ACTA ARITHMETICA
LXVII.3 (1994)

Orders of quadratic extensions of number fields

by

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Let k be a number field of degree n over $\mathbb{Q}$. We denote by $\mathfrak{o}_k$, d and E the ring of integers of k, the discriminant of k and the group of units of k, respectively. Let K be a quadratic extension of k or $K = k \times k$. We call a subring of K with 1 an *order* of K if it is a free $\mathbb{Z}$-module of rank $2n$. In this paper, we consider orders of K with $\mathfrak{o}_k$-module structure. We call such an order a *quadratic order over* $\mathfrak{o}_k$. We shall study the following Dirichlet series:

$$Z_k(s) = |d|^{2s} \sum_{\mathfrak{O}} |D(\mathfrak{O})|^{-s}, \tag{0.1}$$

where $\mathfrak{O}$ runs over all quadratic orders over $\mathfrak{o}_k$ and $D(\mathfrak{O})$ is the absolute discriminant of the order $\mathfrak{O}$. If $k = \mathbb{Q}$, then it is easy to see that

$$Z_{\mathbb{Q}}(s) = (1 - 2^{-s} + 2^{1-2s})\zeta(s), \tag{0.2}$$

where $\zeta(s)$ is the Riemann zeta function. The purpose of this paper is to generalize this formula to an arbitrary number field k. We shall describe $Z_k(s)$ in terms of the partial zeta functions of k. As an application, we shall give another proof of the density formula for the quadratic extensions of k, which was established by D. J. Wright (see Theorem 4.2 of [1]).

1. Quadratic orders. In this section, we shall study the structure of quadratic orders over $\mathfrak{o}_k$. The structure of finitely generated $\mathfrak{o}_k$-modules is well known. We need the following three lemmas (see, for example, Narkiewicz [3], Chapter 1, §3).

LEMMA 1.1. *Every non-zero fractional ideal of k is a projective $\mathfrak{o}_k$-module.*

LEMMA 1.2. *Let A be a finitely generated torsion free $\mathfrak{o}_k$-module. Then there exists a fractional ideal $\mathfrak{a}$ of k and an integer $m \geq 0$ such that*

$$A \cong \mathfrak{o}_k^m \oplus \mathfrak{a}$$

as $\mathfrak{o}_k$-modules.

LEMMA 1.3. *Let I_i, J_j $(1 \le i \le l,\ 1 \le j \le m)$ be non-zero fractional ideals of k and put*

$$A_1 = I_1 \oplus \ldots \oplus I_l, \qquad A_2 = J_1 \oplus \ldots \oplus J_m.$$

Then A_1 and A_2 are isomorphic if and only if $l = m$ and $I_1 \ldots I_l = (a)J_1 \ldots J_m$ for some $a \in k$.

Let $\mathfrak{O}$ be a quadratic order over $\mathfrak{o}_k$. Since $\mathfrak{O}$ is an $\mathfrak{o}_k$-module and contains 1, we have the inclusion $\mathfrak{o}_k \subset \mathfrak{O}$. Hence we have the following short exact sequence of $\mathfrak{o}_k$-modules:

$$0 \to \mathfrak{o}_k \to \mathfrak{O} \to \mathfrak{O}/\mathfrak{o}_k \to 0.$$

Since $\mathfrak{O}/\mathfrak{o}_k$ is a finitely generated torsion free $\mathfrak{o}_k$-module, Lemma 1.2 implies that

$$\mathfrak{O}/\mathfrak{o}_k \cong \mathfrak{o}_k^m \oplus \mathfrak{b}$$

for some fractional ideal $\mathfrak{b}$ of k and some integer $m \ge 0$. Computing the ranks of these modules over $\mathbb{Z}$, we have $m = 0$ and $\mathfrak{O}/\mathfrak{o}_k \cong \mathfrak{b}$. We denote by g and π the isomorphism of $\mathfrak{b}$ onto $\mathfrak{O}/\mathfrak{o}_k$ and the natural homomorphism of $\mathfrak{O}$ onto $\mathfrak{O}/\mathfrak{o}_k$, respectively. Lemma 1.3 implies that the ideal class of $\mathfrak{b}$ is uniquely determined by $\mathfrak{O}$. It also implies that we may assume $\mathfrak{o}_k \subset \mathfrak{b}$. By Lemma 1.1, $\mathfrak{b}$ is a projective $\mathfrak{o}_k$-module. Hence there exists a homomorphism $f : \mathfrak{O}/\mathfrak{o}_k \to \mathfrak{O}$ such that $\pi \cdot f = \mathrm{id}$. Put $\theta = f \cdot g(1)$. Then we have

$$\mathfrak{O} = \mathfrak{o}_k + \mathfrak{b}\theta \qquad \text{(direct sum)}.$$

Since $\mathfrak{O}$ is a ring, θ satisfies a quadratic equation

$$q_a(x) = x^2 + a_1 x + a_2 = 0, \tag{1.1}$$

where $a = (a_1, a_2) \in \mathfrak{b} \times \mathfrak{o}_k$. For any $t, s \in \mathfrak{b}$, we have $t\theta, s\theta \in \mathfrak{O}$. Hence

$$(t\theta)(s\theta) = -a_2 ts - a_1 ts\theta \in \mathfrak{O}.$$

This implies that $a_1 \in \mathfrak{b}^{-1}$ and $a_2 \in \mathfrak{b}^{-2}$. We note that $\mathfrak{b}^{-1}$ is an integral ideal since $\mathfrak{o}_k \subset \mathfrak{b}$. If $\mathfrak{O}$ is an order of a quadratic extension of k, then $q_a(x)$ is irreducible over k. If $\mathfrak{O}$ is an order of $k \times k$, then $q_a(x)$ is reducible over k with two distinct roots.

Let $\mathfrak{a}$ be an integral ideal of k and let $a = (a_1, a_2) \in \mathfrak{a} \times \mathfrak{a}^2$. If $q_a(x)$ is irreducible over k, then we define an associated $\mathfrak{o}_k$-module by

$$\mathfrak{O}(\mathfrak{a}, a) = \mathfrak{o}_k + \mathfrak{a}^{-1}\theta \subset k(\theta),$$

where θ is a root of the quadratic equation (1.1). If $q_a(x)$ is reducible over k with two distinct roots θ_1, θ_2, then we consider the $\mathfrak{o}_k$-module

$$\mathfrak{O}(\mathfrak{a}, a) = \mathfrak{o}_k e + \mathfrak{a}^{-1}\theta \subset k \times k,$$

where $e = (1, 1)$ and $\theta = (\theta_1, \theta_2) \in \mathfrak{o}_k \times \mathfrak{o}_k$. It is easy to see that $\mathfrak{O}(\mathfrak{a}, a)$ is a quadratic order over $\mathfrak{o}_k$ in both cases.

We say that two quadratic orders $\mathfrak{O}_1, \mathfrak{O}_2$ over $\mathfrak{o}_k$ are *$\mathfrak{o}_k$-isomorphic* if there exists a ring isomorphism of $\mathfrak{O}_1$ onto $\mathfrak{O}_2$ which is trivial on $\mathfrak{o}_k$. Let $a = (a_1, a_2)$, $b = (b_1, b_2) \in \mathfrak{a} \times \mathfrak{a}^2$. Assume that there exists an $\mathfrak{o}_k$-isomorphism

$$f : \mathfrak{O}(\mathfrak{a}, b) \to \mathfrak{O}(\mathfrak{a}, a).$$

Then $f(\theta_b) = t\theta_a - s$ for some $t \in \mathfrak{a}^{-1}$ and $s \in \mathfrak{o}_k$, where θ_a and θ_b are the θ for $\mathfrak{O}(\mathfrak{a}, a)$ and $\mathfrak{O}(\mathfrak{a}, b)$, respectively. Since f is an $\mathfrak{o}_k$-isomorphism, we have

$$\mathfrak{o}_k + \mathfrak{a}^{-1}(t\theta_a - s) = \mathfrak{o}_k + \mathfrak{a}^{-1}\theta_a.$$

This implies that $t \in E$ and $s \in \mathfrak{a}$. Further, we have

$$(t^{-1}\theta_b + t^{-1}s)^2 + a_1(t^{-1}\theta_b + t^{-1}s) + a_2 = 0,$$
$$\theta_b^2 + (2s + ta_1)\theta_b + (s^2 + a_1st + a_2t^2) = 0.$$

Hence

$$b_1 = 2s + ta_1, \qquad b_2 = s^2 + a_1st + a_2t^2.$$

So far we have shown the following proposition.

PROPOSITION 1.1. *Let $\mathfrak{a}_i$ $(i = 1, \ldots, h)$ be a complete set of representatives of the ideal classes of k, consisting of integral ideals. Then any quadratic order $\mathfrak{O}$ over $\mathfrak{o}_k$ is $\mathfrak{o}_k$-isomorphic to $\mathfrak{O}(\mathfrak{a}_i, a)$ for some i and some $a \in \mathfrak{a}_i \times \mathfrak{a}_i^2$. Further, $\mathfrak{O}(\mathfrak{a}_i, a) \cong \mathfrak{O}(\mathfrak{a}_j, b)$ if and only if $i = j$, $b_1 = 2s + a_1t$ and $b_2 = s^2 + a_1st + a_2t^2$ for some $t \in E$ and some $s \in \mathfrak{a}_i$.*

Let $\mathfrak{a}$ be an integral ideal of k. We denote by $E(\mathfrak{a})$ the subgroup of E consisting of all units ε with $\varepsilon \equiv 1 \pmod{\mathfrak{a}}$. For any $a = (a_1, a_2) \in \mathfrak{a} \times \mathfrak{a}^2$, put

$$Q_a(x, y) = y^2 q_a(x/y) = x^2 + a_1xy + a_2y^2,$$
$$\Delta(Q_a) = a_1^2 - 4a_2$$

and denote by $\mathcal{Q}(\mathfrak{a})$ the set of all $Q_a(x, y)$ $(a \in \mathfrak{a} \times \mathfrak{a}^2)$ with $\Delta(Q_a) \neq 0$. We denote by $G(\mathfrak{a})$ (resp. $G'(\mathfrak{a})$) the subgroup of $\mathrm{GL}_2(\mathfrak{o}_k)$ consisting of all matrices of the form

$$\gamma = \begin{pmatrix} 1 & 0 \\ s & t \end{pmatrix}, \qquad s \in \mathfrak{a},\ t \in E \ (\text{resp. } t \in E(2)). \tag{1.2}$$

The action of γ on $\mathcal{Q}(\mathfrak{a})$ is defined by

$$(\gamma \cdot Q_a)(x, y) = Q_a(x + sy, ty).$$

We denote by $[Q_a]$ (resp. $[Q_a]'$) the $G(\mathfrak{a})$-equivalence (resp. $G'(\mathfrak{a})$-equivalence) class of Q_a. Then we can rewrite the previous proposition as follows:

PROPOSITION 1.2. *Let $\mathfrak{a}_i$ $(i = 1, \ldots, h)$ be as in Proposition 1.1. Then the mapping $Q_a \mapsto \mathfrak{O}(\mathfrak{a}_i, a)$ induces a bijection of $\bigcup_{i=1}^h G(\mathfrak{a}_i)\backslash\mathcal{Q}(\mathfrak{a}_i)$ onto the set of $\mathfrak{o}_k$-isomorphism classes of quadratic orders over $\mathfrak{o}_k$.*

Now we determine a complete set of representatives for $G(\mathfrak{a})\backslash\mathcal{Q}(\mathfrak{a})$. Take a complete set of representatives $\alpha_1, \ldots, \alpha_{2^n} \in \mathfrak{a} - \{0\}$ of the quotient module $\mathfrak{a}/2\mathfrak{a}$ and put

$$\mathcal{Q}(\mathfrak{a}, \alpha_j) = \{Q_a(x,y) \in \mathcal{Q}(\mathfrak{a}) : a_1 \equiv \alpha_j \pmod{2\mathfrak{a}}\} \quad (j = 1, \ldots, 2^n).$$

Then $\mathcal{Q}(\mathfrak{a}) = \bigcup_{j=1}^{2^n} \mathcal{Q}(\mathfrak{a}, \alpha_j)$ (disjoint union). It is obvious that the subgroup $G'(\mathfrak{a})$ acts on each $\mathcal{Q}(\mathfrak{a}, \alpha_j)$.

Lemma 1.4. *Each $G(\mathfrak{a})$-equivalence class of $\mathcal{Q}(\mathfrak{a})$ consists of exactly $[E : E(2)]$ $G'(\mathfrak{a})$-equivalence classes.*

Proof. If $E = \bigcup_{\nu=1}^{l} t_\nu E(2)$, then $G(\mathfrak{a}) = \bigcup_{\nu=1}^{l} G'(\mathfrak{a})\gamma_\nu$, where

$$\gamma_\nu = \begin{pmatrix} 1 & 0 \\ 0 & t_\nu \end{pmatrix}.$$

Hence for any $Q_a \in \mathcal{Q}(\mathfrak{a})$, we have $[Q_a] = \bigcup_{\nu=1}^{l} [\gamma_\nu \cdot Q_a]'$. Assume $[\gamma_\mu \cdot Q_a]' = [\gamma_\nu \cdot Q_a]'$. Then $\Delta(\gamma_\mu \cdot Q_a) = u^2 \Delta(\gamma_\nu \cdot Q_a)$ for some $u \in E(2)$. This implies that $t_\mu / t_\nu = \pm u \in E(2)$. Hence $\mu = \nu$. ■

Lemma 1.5. *Let $Q_a, Q_b \in \mathcal{Q}(\mathfrak{a})$. Then Q_a is $G'(\mathfrak{a})$-equivalent to Q_b if and only if $Q_a, Q_b \in \mathcal{Q}(\mathfrak{a}, \alpha_j)$ for some j and $\Delta(Q_b) = t^2 \Delta(Q_a)$ for some $t \in E(2)$.*

Proof. The necessity is obvious. To prove the sufficiency, assume $Q_a, Q_b \in \mathcal{Q}(\mathfrak{a}, \alpha_j)$ for some j and $\Delta(Q_b) = t^2 \Delta(Q_a)$ for some $t \in E(2)$. Put $s = (b_1 - t a_1)/2$. Then we have

$$2s = (b_1 - \alpha_j) + (\alpha_j - a_1) + (1 - t) a_1 \in 2\mathfrak{a},$$

hence $s \in \mathfrak{a}$ and $b_1 = t a_1 + 2s$. Since $\Delta(Q_b) = t^2 \Delta(Q_a)$, we have

$$b_2 = \{b_1^2 - t^2(a_1^2 - 4a_2)\}/4 = s^2 + a_1 s t + a_2 t^2.$$

Hence $\gamma \cdot Q_a = Q_b$, where γ is an element of $G'(\mathfrak{a})$ defined by (1.2). ■

Proposition 1.3. *The mapping $Q_a \mapsto \Delta(Q_a)$ induces a bijection*

$$G'(\mathfrak{a})\backslash\mathcal{Q}(\mathfrak{a}, \alpha_j) \leftrightarrow (\alpha_j^2 + 4\mathfrak{a}^2)'/E(2)^2,$$

where $(\alpha_j^2 + 4\mathfrak{a}^2)' = (\alpha_j^2 + 4\mathfrak{a}^2) - \{0\} \subset k$.

Proof. Lemma 1.5 implies the injectivity of the mapping. The surjectivity is obvious. ■

For any integral ideal $\mathfrak{a}$ of k, we denote by $N(\mathfrak{a})$ the absolute norm of $\mathfrak{a}$. Then the following formula is easily deduced from the definition of the discriminant.

Lemma 1.6.

$$D(\mathfrak{O}(\mathfrak{a}, a)) = d^2 N(\Delta(Q_a)\mathfrak{a}^{-2}).$$

This lemma implies that $D(\mathfrak{O})d^{-2}$ is a rational integer for any quadratic order $\mathfrak{O}$ over $\mathfrak{o}_k$. We denote it by $D(\mathfrak{O};\mathfrak{o}_k)$. For any quadratic extension K of k, we denote by $\mathfrak{O}_K$ the ring of integers of K. Then we have

$$|D(\mathfrak{O}_K;\mathfrak{o}_k)| = N(D_{K/k}),$$

where $D_{K/k}$ is the relative discriminant of the quadratic extension K/k. Using this notation, the Dirichlet series $Z_k(s)$ is written as follows:

$$Z_k(s) = \sum_{\mathfrak{O}} |D(\mathfrak{O};\mathfrak{o}_k)|^{-s}. \tag{1.3}$$

Now it follows from Propositions 1.2, 1.3, Lemma 1.4 and the above equation (1.3) that

$$\begin{aligned} Z_k(s) &= \sum_{i=1}^{h} \sum_{Q\in G(\mathfrak{a}_i)\backslash \mathcal{Q}(\mathfrak{a}_i)} N(\Delta(Q)\mathfrak{a}_i^{-2})^{-s} \\ &= \frac{1}{[E:E(2)]} \sum_{i=1}^{h} \sum_{Q\in G'(\mathfrak{a}_i)\backslash \mathcal{Q}(\mathfrak{a}_i)} N(\Delta(Q)\mathfrak{a}_i^{-2})^{-s} \\ &= \frac{1}{[E:E(2)]} \sum_{i=1}^{h} \sum_{j=1}^{2^n} \sum_{Q\in G'(\mathfrak{a}_i)\backslash \mathcal{Q}(\mathfrak{a}_i,\alpha_{ij})} N(\Delta(Q)\mathfrak{a}_i^{-2})^{-s} \\ &= \frac{1}{[E:E(2)]} \sum_{i=1}^{h} \sum_{j=1}^{2^n} \sum_{\delta\in(\alpha_{ij}^2+4\mathfrak{a}_i^2)'/E(2)^2} N((\delta)\mathfrak{a}_i^{-2})^{-s}, \end{aligned} \tag{1.4}$$

where $(\alpha_{ij}^2+4\mathfrak{a}_i^2)' = (\alpha_{ij}^2+4\mathfrak{a}_i^2)-\{0\}$. To calculate the innermost sum, we shall introduce the partial zeta functions of k in the next section.

2. Dirichlet series of discriminants of quadratic orders. We use the same notations as in the previous section. Let r_1, r_2 be the numbers of real and imaginary primes of k, respectively. Hence $r_1 + 2r_2 = n$. We denote by M_r the set of real primes of k. If $v \in M_r$, we denote by σ_v the corresponding embedding of k into $\mathbb{R}$. For any subset S of M_r, we denote by $\mathfrak{h}_S$ the product of the real primes in S. Further, we denote by k_S the subset of $k^\times$ consisting of all elements $\gamma \in k^\times$ satisfying $\sigma_v(\gamma) > 0$ for all $v \in S$. For any subset A of k, put $A_S = A \cap k_S$. If $\mathfrak{O} = \mathfrak{O}(\mathfrak{a}, a)$ is a quadratic order over $\mathfrak{o}_k$, then $\Delta(\mathfrak{O}) = \Delta(Q_a)$ is determined up to multiplication by an element of E^2. Using these notations, we define the Dirichlet series $Z_{k,S}(s)$ as follows:

$$Z_{k,S}(s) = \sum_{\mathfrak{O}} |D(\mathfrak{O};\mathfrak{o}_k)|^{-s}, \tag{2.1}$$

where $\mathfrak{O}$ runs over all quadratic orders over $\mathfrak{o}_k$ with $\Delta(\mathfrak{O}) \in k_S$. We note that if S is the empty set, then $Z_{k,S}(s)$ coincides with $Z_k(s)$.

Let $\mathfrak{f}$ be a non-zero integral ideal of k. We denote by $I(\mathfrak{f})$ the multiplicative group consisting of all non-zero fractional ideals of k which are relatively prime to $\mathfrak{f}$. We denote by $P(\mathfrak{f}\mathfrak{h}_S)$ the subgroup of $I(\mathfrak{f})$ consisting of all principal fractional ideals (δ) with $\delta \in k_S$ and $\delta \equiv 1 \pmod{^* \mathfrak{f}}$. Here $\delta \equiv 1 \pmod{^* \mathfrak{f}}$ means that $\delta = \alpha/\beta$ for some $\alpha, \beta \in \mathfrak{o}_k$ satisfying $(\alpha\beta, \mathfrak{f}) = 1$ and $\alpha \equiv \beta \pmod{\mathfrak{f}}$. We call the quotient group $I(\mathfrak{f})/P(\mathfrak{f}\mathfrak{h}_S)$ the *group of ray classes modulo* $\mathfrak{f}\mathfrak{h}_S$ and denote it by $H(\mathfrak{f}\mathfrak{h}_S)$. For any $\mathfrak{b} \in I(\mathfrak{f})$, we denote by $[\mathfrak{b}, \mathfrak{f}\mathfrak{h}_S]$ the class in $H(\mathfrak{f}\mathfrak{h}_S)$ represented by $\mathfrak{b}$. Now the *partial zeta function* of $c \in H(\mathfrak{f}\mathfrak{h}_S)$ is defined by

$$\zeta_{k,\mathfrak{f}\mathfrak{h}_S}(s, c) = \sum_{\mathfrak{b}} N(\mathfrak{b})^{-s}, \tag{2.2}$$

where $\mathfrak{b}$ runs over all integral ideals belonging to the ray class c. We need some lemmas to give an expression of our Dirichlet series $Z_{k,S}(s)$ in terms of the partial zeta functions of k. We put $E(\mathfrak{f}\mathfrak{h}_S) = E(\mathfrak{f}) \cap k_S$.

LEMMA 2.1. *Let* $\mathfrak{a}$ *be a non-zero integral ideal of* k *and let* α *be a non-zero element of* $\mathfrak{a}$. *Put* $\mathfrak{g} = (\alpha\mathfrak{a}^{-1}, 2)$ *and* $\mathfrak{f} = (2)\mathfrak{g}^{-1}$. *Then*

$$\sum_{\delta \in (\alpha^2 + 4\mathfrak{a}^2)_S / E(2)^2} N((\delta)\mathfrak{a}^{-2})^{-s} = [E(\mathfrak{f}^2\mathfrak{h}_S) : E(2)^2] N(\mathfrak{g})^{-2s} \zeta_{k,\mathfrak{f}^2\mathfrak{h}_S}(s, c^2),$$

where c *is the ray class in* $H(\mathfrak{f}^2\mathfrak{h}_S)$ *represented by the integral ideal* $(\alpha)\mathfrak{a}^{-1}\mathfrak{g}^{-1}$.

Proof. Let $\delta \in (\alpha^2 + 4\mathfrak{a}^2)_S$ and put $\mathfrak{b} = (\delta)\mathfrak{a}^{-2}\mathfrak{g}^{-2}$. Then $\mathfrak{b}$ is an integral ideal prime to $\mathfrak{f}$. Since $\delta\alpha^{-2} \equiv 1 \pmod{^* \mathfrak{f}^2}$ and $\delta\alpha^{-2} \in k_S$, the integral ideal $\mathfrak{b}$ belongs to the ray class c^2. Conversely, if $\mathfrak{b}$ is an integral ideal belonging to the ray class c^2, then $\mathfrak{b} = (\beta)(\alpha^2)\mathfrak{a}^{-2}\mathfrak{g}^{-2}$ for some $\beta \in k_S$ with $\beta \equiv 1 \pmod{^* \mathfrak{f}^2}$. Hence $\mathfrak{b} = (\delta)\mathfrak{a}^{-2}\mathfrak{g}^{-2}$ with $\delta = \beta\alpha^2 \in (\alpha^2 + 4\mathfrak{a}^2)_S$. Let $\delta_1, \delta_2 \in (\alpha^2 + 4\mathfrak{a}^2)_S$ and denote by $\mathfrak{b}_1$ and $\mathfrak{b}_2$ the ideals corresponding to δ_1 and δ_2, respectively. Then $\mathfrak{b}_1 = \mathfrak{b}_2$ if and only if $\delta_1/\delta_2 \in E(\mathfrak{f}^2\mathfrak{h}_S)$. Now the desired formula follows immediately. ∎

For any $\mathfrak{f} \mid 2$, and for any fractional ideal $\mathfrak{c}$ of k relatively prime to $\mathfrak{f}$, put

$$\varrho_{\mathfrak{f}}([\mathfrak{c}, \mathfrak{f}]) = [\mathfrak{c}^2, \mathfrak{f}^2\mathfrak{h}_S].$$

Then $\varrho_{\mathfrak{f}}$ is a well defined homomorphism of $H(\mathfrak{f})$ to $H(\mathfrak{f}^2\mathfrak{h}_S)$.

LEMMA 2.2. *If the order of the group* $H(\mathfrak{f}^2\mathfrak{h}_S)$ *is odd*, *then* $\varrho_{\mathfrak{f}}$ *is an isomorphism of* $H(\mathfrak{f})$ *onto* $H(\mathfrak{f}^2\mathfrak{h}_S)$.

Proof. The assumption implies that any element of $H(\mathfrak{f}^2\mathfrak{h}_S)$ can be written as $[\mathfrak{c}, \mathfrak{f}^2\mathfrak{h}_S]^2$ for some ideal $\mathfrak{c}$. Hence $\varrho_{\mathfrak{f}}$ is surjective. On the other hand, we have the natural surjective homomorphism of $H(\mathfrak{f}^2\mathfrak{h}_S)$ onto $H(\mathfrak{f})$. Hence $\varrho_{\mathfrak{f}}$ must be an isomorphism. ■

The order of the group $H(\mathfrak{f}\mathfrak{h}_S)$ is given by the following lemma (see Lang [2], Chapter VI, Theorem 1).

LEMMA 2.3. *For any non-zero integral ideal* $\mathfrak{f}$,

$$\#H(\mathfrak{f}\mathfrak{h}_S) = \frac{h\varphi(\mathfrak{f})2^{\#S}}{[E : E(\mathfrak{f}\mathfrak{h}_S)]},$$

where φ *is the Euler function of* k.

Now we are ready to prove our main theorem.

THEOREM 1.

$$Z_{k,S}(s) = 2^{r_1+r_2-2ns} \sum_{\mathfrak{f}\mid 2} \frac{N(\mathfrak{f})^{2s}}{[E(\mathfrak{f}) : E(\mathfrak{f}^2\mathfrak{h}_S)]} \sum_{c\in H(\mathfrak{f})} \zeta_{k,\mathfrak{f}^2\mathfrak{h}_S}(s, \varrho_{\mathfrak{f}}(c)).$$

In particular, *if the order of* $H(4\mathfrak{h}_S)$ *is odd*, *then*

$$Z_{k,S}(s) = 2^{r_1+r_2-\#S-2ns}\zeta_k(s) \sum_{\mathfrak{f}\mid 2} N(\mathfrak{f})^{2s-1} \prod_{\mathfrak{p}\mid\mathfrak{f}} (1 - N(\mathfrak{p})^{-s}),$$

where $\zeta_k(s)$ *is the Dedekind zeta function of* k.

Proof. Let $\alpha_{i1}, \ldots, \alpha_{i2^n} \in \mathfrak{a}_i - \{0\}$ be a complete set of representatives of the quotient module $\mathfrak{a}_i/2\mathfrak{a}_i$. Then in the same way as when deducing the equation (1.4), we get

$$(2.3) \qquad Z_{k,S}(s) = \frac{1}{[E : E(2)]} \sum_{i=1}^{h} \sum_{j=1}^{2^n} \sum_{\delta\in(\alpha_{ij}^2+4\mathfrak{a}_i^2)_S/E(2)^2} N((\delta)\mathfrak{a}_i^{-2})^{-s}.$$

By Lemma 2.1 and the above equation (2.3), we have

$$Z_{k,S}(s) = \sum_{i=1}^{h} \sum_{j=1}^{2^n} \frac{[E(\mathfrak{f}^2\mathfrak{h}_S) : E(2)^2]}{[E : E(2)]} N(\mathfrak{g})^{-2s} \zeta_{k,\mathfrak{f}^2\mathfrak{h}_S}(s, \varrho_{\mathfrak{f}}(c_{ij})),$$

where $\mathfrak{g} = (\alpha_{ij}\mathfrak{a}_i^{-1}, 2)$, $\mathfrak{f} = (2)\mathfrak{g}^{-1}$ and $c_{ij} = [(\alpha_{ij})\mathfrak{a}_i^{-1}\mathfrak{g}^{-1}, \mathfrak{f}]$. For any $\mathfrak{g}\mid 2$, we consider the sum

$$T_{\mathfrak{g}} = \sum \zeta_{k,\mathfrak{f}^2}(s, c_{ij}^2),$$

where the summation is taken over all i, j with $(\alpha_{ij}\mathfrak{a}_i^{-1}, 2) = \mathfrak{g}$. Then

$$(2.4) \qquad Z_{k,S}(s) = \sum_{\mathfrak{g}\mid 2} \frac{[E(\mathfrak{f}^2\mathfrak{h}_S) : E(2)^2]}{[E : E(2)]} N(\mathfrak{g})^{-2s} T_{\mathfrak{g}}.$$

Now we claim that the ray class c_{ij} in $T_{\mathfrak{g}}$ represents every element of $H(\mathfrak{f})$ exactly $[E : E(\mathfrak{f})]$ times. To prove this, for any $c \in H(\mathfrak{f})$, take an integral ideal $\mathfrak{b}$ relatively prime to $\mathfrak{f}$ such that $c = [\mathfrak{b}, \mathfrak{f}]$. Since $\mathfrak{a}_1^{-1}\mathfrak{g}^{-1}, \ldots, \mathfrak{a}_h^{-1}\mathfrak{g}^{-1}$ is a complete set of representatives of the ideal classes of k, $\mathfrak{b} = (\gamma)\mathfrak{a}_i^{-1}\mathfrak{g}^{-1}$ for some i and $\gamma \in k^\times$. Since $\mathfrak{b}$ is integral, we have $\gamma \in \mathfrak{a}_i\mathfrak{g} \subset \mathfrak{a}_i$. Hence $\gamma \equiv \alpha_{ij}$ (mod $2\mathfrak{a}_i$) for some j. Now the fact that $(\mathfrak{b}, \mathfrak{f}) = 1$ implies $(\alpha_{ij}\mathfrak{a}_i^{-1}, 2) = \mathfrak{g}$ and $\gamma\alpha_{ij}^{-1} \equiv 1$ (mod* $\mathfrak{f}$). Hence $[\mathfrak{b}, \mathfrak{f}] = c_{ij}$. It is easy to see that $c_{i'j'} = c_{ij}$ if and only if $i' = i$ and $\alpha_{i'j'}\alpha_{ij}^{-1} \equiv \varepsilon$ (mod* $\mathfrak{f}$) for some $\varepsilon \in E$. This proves our claim, and hence we have established the following equation:

$$T_{\mathfrak{g}} = [E : E(\mathfrak{f})] \sum_{c \in H(\mathfrak{f})} \zeta_{k,\mathfrak{f}^2\mathfrak{h}_S}(s, \varrho_{\mathfrak{f}}(c)). \tag{2.5}$$

It is clear that

$$\frac{[E(\mathfrak{f}^2\mathfrak{h}_S) : E(2)^2][E : E(\mathfrak{f})]}{[E : E(2)]} = \frac{[E(2) : E(2)^2]}{[E(\mathfrak{f}) : E(\mathfrak{f}^2\mathfrak{h}_S)]}. \tag{2.6}$$

Dirichlet's unit theorem and the fact that $\pm 1 \in E(2)$ imply

$$[E(2) : E(2)^2] = 2^{r_1+r_2}. \tag{2.7}$$

By (2.4)–(2.7), we have

$$\begin{aligned} Z_{k,S}(s) &= 2^{r_1+r_2} \sum_{\mathfrak{g}|2} \frac{N(\mathfrak{g})^{-2s}}{[E(\mathfrak{f}) : E(\mathfrak{f}^2\mathfrak{h}_S)]} \sum_{c \in H(\mathfrak{f})} \zeta_{k,\mathfrak{f}^2\mathfrak{h}_S}(s, \varrho_{\mathfrak{f}}(c)) \\ &= 2^{r_1+r_2-2ns} \sum_{\mathfrak{f}|2} \frac{N(\mathfrak{f})^{2s}}{[E(\mathfrak{f}) : E(\mathfrak{f}^2\mathfrak{h}_S)]} \sum_{c \in H(\mathfrak{f})} \zeta_{k,\mathfrak{f}^2\mathfrak{h}_S}(s, \varrho_{\mathfrak{f}}(c)). \end{aligned} \tag{2.8}$$

Now we assume that the order of $H(4\mathfrak{h}_S)$ is odd. Then the order of $H(\mathfrak{f}^2\mathfrak{h}_S)$ is odd for any $\mathfrak{f} \,|\, 2$. By Lemma 2.2, $\varrho_{\mathfrak{f}}$ is an isomorphism of $H(\mathfrak{f})$ onto $H(\mathfrak{f}^2\mathfrak{h}_S)$. Hence the inner sum of the right hand side of (2.8) is equal to

$$\sum_{c \in H(\mathfrak{f}^2\mathfrak{h}_S)} \zeta_{k,\mathfrak{f}^2\mathfrak{h}_S}(s, c) = \zeta_k(s) \prod_{\mathfrak{p}|\mathfrak{f}} (1 - N(\mathfrak{p})^{-s}). \tag{2.9}$$

On the other hand, the fact that $H(\mathfrak{f}) \cong H(\mathfrak{f}^2\mathfrak{h}_S)$ and Lemma 2.3 imply

$$[E(\mathfrak{f}) : E(\mathfrak{f}^2\mathfrak{h}_S)] = 2^{\#S} N(\mathfrak{f}). \tag{2.10}$$

Now the second formula of the theorem follows from (2.8)–(2.10). ∎

Corollary 1. *The Dirichlet series $Z_{k,S}(s)$ converges absolutely for* $\operatorname{Re} s > 1$ *and can be analytically continued to a meromorphic function on the whole complex plane. Its only singularity is a simple pole at $s = 1$ with residue*

$$\frac{2^{r_1-\#S}\pi^{r_2} Rh}{w\sqrt{|d|}},$$

where R is the regulator of k and w is the number of roots of unity contained in k.

P r o o f. The first statement is obvious because the corresponding one for the partial zeta functions holds. It is well known that the residue of the partial zeta function $\zeta_{k,\mathfrak{f}^2}(s,c)$ at $s=1$ does not depend on the ray class c. Hence

$$\operatorname{Res}_{s=1}\zeta_{k,\mathfrak{f}^2\mathfrak{h}_S}(s,c)=\frac{\operatorname{Res}_{s=1}\zeta_k(s)}{\#H(\mathfrak{f}^2\mathfrak{h}_S)}\prod_{\mathfrak{p}\mid\mathfrak{f}}(1-N(\mathfrak{p})^{-1}). \tag{2.11}$$

By Theorem 1, Lemma 2.3 and the above equation (2.11), we have

$$\begin{aligned}\operatorname{Res}_{s=1}Z_{k,S}(s)&=2^{r_1+r_2-2n}\sum_{\mathfrak{f}\mid 2}\frac{N(\mathfrak{f})^2}{[E(\mathfrak{f}):E(\mathfrak{f}^2\mathfrak{h}_S)]}\frac{\#H(\mathfrak{f})}{\#H(\mathfrak{f}^2\mathfrak{h}_S)}\\ &\quad\times\operatorname{Res}_{s=1}\zeta_k(s)\prod_{\mathfrak{p}\mid\mathfrak{f}}(1-N(\mathfrak{p})^{-1})\\ &=2^{r_1+r_2-\#S-2n}\operatorname{Res}_{s=1}\zeta_k(s)\sum_{\mathfrak{f}\mid 2}\varphi(\mathfrak{f})\\ &=2^{-r_2-\#S}\operatorname{Res}_{s=1}\zeta_k(s).\end{aligned} \tag{2.12}$$

It is well known that

$$\operatorname{Res}_{s=1}\zeta_k(s)=\frac{2^{r_1+r_2}\pi^{r_2}Rh}{w\sqrt{|d|}} \tag{2.13}$$

(see, for example, [2], Chapter VIII, Theorem 5). Now the desired formula for the residue of $Z_{k,S}(s)$ at $s=1$ follows from (2.12) and (2.13). ∎

Corollary 1 and the Ikehara theorem imply

COROLLARY 2.

$$\#\{\mathfrak{O}:\Delta(\mathfrak{O})\in k_S,|D(\mathfrak{O};\mathfrak{o}_k)|\le X\}\sim\frac{2^{r_1-\#S}\pi^{r_2}Rh}{w\sqrt{|d|}}X\quad \textit{as } X\to\infty.$$

COROLLARY 3. *Assume that the order of the group $H(4\mathfrak{h}_S)$ is odd. Put*

$$A=2^{-r_1}\pi^{-n/2}|d|^{1/2},$$
$$G_{k,S}(s)=2^{ns}A^s\Gamma\left(\frac{s}{2}\right)^{r_1}\Gamma(s)^{r_2}Z_{k,S}(s).$$

Then $G_{k,S}(s)$ satisfies the functional equation

$$G_{k,S}(1-s)=G_{k,S}(s).$$

P r o o f. Let $(2)=\mathfrak{p}_1^{e_1}\ldots\mathfrak{p}_g^{e_g}$ be the prime ideal factorization of 2 in k. For any $\mathfrak{f}\mid 2$, put

$$\psi(s,\mathfrak{f})=N(\mathfrak{f})^{2s-1}\prod_{\mathfrak{p}\mid\mathfrak{f}}(1-N(\mathfrak{p})^{-s})$$

and

$$f(s) = \sum_{\mathfrak{f}|2} \psi(s, \mathfrak{f}).$$

Since $\psi(s, \mathfrak{f})$ is multiplicative, we have

$$f(s) = \prod_{i=1}^{g} f_i(s),$$

where $f_i(s) = \sum_{r=0}^{e_i} \psi(s, \mathfrak{p}_i^r)$. Then

$$f_i(1-s) = N(\mathfrak{p}_i)^{e_i(1-2s)} f_i(s), \quad i = 1, \ldots, g.$$

Hence $f(s)$ satisfies

$$f(1-s) = f(s) \prod_{i=1}^{g} N(\mathfrak{p}_i)^{e_i(1-2s)} = f(s) 2^{n(1-2s)}. \tag{2.14}$$

Now the functional equation of the Dedekind zeta function (see [2], Chapter XIII, Theorem 2) and (2.14) imply $G_{k,S}(1-s) = G_{k,S}(s)$. ■

3. Quadratic extensions. Let S be a subset of M_r. In this section, we study the following Dirichlet series:

$$\xi_{k,S}(s) = \sum_{K} N(D_{K/k})^{-s},$$

where K runs over all quadratic extensions of k which are unramified at any $v \in S$. Wright studied this Dirichlet series in [4] and [5] by class field theory and by developing the theory of Iwasawa–Tate zeta function, respectively.

Let K be a quadratic extension of k. Then $\mathfrak{O}_K$ is a quadratic order over $\mathfrak{o}_k$. Hence $\mathfrak{O}_K = \mathfrak{O}(\mathfrak{a}_i, a)$ for some i and $a \in \mathfrak{a}_i \times \mathfrak{a}_i^2$. If θ is a root of the quadratic equation $q_a(x) = 0$, then $\mathfrak{O}_K = \mathfrak{o}_k + \mathfrak{a}_i^{-1}\theta$. Let $\mathfrak{O}$ be a quadratic order over $\mathfrak{o}_k$ contained in K. Since $\mathfrak{O} \subset \mathfrak{O}_K$, $\{\lambda \in \mathfrak{a}_i^{-1} : \lambda\theta \in \mathfrak{O}\}$ is a fractional ideal of k contained in $\mathfrak{a}_i^{-1}$. Hence $\mathfrak{O}$ can be written

$$\mathfrak{O} = \mathfrak{o}_k + \mathfrak{a}_i^{-1}\mathfrak{b}\theta \tag{3.1}$$

for some integral ideal $\mathfrak{b}$ of k. Conversely, the $\mathfrak{o}_k$-module defined by (3.1) is obviously a quadratic order over $\mathfrak{o}_k$ contained in K. Hence

$$\sum_{\mathfrak{O} \subset K} |D(\mathfrak{O}; \mathfrak{o}_k)|^{-s} = \sum_{\mathfrak{b}} N(D_{K/k})^{-s} N(\mathfrak{b})^{-2s} = N(D_{K/k})^{-s} \zeta_k(2s). \tag{3.2}$$

Let $K = k \times k$ and denote by $\mathfrak{O}_K$ the maximal order of K. Then $\mathfrak{O}_K = \mathfrak{o}_k e + \mathfrak{o}_k \theta$ with $e = (1,1)$ and $\theta = (0,1)$. Any quadratic order contained in K is of the form $\mathfrak{o}_k e + \mathfrak{b}\theta$ for some integral ideal $\mathfrak{b}$ of k. Hence

$$\sum_{\mathfrak{O} \subset K} |D(\mathfrak{O}; \mathfrak{o}_k)|^{-s} = \sum_{\mathfrak{b}} N(\mathfrak{b})^{-2s} = \zeta_k(2s). \tag{3.3}$$

By Corollary 1, the Dirichlet series $Z_k(s)$ converges absolutely for $\operatorname{Re} s > 1$. Hence the equations (3.2) and (3.3) imply

$$Z_{k,S}(s) = \zeta_k(2s) + \zeta_k(2s)\xi_{k,S}(s). \tag{3.4}$$

By (3.4) and Theorem 1, we have given another proof of the following theorem which is a special case of Wright's theorem.

THEOREM 2.

$$\xi_{k,S}(s) = \frac{2^{r_1+r_2-2ns}}{\zeta_k(2s)} \sum_{\mathfrak{f}|2} \frac{N(\mathfrak{f})^{2s}}{[E(\mathfrak{f}) : E(\mathfrak{f}^2\mathfrak{h}_S)]} \sum_{c\in H(\mathfrak{f})} \zeta_{k,\mathfrak{f}^2\mathfrak{h}_S}(s, \varrho_{\mathfrak{f}}(c)) - 1.$$

COROLLARY 4. *Denote by $c_S(X)$ the number of quadratic extensions K of k with $|N(D_{K/k})| \le X$ which are unramified at any $v \in S$. Then*

$$c_S(X) \sim \frac{2^{r_1-\#S}\pi^{r_2}Rh}{w\sqrt{|d|}\zeta_k(2)}X \quad \text{as } X \to \infty.$$

By Corollary 4 and an elementary argument on counting cardinality, we have

COROLLARY 5. *Denote by $c'_S(X)$ the number of quadratic extensions K of k with $|N(D_{K/k})| \le X$ which are unramified at any $v \in S$ and ramified at any $v \in M_r - S$. Then*

$$c'_S(X) \sim \frac{\pi^{r_2}Rh}{w\sqrt{|d|}\zeta_k(2)}X \quad \text{as } X \to \infty.$$

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Received on 17. 6. 1993 (2447)