Orders of quadratic extensions
of number fields

by

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Let $k$ be a number field of degree $n$ over $\mathbb{Q}$. We denote by $\mathcal{O}_k$, $d$ and $E$ the ring of integers of $k$, the discriminant of $k$ and the group of units of $k$, respectively. Let $K$ be a quadratic extension of $k$ or $K = k \times k$. We call a subring of $K$ with 1 an order of $K$ if it is a free $\mathbb{Z}$-module of rank $2n$. In this paper, we consider orders of $K$ with $\mathcal{O}_k$-module structure. We call such an order a quadratic order over $\mathcal{O}_k$. We shall study the following Dirichlet series:

$$Z_k(s) = |d|^{2s} \sum_{\mathcal{O}} |D(\mathcal{O})|^{-s},$$

where $\mathcal{O}$ runs over all quadratic orders over $\mathcal{O}_k$ and $D(\mathcal{O})$ is the absolute discriminant of the order $\mathcal{O}$. If $k = \mathbb{Q}$, then it is easy to see that

$$Z_{\mathbb{Q}}(s) = (1 - 2^{-s} + 2^{1-2s}) \zeta(s),$$

where $\zeta(s)$ is the Riemann zeta function. The purpose of this paper is to generalize this formula to an arbitrary number field $k$. We shall describe $Z_k(s)$ in terms of the partial zeta functions of $k$. As an application, we shall give another proof of the density formula for the quadratic extensions of $k$, which was established by D. J. Wright (see Theorem 4.2 of [1]).

1. Quadratic orders. In this section, we shall study the structure of quadratic orders over $\mathcal{O}_k$. The structure of finitely generated $\mathcal{O}_k$-modules is well known. We need the following three lemmas (see, for example, Narkiewicz [3], Chapter 1, §3).

**Lemma 1.1.** Every non-zero fractional ideal of $k$ is a projective $\mathcal{O}_k$-module.

**Lemma 1.2.** Let $A$ be a finitely generated torsion free $\mathcal{O}_k$-module. Then there exists a fractional ideal $a$ of $k$ and an integer $m \geq 0$ such that

$$A \cong \mathcal{O}_k^m \oplus a$$

as $\mathcal{O}_k$-modules.
Lemma 1.3. Let $I_i, J_j$ (1 $\leq i \leq l$, 1 $\leq j \leq m$) be non-zero fractional ideals of $k$ and put

$$A_1 = I_1 \oplus \ldots \oplus I_l, \quad A_2 = J_1 \oplus \ldots \oplus J_m.$$ 

Then $A_1$ and $A_2$ are isomorphic if and only if $l = m$ and $I_1 \cdots I_l = (a)J_1 \cdots J_m$ for some $a \in k$.

Let $\mathcal{D}$ be a quadratic order over $\mathfrak{o}_k$. Since $\mathcal{D}$ is an $\mathfrak{o}_k$-module and contains 1, we have the inclusion $\mathfrak{o}_k \subset \mathcal{D}$. Hence we have the following short exact sequence of $\mathfrak{o}_k$-modules:

$$0 \to \mathfrak{o}_k \to \mathcal{D} \to \mathcal{D}/\mathfrak{o}_k \to 0.$$ 

Since $\mathcal{D}/\mathfrak{o}_k$ is a finitely generated torsion free $\mathfrak{o}_k$-module, Lemma 1.2 implies that

$$\mathcal{D}/\mathfrak{o}_k \cong \mathfrak{o}_k^m \oplus \mathfrak{b}$$ 

for some fractional ideal $\mathfrak{b}$ of $k$ and some integer $m \geq 0$. Computing the ranks of these modules over $\mathbb{Z}$, we have $m = 0$ and $\mathcal{D}/\mathfrak{o}_k \cong \mathfrak{b}$. We denote by $g$ and $\pi$ the isomorphism of $\mathfrak{b}$ onto $\mathcal{D}/\mathfrak{o}_k$ and the natural homomorphism of $\mathcal{D}$ onto $\mathcal{D}/\mathfrak{o}_k$, respectively. Lemma 1.3 implies that the ideal class of $\mathfrak{b}$ is uniquely determined by $\mathcal{D}$. It also implies that we may assume $\mathfrak{o}_k \subset \mathfrak{b}$. By Lemma 1.1, $\mathfrak{b}$ is a projective $\mathfrak{o}_k$-module. Hence there exists a homomorphism $f : \mathcal{D}/\mathfrak{o}_k \to \mathcal{D}$ such that $\pi \cdot f = \text{id}$. Put $\theta = f \cdot g(1)$. Then we have

$$\mathcal{D} = \mathfrak{o}_k + \mathfrak{b}\theta \quad \text{(direct sum)}.$$ 

Since $\mathcal{D}$ is a ring, $\theta$ satisfies a quadratic equation

$$(1.1) \quad q_\alpha(x) = x^2 + a_1 x + a_2 = 0,$$ 

where $a = (a_1, a_2) \in \mathfrak{b} \times \mathfrak{o}_k$. For any $t, s \in \mathfrak{b}$, we have $t \theta, s \theta \in \mathcal{D}$. Hence

$$(t \theta)(s \theta) = -a_2 ts - a_1 ts \theta \in \mathcal{D}.$$ 

This implies that $a_1 \in \mathfrak{b}^{-1}$ and $a_2 \in \mathfrak{b}^{-2}$. We note that $\mathfrak{b}^{-1}$ is an integral ideal since $\mathfrak{o}_k \subset \mathfrak{b}$. If $\mathcal{D}$ is an order of a quadratic extension of $k$, then $q_\alpha(x)$ is irreducible over $k$. If $\mathcal{D}$ is an order of $k \times k$, then $q_\alpha(x)$ is reducible over $k$ with two distinct roots.

Let $\mathfrak{a}$ be an integral ideal of $k$ and let $a = (a_1, a_2) \in \mathfrak{a} \times \mathfrak{a}^2$. If $q_\alpha(x)$ is irreducible over $k$, then we define an associated $\mathfrak{o}_k$-module by

$$\mathcal{D}(\mathfrak{a}, \mathfrak{a}) = \mathfrak{o}_k + \mathfrak{a}^{-1} \theta \subset k(\theta),$$ 

where $\theta$ is a root of the quadratic equation (1.1). If $q_\alpha(x)$ is reducible over $k$ with two distinct roots $\theta_1, \theta_2$, then we consider the $\mathfrak{o}_k$-module

$$\mathcal{D}(\mathfrak{a}, \mathfrak{a}) = \mathfrak{o}_k e + \mathfrak{a}^{-1} \theta \subset k \times k,$$ 

where $e = (1, 1)$ and $\theta = (\theta_1, \theta_2) \in \mathfrak{o}_k \times \mathfrak{o}_k$. It is easy to see that $\mathcal{D}(\mathfrak{a}, \mathfrak{a})$ is a quadratic order over $\mathfrak{o}_k$ in both cases.
We say that two quadratic orders $\mathcal{O}_1, \mathcal{O}_2$ over $\mathfrak{o}_k$ are $\mathfrak{o}_k$-isomorphic if there exists a ring isomorphism of $\mathcal{O}_1$ onto $\mathcal{O}_2$ which is trivial on $\mathfrak{o}_k$. Let $a = (a_1, a_2), b = (b_1, b_2) \in \mathfrak{a} \times \mathfrak{a}^2$. Assume that there exists an $\mathfrak{o}_k$-isomorphism

$$f : \mathcal{O}(a, b) \to \mathcal{O}(a, a).$$

Then $f(\theta_b) = t\theta_a - s$ for some $t \in \mathfrak{a}^{-1}$ and $s \in \mathfrak{o}_k$, where $\theta_a$ and $\theta_b$ are the $\theta$ for $\mathcal{O}(a, a)$ and $\mathcal{O}(a, b)$, respectively. Since $f$ is an $\mathfrak{o}_k$-isomorphism, we have

$$\mathfrak{o}_k + a^{-1}(t\theta_a - s) = \mathfrak{o}_k + a^{-1}\theta_a.$$

This implies that $t \in E$ and $s \in \mathfrak{a}$. Further, we have

$$(t^{-1}\theta_b + t^{-1}s)^2 + a_1(t^{-1}\theta_b + t^{-1}s) + a_2 = 0,$$

$$\theta_b^2 + (2s + ta_1)\theta_b + (s^2 + a_1st + a_2t^2) = 0.$$

Hence

$$b_1 = 2s + ta_1, \quad b_2 = s^2 + a_1st + a_2t^2.$$

So far we have shown the following proposition.

**Proposition 1.1.** Let $\mathfrak{a}_i$ ($i = 1, \ldots, h$) be a complete set of representatives of the ideal classes of $k$, consisting of integral ideals. Then any quadratic order $\mathcal{O}$ over $\mathfrak{o}_k$ is $\mathfrak{o}_k$-isomorphic to $\mathcal{O}(\mathfrak{a}_i, a)$ for some $i$ and some $a \in \mathfrak{a}_i \times \mathfrak{a}_i^2$. Further, $\mathcal{O}(\mathfrak{a}_i, a) \cong \mathcal{O}(\mathfrak{a}_j, b)$ if and only if $i = j$, $b_1 = 2s + a_1t$ and $b_2 = s^2 + a_1st + a_2t^2$ for some $t \in E$ and some $s \in \mathfrak{a}_i$.

Let $\mathfrak{a}$ be an integral ideal of $k$. We denote by $E(\mathfrak{a})$ the subgroup of $E$ consisting of all units $\varepsilon$ with $\varepsilon \equiv 1 \pmod{\mathfrak{a}}$. For any $a = (a_1, a_2) \in \mathfrak{a} \times \mathfrak{a}^2$, put

$$Q_a(x, y) = y^2q_a(x/y) = x^2 + a_1xy + a_2y^2,$$

$$\Delta(Q_a) = a_1^2 - 4a_2$$

and denote by $Q(\mathfrak{a})$ the set of all $Q_a(x, y)$ ($a \in \mathfrak{a} \times \mathfrak{a}^2$) with $\Delta(Q_a) \neq 0$. We denote by $G(\mathfrak{a})$ (resp. $G'(\mathfrak{a})$) the subgroup of $\text{GL}_2(\mathfrak{a}_k)$ consisting of all matrices of the form

$$(1.2) \quad \gamma = \begin{pmatrix} 1 & 0 \\ s & t \end{pmatrix}, \quad s \in \mathfrak{a}, \ t \in E \ (\text{resp.} \ t \in E(2)).$$

The action of $\gamma$ on $Q(\mathfrak{a})$ is defined by

$$(\gamma \cdot Q_a)(x, y) = Q_a(x + sy, ty).$$

We denote by $[Q_a]$ (resp. $[Q_a]'$) the $G(\mathfrak{a})$-equivalence (resp. $G'(\mathfrak{a})$-equivalence) class of $Q_a$. Then we can rewrite the previous proposition as follows:

**Proposition 1.2.** Let $\mathfrak{a}_i$ ($i = 1, \ldots, h$) be as in Proposition 1.1. Then the mapping $Q_a \mapsto \mathcal{O}(\mathfrak{a}_i, a)$ induces a bijection of $\bigcup_{i=1}^{h} G(\mathfrak{a}_i) \backslash Q(\mathfrak{a}_i)$ onto the set of $\mathfrak{o}_k$-isomorphism classes of quadratic orders over $\mathfrak{o}_k$. 
Now we determine a complete set of representatives for \( G(\mathfrak{a}) \setminus Q(\mathfrak{a}) \). Take a complete set of representatives \( \alpha_1, \ldots, \alpha_{2^n} \in \mathfrak{a} - \{0\} \) of the quotient module \( \mathfrak{a}/2\mathfrak{a} \) and put

\[
Q(\mathfrak{a}, \alpha_j) = \{ Q_a(x, y) \in Q(\mathfrak{a}) : a_1 \equiv \alpha_j \pmod{2\mathfrak{a}} \} \quad (j = 1, \ldots, 2^n).
\]

Then \( Q(\mathfrak{a}) = \bigcup_{j=1}^{2^n} Q(\mathfrak{a}, \alpha_j) \) (disjoint union). It is obvious that the subgroup \( G'(\mathfrak{a}) \) acts on each \( Q(\mathfrak{a}, \alpha_j) \).

**Lemma 1.4.** Each \( G(\mathfrak{a}) \)-equivalence class of \( Q(\mathfrak{a}) \) consists of exactly \( [E : E(2)] \) \( G'(\mathfrak{a}) \)-equivalence classes.

**Proof.** If \( E = \bigcup_{\nu=1}^I t_{\nu} E(2) \), then \( G(\mathfrak{a}) = \bigcup_{\nu=1}^I G'(\mathfrak{a}) \gamma_{\nu} \), where

\[
\gamma_{\nu} = \begin{pmatrix} 1 & 0 \\ 0 & t_{\nu} \end{pmatrix}.
\]

Hence for any \( Q_a \in Q(\mathfrak{a}) \), we have \([Q_a] = \bigcup_{\nu=1}^I [\gamma_{\nu} \cdot Q_a]'\). Assume \([\gamma_{\mu} \cdot Q_a]' = [\gamma_{\nu} \cdot Q_a]'\). Then \( \Delta(\gamma_{\mu} \cdot Q_a) = u^2 \Delta(\gamma_{\nu} \cdot Q_a) \) for some \( u \in E(2) \). This implies that \( t_{\mu}/t_{\nu} = \pm u \in E(2) \). Hence \( \mu = \nu \). \( \blacksquare \)

**Lemma 1.5.** Let \( Q_{a_1}, Q_{a_2} \in Q(\mathfrak{a}) \). Then \( Q_{a_1} \) is \( G'(\mathfrak{a}) \)-equivalent to \( Q_{a_2} \) if and only if \( Q_{a_1}, Q_{a_2} \in Q(\mathfrak{a}, \alpha_j) \) for some \( j \) and \( \Delta(Q_{a_1}) = t^2 \Delta(Q_{a_2}) \) for some \( t \in E(2) \).

**Proof.** The necessity is obvious. To prove the sufficiency, assume \( Q_{a_1}, Q_{a_2} \in Q(\mathfrak{a}, \alpha_j) \) for some \( j \) and \( \Delta(Q_{a_1}) = t^2 \Delta(Q_{a_2}) \) for some \( t \in E(2) \). Put \( s = (b_1 - ta_1)/2 \). Then we have

\[
2s = (b_1 - \alpha_j) + (\alpha_j - a_1) + (1 - t)a_1 \in 2\mathfrak{a},
\]

hence \( s \in \mathfrak{a} \) and \( b_1 = ta_1 + 2s \). Since \( \Delta(Q_{a_1}) = t^2 \Delta(Q_{a_2}) \), we have

\[
b_2 = \left( b_1^2 - t^2 (a_1^2 - 4a_2) \right)/4 = s^2 + a_1 st + a_2 t^2.
\]

Hence \( \gamma \cdot Q_{a_1} = Q_{a_2} \), where \( \gamma \) is an element of \( G'(\mathfrak{a}) \) defined by (1.2). \( \blacksquare \)

**Proposition 1.3.** The mapping \( Q_{a_1} \mapsto \Delta(Q_{a_1}) \) induces a bijection

\[
G'(\mathfrak{a}) \setminus Q(\mathfrak{a}, \alpha_j) \leftrightarrow (\alpha_j^2 + 4\mathfrak{a}^2)'/E(2)^2,
\]

where \( (\alpha_j^2 + 4\mathfrak{a}^2)' = (\alpha_j^2 + 4\mathfrak{a}^2) - \{0\} \subset k \).

**Proof.** Lemma 1.5 implies the injectivity of the mapping. The surjectivity is obvious. \( \blacksquare \)

For any integral ideal \( \mathfrak{a} \) of \( k \), we denote by \( N(\mathfrak{a}) \) the absolute norm of \( \mathfrak{a} \). Then the following formula is easily deduced from the definition of the discriminant.

**Lemma 1.6.**

\[
D(\mathfrak{O}(\mathfrak{a}, a)) = d^2 N(\Delta(Q_{a})a^{-2}).
\]
This lemma implies that $D(\mathfrak{O})d^{-2}$ is a rational integer for any quadratic order $\mathfrak{O}$ over $\mathfrak{o}_k$. We denote it by $D(\mathfrak{O}; \mathfrak{o}_k)$. For any quadratic extension $K$ of $k$, we denote by $\mathfrak{O}_K$ the ring of integers of $K$. Then we have

$$|D(\mathfrak{O}_K; \mathfrak{o}_k)| = N(D_{K/k}),$$

where $D_{K/k}$ is the relative discriminant of the quadratic extension $K/k$. Using this notation, the Dirichlet series $Z_k(s)$ is written as follows:

$$(1.3) Z_k(s) = \sum_{\mathfrak{O}} |D(\mathfrak{O}; \mathfrak{o}_k)|^{-s}.$$ 

Now it follows from Propositions 1.2, 1.3, Lemma 1.4 and the above equation (1.3) that

$$(1.4) Z_k(s) = \frac{1}{|E : E(2)|} \sum_{\mathfrak{O} \in \mathfrak{G}(a_i)} \sum_{Q \in \mathfrak{G}(a_i)} N(\Delta(Q)\alpha_i^{-2})^{-s}$$

$$= \frac{1}{|E : E(2)|} \sum_{\mathfrak{O} \in \mathfrak{G}(a_i,\alpha_{ij})} N(\Delta(Q)\alpha_i^{-2})^{-s}$$

$$= \frac{1}{|E : E(2)|} \sum_{\mathfrak{O} \in \mathfrak{G}(a_i,\alpha_{ij})} N((\delta)\alpha_i^{-2})^{-s},$$

where $((\alpha_{ij}^2 + 4a_i^2)') = (\alpha_{ij}^2 + 4a_i^2) - \{0\}$. To calculate the innermost sum, we shall introduce the partial zeta functions of $k$ in the next section.

2. Dirichlet series of discriminants of quadratic orders. We use the same notations as in the previous section. Let $r_1, r_2$ be the numbers of real and imaginary primes of $k$, respectively. Hence $r_1 + 2r_2 = n$. We denote by $M_r$ the set of real primes of $k$. If $v \in M_r$, we denote by $\sigma_v$ the corresponding embedding of $k$ into $\mathbb{R}$. For any subset $S$ of $M_r$, we denote by $\mathfrak{h}_S$ the product of the real primes in $S$. Further, we denote by $k_S$ the subset of $k^\times$ consisting of all elements $\gamma \in k^\times$ satisfying $\sigma_v(\gamma) > 0$ for all $v \in S$. For any subset $A$ of $k$, put $A_S = A \cap k_S$. If $\mathfrak{O} = \mathfrak{O}(a, a)$ is a quadratic order over $\mathfrak{o}_k$, then $\Delta(\mathfrak{O}) = \Delta(Q_a)$ is determined up to multiplication by an element of $E^2$. Using these notations, we define the Dirichlet series $Z_{k,S}(s)$ as follows:

$$(2.1) Z_{k,S}(s) = \sum_{\mathfrak{O}} |D(\mathfrak{O}; \mathfrak{o}_k)|^{-s},$$
where $\mathcal{O}$ runs over all quadratic orders over $\mathcal{O}_k$ with $\Delta(\mathcal{O}) \in k_S$. We note that if $S$ is the empty set, then $Z_{k,S}(s)$ coincides with $Z_k(s)$.

Let $f$ be a non-zero integral ideal of $k$. We denote by $I(f)$ the multiplicative group consisting of all non-zero fractional ideals of $k$ which are relatively prime to $f$. We denote by $P(f\mathcal{O}_S)$ the subgroup of $I(f)$ consisting of all principal fractional ideals $(\delta)$ with $\delta \in k_S$ and $\delta \equiv 1 \pmod{f}$. Here $\delta \equiv 1 \pmod{f}$ means that $\delta = \alpha/\beta$ for some $\alpha, \beta \in \mathcal{O}_k$ satisfying $(\alpha\beta, f) = 1$ and $\alpha \equiv \beta \pmod{f}$. We call the quotient group $I(f)/P(f\mathcal{O}_S)$ the group of ray classes modulo $f\mathcal{O}_S$ and denote it by $H(f\mathcal{O}_S)$. For any $b \in I(f)$, we denote by $[b, f\mathcal{O}_S]$ the class in $H(f\mathcal{O}_S)$ represented by $b$. Now the partial zeta function of $c \in H(f\mathcal{O}_S)$ is defined by

$$
\zeta_{k,f\mathcal{O}_S}(s, c) = \sum_b N(b)^{-s},
$$

where $b$ runs over all integral ideals belonging to the ray class $c$. We need some lemmas to give an expression of our Dirichlet series $Z_{k,S}(s)$ in terms of the partial zeta functions of $f$. We put $E(f\mathcal{O}_S) = E(f) \cap k_S$.

**Lemma 2.1.** Let $a$ be a non-zero integral ideal of $k$ and let $\alpha$ be a non-zero element of $a$. Put $g = (\alpha a^{-1}, 2)$ and $f = (2)g^{-1}$. Then

$$
\sum_{\delta \in (\alpha^2 + 4a^2)_S/E(2)^2} N((\delta)a^{-2})^{-s} = [E(\mathcal{O}_S^2)/E(2)] N(g)^{-2s} \zeta_{k,f\mathcal{O}_S}(s, c^2),
$$

where $c$ is the ray class in $H(f^2\mathcal{O}_S)$ represented by the integral ideal $(\alpha)a^{-1}g^{-1}$.

**Proof.** Let $\delta \in (\alpha^2 + 4a^2)_S$ and put $b = (\delta)a^{-2}g^{-2}$. Then $b$ is an integral ideal prime to $f$. Since $\delta a^{-2} \equiv 1 \pmod{f^2}$ and $\delta a^{-2} \in k_S$, the integral ideal $b$ belongs to the ray class $c^2$. Conversely, if $b$ is an integral ideal belonging to the ray class $c^2$, then $b = (\beta)(\alpha^2)a^{-2}g^{-2}$ for some $\beta \in k_S$ with $\beta \equiv 1 \pmod{f^2}$. Hence $b = (\delta)a^{-2}g^{-2}$ with $\delta = \beta a^2 \in (\alpha^2 + 4a^2)_S$. Let $\delta_1, \delta_2 \in (\alpha^2 + 4a^2)_S$ and denote by $b_1$ and $b_2$ the ideals corresponding to $\delta_1$ and $\delta_2$, respectively. Then $b_1 = b_2$ if and only if $\delta_1/\delta_2 \in E(f^2\mathcal{O}_S)$. Now the desired formula follows immediately. □

For any $f | 2$, and for any fractional ideal $c$ of $k$ relatively prime to $f$, put $\varrho_f([c, f]) = [c^2, f^2\mathcal{O}_S]$. Then $\varrho_f$ is a well defined homomorphism of $H(f)$ to $H(f^2\mathcal{O}_S)$.

**Lemma 2.2.** If the order of the group $H(f^2\mathcal{O}_S)$ is odd, then $\varrho_f$ is an isomorphism of $H(f)$ onto $H(f^2\mathcal{O}_S)$.

**Proof.** The assumption implies that any element of $H(f^2\mathcal{O}_S)$ can be written as $[c, f^2\mathcal{O}_S]^2$ for some ideal $c$. Hence $\varrho_f$ is surjective. On the other
hand, we have the natural surjective homomorphism of $H(f^2b_S)$ onto $H(f)$. Hence $\varphi_f$ must be an isomorphism. \[ \]

The order of the group $H(f^2b_S)$ is given by the following lemma (see Lang [2], Chapter VI, Theorem 1).

**Lemma 2.3.** For any non-zero integral ideal $f$, 

$$\#H(f^2b_S) = \frac{h \varphi_f(f)2^{2s}}{[E : E(f^2b_S)]},$$

where $\varphi$ is the Euler function of $k$.

Now we are ready to prove our main theorem.

**Theorem 1.**

$$Z_{k,S}(s) = 2^{r_1+r_2-2ns} \sum_{\ell | 2} \frac{N(f)^{2s}}{[E(f) : E(f^2b_S)]} \sum_{c \in H(f)} \zeta_{k,f^2b_S}(s, \varphi_f(c)).$$

In particular, if the order of $H(4b_S)$ is odd, then

$$Z_{k,S}(s) = 2^{r_1+r_2-2ns} \zeta_k(s) \sum_{\ell | 2} N(f)^{2s-1} \prod_{p | \ell} (1 - N(p)^{-s}),$$

where $\zeta_k(s)$ is the Dedekind zeta function of $k$.

**Proof.** Let $\alpha_i, \ldots, \alpha_{i2^n} \in a_i - \{0\}$ be a complete set of representatives of the quotient module $a_i/2a_i$. Then in the same way as when deducing the equation (1.4), we get

$$Z_{k,S}(s) = 2^{r_1+r_2-2ns} \sum_{\ell | 2} \frac{N(f)^{2s}}{[E(f) : E(f^2b_S)]} \sum_{c \in H(f)} \zeta_{k,f^2b_S}(s, \varphi_f(c)).$$

By Lemma 2.1 and the above equation (2.3), we have

$$Z_{k,S}(s) = \sum_{i=1}^h \sum_{j=1}^{2^n} \frac{N(f)^{2s}}{[E(f) : E(2)]} N(g)^{-2s} \zeta_{k,f^2b_S}(s, \varphi_f(c_{ij})).$$

where $g = (\alpha_{ij}a_i^{-1}, 2)$, $f = (2)g^{-1}$ and $c_{ij} = [(\alpha_{ij})a_i^{-1}g^{-1}, f]$. For any $g | 2$, we consider the sum

$$T_g = \sum \zeta_{k,f^2b_S}(s, c_{ij}^2),$$

where the summation is taken over all $i, j$ with $(\alpha_{ij}a_i^{-1}, 2) = g$. Then

$$Z_{k,S}(s) = \sum_{g | 2} \frac{N(f)^{2s}}{[E(f) : E(2)]} N(g)^{-2s} T_g.$$

Now we claim that the ray class $c_{ij}$ in $T_g$ represents every element of $H(f)$ exactly $[E : E(f)]$ times. To prove this, for any $c \in H(f)$, take an integral ideal $b$ relatively prime to $f$ such that $c = [b, f]$. Since $a_i^{-1}g^{-1}, \ldots, a_i^{-1}g^{-1}$ is
a complete set of representatives of the ideal classes of $k$, $b = (\gamma)a_i^{-1}g^{-1}$ for some $i$ and $\gamma \in k^\times$. Since $b$ is integral, we have $\gamma \in a_i g \subset a_i$. Hence $\gamma \equiv \alpha_{ij}$ (mod $2a_i$) for some $j$. Now the fact that $(b, f) = 1$ implies $(\alpha_{ij}a_i^{-1}, 2) = g$ and $\gamma\alpha_{ij}^{-1} \equiv 1$ (mod $f$). Hence $[b, f] = c_{ij}$. It is easy to see that $c_{ij'} = c_{ij}$ if and only if $i' = i$ and $\alpha_{i'j'}\alpha_{ij}^{-1} \equiv \varepsilon$ (mod $f$) for some $\varepsilon \in E$. This proves our claim, and hence we have established the following equation:

\begin{equation}
T_g = [E : E(f)] \sum_{c \in H(f)} \zeta_{k, f^2 h_S}(s, q_\ell(c)).
\end{equation}

It is clear that

\begin{equation}
\frac{[E(f^2 h_S) : E(2)^2][E : E(f)]}{[E : E(2)]} = \frac{[E(2) : E(2)^2]}{[E(f) : E(f^2 h_S)]}.
\end{equation}

Dirichlet’s unit theorem and the fact that $\pm 1 \in E(2)$ imply

\begin{equation}
[E(2) : E(2)^2] = 2^{r_1 + r_2}.
\end{equation}

By (2.4)–(2.7), we have

\begin{equation}
Z_{k, S}(s) = 2^{r_1 + r_2} \sum_{g \mid 2} \frac{N(g)^{-2s}}{[E(f) : E(f^2 h_S)]} \sum_{c \in H(f)} \zeta_{k, f^2 h_S}(s, q_\ell(c))
\end{equation}

\begin{equation}
= 2^{r_1 + r_2 - 2ns} \sum_{g \mid 2} \frac{N(f)^{2s}}{[E(f) : E(f^2 h_S)]} \sum_{c \in H(f)} \zeta_{k, f^2 h_S}(s, q_\ell(c)).
\end{equation}

Now we assume that the order of $H(4h_S)$ is odd. Then the order of $H(f^2 h_S)$ is odd for any $f \mid 2$. By Lemma 2.2, $q_\ell$ is an isomorphism of $H(f)$ onto $H(f^2 h_S)$. Hence the inner sum of the right hand side of (2.8) is equal to

\begin{equation}
\sum_{c \in H(f^2 h_S)} \zeta_{k, f^2 h_S}(s, c) = \zeta_k(s) \prod_{p \mid f} (1 - N(p)^{-s}).
\end{equation}

On the other hand, the fact that $H(f) \cong H(f^2 h_S)$ and Lemma 2.3 imply

\begin{equation}
[E(f) : E(f^2 h_S)] = 2^s N(f).
\end{equation}

Now the second formula of the theorem follows from (2.8)–(2.10).

**Corollary 1.** The Dirichlet series $Z_{k, S}(s)$ converges absolutely for $\Re s > 1$ and can be analytically continued to a meromorphic function on the whole complex plane. Its only singularity is a simple pole at $s = 1$ with residue

\[
\frac{2^{r_1 - s} \pi^{r_2} R h}{w \sqrt{|d|}},
\]

where $R$ is the regulator of $k$ and $w$ is the number of roots of unity contained in $k$. 
Proof. The first statement is obvious because the corresponding one for the partial zeta functions holds. It is well known that the residue of the partial zeta function \( \zeta_{k,f^2}(s,c) \) at \( s = 1 \) does not depend on the ray class \( c \). Hence

\[
(2.11) \quad \text{Res}_{s=1} \zeta_{k,f^2}(s,c) = \frac{\text{Res}_{s=1} \zeta_k(s)}{\#H(f^2h_S)} \prod_{p \mid f} (1 - N(p)^{-1}).
\]

By Theorem 1, Lemma 2.3 and the above equation (2.11), we have

\[
(2.12) \quad \text{Res}_{s=1} Z_{k,S}(s) = 2^{r_1 + r_2 - 2n} \sum_{f \mid 2} \frac{N(f)^2}{[E(f) : E(f^2h_S)]} \frac{\#H(f)}{\#H(f^2h_S)} \times \text{Res}_{s=1} \zeta_k(s) \prod_{p \mid f} (1 - N(p)^{-1})
\]

\[
= 2^{r_1 + r_2 - \#S - 2n} \text{Res}_{s=1} \zeta_k(s) \sum_{f \mid 2} \varphi(f)
\]

\[
= 2^{-r_2 - \#S} \text{Res}_{s=1} \zeta_k(s).
\]

It is well known that

\[
(2.13) \quad \text{Res}_{s=1} \zeta_k(s) = \frac{2^{r_1 + r_2} \pi^{r_2} Rh}{w \sqrt{|d|}}
\]

(see, for example, [2], Chapter VIII, Theorem 5). Now the desired formula for the residue of \( Z_{k,S}(s) \) at \( s = 1 \) follows from (2.12) and (2.13).

Corollary 1 and the Ikehara theorem imply

**Corollary 2.**

\[
\#\{\mathfrak{D} : \Delta(\mathfrak{D}) \in k_S, |D(\mathfrak{D}; o_k)| \leq X\} \sim \frac{2^{r_1 - \#S} \pi^{r_2} Rh}{w \sqrt{|d|}} X \quad \text{as} \quad X \to \infty.
\]

**Corollary 3.** Assume that the order of the group \( H(4h_S) \) is odd. Put

\[
A = 2^{-r_1} \pi^{-n/2} |d|^{1/2},
\]

\[
G_{k,S}(s) = 2^{ns} A^s \Gamma(\frac{s}{2})^{r_1} \Gamma(s)^{r_2} Z_{k,S}(s).
\]

Then \( G_{k,S}(s) \) satisfies the functional equation

\[
G_{k,S}(1 - s) = G_{k,S}(s).
\]

Proof. Let \( (2) = p_1^{e_1} \cdots p_g^{e_g} \) be the prime ideal factorization of \( 2 \) in \( k \). For any \( f \mid 2 \), put

\[
\psi(s, f) = N(f)^{2s-1} \prod_{p \mid f} (1 - N(p)^{-s})
\]
and
\[ f(s) = \sum_{f | 2} \psi(s, f). \]

Since \( \psi(s, f) \) is multiplicative, we have
\[ f(s) = \prod_{i=1}^{g} f_i(s), \]
where \( f_i(s) = \sum_{r=0}^{e_i} \psi(s, p_i^r) \). Then
\[ f_i(1-s) = N(p_i)^{e_i(1-2s)} f_i(s), \quad i = 1, \ldots, g. \]

Hence \( f(s) \) satisfies
\[
(2.14) \quad f(1-s) = f(s) \prod_{i=1}^{g} N(p_i)^{e_i(1-2s)} = f(s)2^{\sigma(1-2s)}. 
\]

Now the functional equation of the Dedekind zeta function (see [2], Chapter XIII, Theorem 2) and (2.14) imply
\[ G_{k,S}(1-s) = G_{k,S}(s). \]

3. Quadratic extensions. Let \( S \) be a subset of \( M_r \). In this section, we study the following Dirichlet series:
\[ \xi_{k,S}(s) = \sum_{K} N(D_{K/k})^{-s}, \]
where \( K \) runs over all quadratic extensions of \( k \) which are unramified at any \( v \in S \). Wright studied this Dirichlet series in [4] and [5] by class field theory and by developing the theory of Iwasawa–Tate zeta function, respectively.

Let \( K \) be a quadratic extension of \( k \). Then \( \mathfrak{O}_K \) is a quadratic order over \( \mathfrak{o}_k \). Hence \( \mathfrak{O}_K = \mathfrak{O}(a_i, a) \) for some \( i \) and \( a \in a_i \times a_i^2 \). If \( \theta \) is a root of the quadratic equation \( q_\alpha(x) = 0 \), then \( \mathfrak{O}_K = \mathfrak{o}_k + a_i^{-1} \theta \). Let \( \mathfrak{O} \) be a quadratic order over \( \mathfrak{o}_k \) contained in \( K \). Since \( \mathfrak{O} \subset \mathfrak{O}_K \), \( \{ \lambda \in a_i^{-1} : \lambda \theta \in \mathfrak{O} \} \) is a fractional ideal of \( k \) contained in \( a_i^{-1} \). Hence \( \mathfrak{O} \) can be written
\[
(3.1) \quad \mathfrak{O} = \mathfrak{o}_k + a_i^{-1} b \theta
\]
for some integral ideal \( b \) of \( k \). Conversely, the \( \mathfrak{o}_k \)-module defined by (3.1) is obviously a quadratic order over \( \mathfrak{o}_k \) contained in \( K \). Hence
\[
(3.2) \quad \sum_{\mathfrak{D} \subset K} |D(\mathfrak{D}; \mathfrak{o}_k)|^{-s} = \sum_{\mathfrak{b}} N(D_{K/k})^{-s} N(\mathfrak{b})^{-2s} = N(D_{K/k})^{-s} \zeta_k(2s).
\]

Let \( K = k \times k \) and denote by \( \mathfrak{O}_K \) the maximal order of \( K \). Then \( \mathfrak{O}_K = \mathfrak{o}_k e + \mathfrak{o}_k \theta \) with \( e = (1, 1) \) and \( \theta = (0, 1) \). Any quadratic order contained in \( K \) is of the form \( \mathfrak{o}_k e + \mathfrak{b} \theta \) for some integral ideal \( \mathfrak{b} \) of \( k \). Hence
\[
(3.3) \quad \sum_{\mathfrak{D} \subset K} |D(\mathfrak{D}; \mathfrak{o}_k)|^{-s} = \sum_{\mathfrak{b}} N(\mathfrak{b})^{-2s} = \zeta_k(2s).
\]
By Corollary 1, the Dirichlet series $Z_k(s)$ converges absolutely for $\text{Re} \, s > 1$. Hence the equations (3.2) and (3.3) imply

$$Z_{k,S}(s) = \zeta_k(2s) + \zeta_k(2s)\xi_{k,S}(s).$$

By (3.4) and Theorem 1, we have given another proof of the following theorem which is a special case of Wright’s theorem.

**Theorem 2.**

$$\xi_{k,S}(s) = 2^{r_1 + r_2 - 2ns/\zeta_k(2s)} \sum_{f \mid 2} \frac{N(f)^{2s}}{[E(f) : E(f^2h_S)]} \sum_{c \in H(f)} \zeta_{k,f^2h_S}(s, \vartheta_f(c)) - 1.$$  

**Corollary 4.** Denote by $c_S(X)$ the number of quadratic extensions $K$ of $k$ with $|N(D_{K/k})| \leq X$ which are unramified at any $v \in S$. Then

$$c_S(X) \sim \frac{2^{r_1} - \#S \pi r_2 Rh}{w \sqrt{|d| \zeta_k(2)}} X \quad \text{as } X \to \infty.$$  

By Corollary 4 and an elementary argument on counting cardinality, we have

**Corollary 5.** Denote by $c'_S(X)$ the number of quadratic extensions $K$ of $k$ with $|N(D_{K/k})| \leq X$ which are unramified at any $v \in S$ and ramified at any $v \in M_r - S$. Then

$$c'_S(X) \sim \frac{\pi r_2 Rh}{w \sqrt{|d| \zeta_k(2)}} X \quad \text{as } X \to \infty.$$  

**References**


