi^2 a\widehat{w}^2 < \log a$, and

$$\widehat{w} < \pi^{-1}(a^{-1}\log a)^{1/2}. \tag{4.3}$$

Therefore $\widehat{w}$ small for large a. Due to the Diophantine condition this implies that for some $\zeta > 0$ in the circle

$$k^2 + l^2 \le b^\zeta \tag{4.4}$$

there is no point from M_6. Indeed, in M_6, due to (1.5) and (4.3),

$$C(k^2+l^2)^{-D} \le \widehat{w} \le \pi^{-1}((k^2+l^2)/b)^{1/2}|\log((k^2+l^2)/b)|^{1/2}. \tag{4.5}$$

This implies that for large b,

$$k^2 + l^2 > b^\zeta \tag{4.6}$$

with $\zeta = (2D+1)^{-1} + \varepsilon$, $\varepsilon > 0$, hence in the circle (4.4) there is no point from M_6.

Let us divide M_6 into annular parts $M_{6j} = M_6 \cap A_j$ with

$$A_j = \{2^{j-1}\delta^{-1/3} < a \le 2^j\delta^{-1/3}\} = \{2^{-j}\delta^{1/3}b \le k^2+l^2 < 2^{-j+1}\delta^{1/3}b\},$$

$j = 1,\dots,J$, where J is the least integer number with $2^{-J}\delta^{1/3}b < b^\zeta$. Let us fix some j, $1 \le j \le J$, and estimate

$$S_{6j} = \sum_{M_{6j}} |F(w)F(0) - 1| \le Ca|M_{6j}|$$

where $a = b/(k^2+l^2)$ refers to an arbitrary point inside M_{6j}.

Let s be the width of the annulus A_j. For $(k,l) \in A_j$,

$$C_0 s < (k^2+l^2)^{1/2} < C_1 s.$$

For a fixed k, the number of l with $\widehat{w} < \lambda = \pi^{-1}(a^{-1}\log a)^{1/2}$ is estimated by (see (B.47) of [BCDL])

$$(s/N+1)(\lambda N+2) = \lambda s + \lambda N + s/N + 2,$$

where $N < s$ is the denominator of an approximant M/N of $2\alpha_2$. So

$$|M_{6j}|/|A_j| \le C(\lambda + 1/N),$$

and

$$|S_{6j}| \le C\sum_{A_j} a(\lambda + 1/N). \tag{4.7}$$

Let $N_i \le s < N_{i+1}$, where N_i are the denominators of subsequent approximants. The Diophantine condition implies

$$CN_i^{-D} \le |M_i/N_i - \alpha_2| \le |M_i/N_i - M_{i+1}/N_{i+1}| = (N_iN_{i+1})^{-1},$$

hence $N_i \geq (CN_{i+1})^{(D-1)^{-1}} \geq (Cs)^{(D-1)^{-1}}$ and $N_i^{-1} \leq C_0 s^{-(D-1)^{-1}}$. Therefore from (4.7),

$$|S_{6j}| \leq C \sum_{A_j} a(\lambda + s^{-\gamma})$$

with $\gamma = (D-1)^{-1}$. Hence

$$|S_{6j}| \leq C_0 \sum_{A_j} a((a^{-1}\log a)^{1/2} + (k^2+l^2)^{-\gamma/2})$$

or

$$|S_6| \leq C_0 \sum_{(1/2)b^{-\zeta} \leq k^2+l^2 \leq b\delta^{1/3}} a((a^{-1}\log a)^{1/2} + (k^2+l^2)^{-\gamma/2}).$$

Now,

$$\sum_{k^2+l^2 \leq b\delta^{1/3}} (a\log a)^{1/2} = \sum_{k^2+l^2 \leq b\delta^{1/3}} (b/(k^2+l^2))^{1/2} \log^{1/2}(b/(k^2+l^2))$$
$$\leq C\delta^{-1/6}\log^{1/2}\delta^{-1/3} b\delta^{1/3} = (C/3) b\delta^{1/6}|\log\delta|^{1/2},$$

and

$$\sum_{(1/2)b^{-\zeta} \leq k^2+l^2} a(k^2+l^2)^{-\gamma/2} = \sum_{(1/2)b^{-\zeta} \leq k^2+l^2} b(k^2+l^2)^{-1-\gamma/2} \leq Cb^{1-\zeta\gamma},$$

which implies

$$|S_6| \leq Cb(\delta^{1/6}|\log\delta|^{1/2} + b^{-\zeta\gamma}).$$

Therefore S_6 satisfies (4.2), and Lemma 4.1 is proved.

Proof of Theorem 1.4. In virtue of (4.1), Theorem 1.4 will be proved if we prove the following lemma:

LEMMA 4.2. *Assume that* $\alpha = (\alpha_1, \alpha_2)$ *satisfies* (1.6), (1.7). *Then*

$$\lim_{\delta\to 0} \limsup_{b\to\infty} |b^{-1} S_2 - \varepsilon(pq)(p^2+q^2)| = 0, \tag{4.8}$$

with $\varepsilon(n) = (n \bmod 2) + 1$.

Proof. The proof of Lemma 4.2 repeats word for word the one of Lemma 4.1 excepting one point: we proved in Lemma 4.1 that if α is Diophantine then in the circle (4.4) there is no point from M_6; now we state that if α satisfies (1.6), (1.7) then in the circle (4.4) there are exactly two points from M_6,

$$(k, l) = \pm(-q, p). \tag{4.9}$$

Notice that due to (1.6), if (4.9) holds then

$$w = 2(l\alpha_1 - k\alpha_2) = \pm 2(p\alpha_1 + q\alpha_2) = \pm 2r,$$

hence $\widehat{w} = 0$, so that these two points contribute to S_e the term

$$F^2(0) - 1 = \pi b(k^2 + l^2)^{-1} + O(1) = \pi b(p^2 + q^2)^{-1} + O(1).$$

If pq is odd, then these two points contribute a similar term to S_o. Therefore totally they contribute to $S_\alpha(b)$ the term $\pi b\varepsilon(pq)(p^2 + q^2)^{-1} + O(1)$.

These considerations show that (4.8) will be proved if we prove that (4.9) are the only points from M_6 in the circle (4.4). Without loss of generality we may assume $p \neq 0$. Assume

$$(k, l) \neq \pm(-q, p). \tag{4.10}$$

We have

$$l\alpha_1 - k\alpha_2 = (l/p)(\alpha_1 p + \alpha_2 q) - \alpha_2(k + lq/p) = (l/p)r - \alpha_2(k + lq/p). \tag{4.11}$$

Note that

$$k + lq/p \neq 0. \tag{4.12}$$

Indeed, otherwise $lq = -kp$, and since the pairs (k, l) and (p, q) are coprime, $(k, l) = \pm(-q, p)$, which contradicts (4.10).

(4.11), (4.12) and (1.7) imply that for every integer n,

$$|2p(l\alpha_1 - k\alpha_2) - n| = |-2\alpha_2(kp + lq) + 2lr - n| \geq C(2|kp + lq|)^{-2D},$$

hence if m is the closest integer to w, then

$$p\widehat{w} = p|2(l\alpha_1 - k\alpha_2) - m| \geq C(2|kp + lq|)^{-2D}.$$

Hence

$$\widehat{w} \geq C_{pq}(k^2 + l^2)^{-D}. \tag{4.13}$$

This proves that (1.6), (1.7) and (4.10) imply (4.13). If we assume in addition that $(k, l) \in M_6$, then (4.5) holds. Since (4.5) implies (4.6), the point (k, l) lies outside of the circle (4.4). This means that (4.9) are the only points from M_6 in this circle. Lemma 4.2 is proved.

Appendix. Proof of Corollary 2 of Theorem 1.5. The proof of Corollary 2 is the same as the proof of Theorem 1.5 = Theorem B.3 in [BCDL], except that $w = 2(l\alpha_1 - k\alpha_2)$ is now an approximate integer instead of an exact integer when

$$lP_{j1} - kP_{j2} \equiv 0 \bmod Q_j. \tag{A.1}$$

From (2.14),

$$F(w)F(0) - 1 > \pi a \exp(-1), \tag{A.2}$$

with the particular choice of b given by

$$b = b_j = \pi|\varepsilon_j|^{-2}(Q_j)^2. \tag{A.3}$$

Instead of (1.11) we now have

$$S_\alpha(b_j) > C\exp(-1)b_j(Q_j r(Q_j))^{-1}\log(b_j/Q_j) + O(b_j). \tag{A.4}$$

From (1.15), (A.3) and (A.4), (1.16) follows immediately.

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