ACTA ARITHMETICA
LXVII.3 (1994)

On CM-fields with the same maximal real subfield

by

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We shall mean by a *number field* a finite extension over the rational field $\mathbb{Q}$ contained in the complex field $\mathbb{C}$, and by a *CM-field* a totally imaginary quadratic extension in $\mathbb{C}$ over a totally real number field. Let k be a totally real number field. Let Γ denote the set of all CM-fields that are quadratic extensions over k, so that Γ is an infinite set. In this paper, giving a characterization of CM-fields with odd relative class number, we shall prove that there exist infinitely many CM-fields in Γ with odd relative class number if and only if the class number of k in the narrow sense is odd. We shall also find out, by virtue of formulae of Kida [7], when Γ contains infinitely many CM-fields K such that $\mu_K^- = \lambda_K^- = 0$. Here, for each CM-field K, μ_K^- and λ_K^- denote respectively the Iwasawa μ^-- and λ^--invariants associated with the basic $\mathbb{Z}_2$-extension over K, $\mathbb{Z}_2$ being of course the additive group of the 2-adic integer ring. An additional remark will be made in the last section.

Part of the notation used in the paper is as follows. For any number field F, we let C_F denote the ideal class group of F, A_F the Sylow 2-subgroup of C_F, and h_F the class number of F; $h_F = |C_F|$. Moreover, h_F^* will denote the class number of F in the narrow sense, F^+ the maximal real subfield of F, I_F the ideal group of F, E_F the unit group of F, E_F^* the subgroup of E_F consisting of all totally positive elements of F in E_F, and $N_{F/H}$, for each subfield H of F, the norm map from the multiplicative group $F^\times = F \setminus \{0\}$ to the multiplicative group $H^\times$. Given arbitrary algebraic numbers $\alpha_1, \ldots, \alpha_n$ in $\mathbb{C}$, we write $(\alpha_1, \ldots, \alpha_n)$ for the fractional ideal of $\mathbb{Q}(\alpha_1, \ldots, \alpha_n)$ generated by $\alpha_1, \ldots, \alpha_n$. It is therefore understood that $(\alpha_1, \ldots, \alpha_n)$ lies in I_F whenever the number field F contains $\alpha_1, \ldots, \alpha_n$. Now, let K be any CM-field. We then denote by A_K^- the Sylow 2-subgroup of the kernel of the norm map $C_K \to C_{K^+}$ and by h_K^- the relative class number of K; $h_K^- = h_K/h_{K^+}$. As is well known, the norm map $C_K \to C_{K^+}$ is surjective so that h_K^- equals the order of the kernel of this norm map. We let t_K denote the number of prime ideals of K ramified for K/K^+. The rank of each finite abelian group G will be denoted by $r(G)$.

1. We first give a brief proof of the following fact which might be essentially well known.

LEMMA 1. *Let K be a CM-field such that $2 \nmid h_{K^+}$. Then*

(i) $t_K - 1 \le r(A_K) \le t_K - 1 + [K^+ : \mathbb{Q}] - r(E_{K^+}/E^*_{K^+})$,

(ii) $r(A_K) = t_K - 1 + [K^+ : \mathbb{Q}] - r(E_{K^+}/E^*_{K^+})$ *if* $t_K = 0$ *or* 1.

P r o o f. By $2 \nmid h_{K^+}$, the ambiguous ideal classes for K/K^+ in A_K coincide with the ideal classes in A_K of order at most 2. The ambiguous class number formula (cf. Satz 13 in Ia of [4]) therefore implies

$$(1.1) \qquad r(A_K) = t_K - 1 + [K^+ : \mathbb{Q}] - r(E_{K^+}/(N_{K/K^+}(K^\times) \cap E_{K^+})).$$

Thus (i) follows from

$$E^*_{K^+} \supseteq N_{K/K^+}(K^\times) \cap E_{K^+} \supseteq E^2_{K^+} = \{\varepsilon^2 \mid \varepsilon \in E_{K^+}\}.$$

Next assume $t_K = 1$. The product formula for the Hasse norm residue symbol then shows that every element of $E^*_{K^+}$ is a norm residue for K/K^+ modulo the conductor of K/K^+, whence, by the Hasse norm theorem for K/K^+,

$$E^*_{K^+} \subseteq N_{K/K^+}(K^\times), \quad \text{namely} \quad E^*_{K^+} = N_{K/K^+}(K^\times) \cap E_{K^+}.$$

Therefore from (1.1) we obtain

$$r(A_K) = [K^+ : \mathbb{Q}] - r(E_{K^+}/E^*_{K^+})$$

as stated in (ii).

In the case $t_K = 0$, the assertion (ii) is an immediate consequence of (1.1) and the Hasse norm theorem for K/K^+.

By the *Hilbert 2-class field* over a number field F, we shall mean as usual the maximal unramified abelian 2-extension over F in $\mathbb{C}$.

THEOREM 1. *Let K be a CM-field. Let M denote the Hilbert 2-class field over K^+. Then h^-_K is odd if and only if the following conditions are satisfied*:

(1-i) *M is cyclic over K^+, i.e., A_{K^+} is cyclic*,

(1-ii) *$t_K = 0$ or 1 and, in the case $t_K = 1$, the prime ideal of K^+ ramified in K remains prime in M*,

(1-iii) $r(E_M/E^*_M) = [M : \mathbb{Q}] + t_K - 1$.

P r o o f. We note first of all that the inequality $r(A^-_K) + 1 \ge r(A_{K^+})$ holds in general (cf. Proposition 10.12 of [12]).

Now, assume that h^-_K is odd, so that (1-i) certainly holds by the above inequality. Obviously the composite KM is a CM-field whose maximal real subfield is M. Since h_M is odd by (1-i), we see from Lemma 1 that $1 \ge$

$t_{KM} = t_K$, which implies (1-ii). Lemma 1 further shows us that

$$t_K - 1 + [M : \mathbb{Q}] - r(E_M/E_M^*) = r(A_{KM}).$$

However, the right hand side here equals 0 by $A_K^- = \{1\}$ or equivalently by $[KM : K] = |A_K|$, because KM is the Hilbert 2-class field over K as well as a cyclic extension over K. Thus we also have (1-iii).

Let us next assume (1-i)–(1-iii). As h_M is odd by (1-i) and as $t_K = t_{KM} \le 1$ by (1-ii), it follows from Lemma 1 and (1-iii) that

$$r(A_{KM}) = t_K - 1 + [M : \mathbb{Q}] - r(E_M/E_M^*) = 0.$$

Hence we have $r(A_K^-) = 0$, i.e., $2 \nmid h_K^-$. Theorem 1 is therefore proved.

Example. Let p be a prime number $\equiv 5 \pmod 8$. Then $\mathbb{Q}(\sqrt{2}, \sqrt{p})$ is the Hilbert 2-class field over $\mathbb{Q}(\sqrt{2p})$ (cf. [11]) and $\mathbb{Q}(\sqrt{2p})$ is the maximal real subfield of the CM-field $\mathbb{Q}(\sqrt{-1}, \sqrt{2p})$; so we let

$$K = \mathbb{Q}(\sqrt{-1}, \sqrt{2p}), \qquad M = \mathbb{Q}(\sqrt{2}, \sqrt{p}).$$

The only prime ideal $(2, \sqrt{2p})$ of $K^+ = \mathbb{Q}(\sqrt{2p})$ ramified in K remains prime in M. Take any $\mathfrak{a} \in I_M$ satisfying $\mathfrak{a}^2 \in I_{\mathbb{Q}}$. Clearly, $\mathfrak{a}$ is an ambiguous ideal for the quadratic extension $M/\mathbb{Q}(\sqrt{2p})$. As M is unramified over $\mathbb{Q}(\sqrt{2p})$, $\mathfrak{a}$ has generators in $\mathbb{Q}(\sqrt{2p})$: $\mathfrak{a} \in I_{\mathbb{Q}(\sqrt{2p})}$. Therefore, in I_M,

$$\mathfrak{a} = (\alpha) \text{ or } (\sqrt{2})(\alpha) \qquad \text{for some } \alpha \in \mathbb{Q}(\sqrt{2p}).$$

Let ε_1, ε_2 and ε_3 denote respectively the fundamental units > 1 of $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{p})$ and $\mathbb{Q}(\sqrt{2p})$ (so that $\varepsilon_1 = 1 + \sqrt{2}$). Since

$$N_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}(\varepsilon_1) = N_{\mathbb{Q}(\sqrt{p})/\mathbb{Q}}(\varepsilon_2) = N_{\mathbb{Q}(\sqrt{2p})/\mathbb{Q}}(\varepsilon_3) = -1,$$

it now follows from Hilfssatz 6 of [8] that the three numbers $\sqrt{\varepsilon_1\varepsilon_2\varepsilon_3}$, ε_2, ε_3 form a system of fundamental units of M. Hence, by Hilfssatz 3 of [8], we easily have

$$r(E_M/E_M^*) = 4 = [M : \mathbb{Q}].$$

The CM-field $K = \mathbb{Q}(\sqrt{-1}, \sqrt{2p})$ thus satisfies the conditions (1-i)–(1-iii) of Theorem 1. Hence, by Theorem 1, h_K^- is odd, the Hilbert 2-class field over K being $KM = \mathbb{Q}(\sqrt{-1}, \sqrt{2}, \sqrt{p})$.

The next lemma supplements Theorem 1 and will be useful to prove Theorem 2.

Lemma 2. *Let K be a CM-field and M the Hilbert 2-class field over K^+. Assume that $2 \mid h_{K^+}$, $t_{KM} = 1$, and hence $t_K = 1$. Then the prime ideal of K^+ ramified in K divides 2.*

Proof. We take an algebraic integer α in K^+ with $K = K^+(\sqrt{\alpha})$. Let $\mathfrak{p}$ be the unique prime ideal of K^+ ramified in K, so that

$$(\alpha) = \mathfrak{p}^n \mathfrak{a}^2$$

for some integer $n \ge 0$ and for some integral ideal $\mathfrak{a}$ in I_{K^+} prime to $\mathfrak{p}$. It follows from $t_{KM} = 1$ that $\mathfrak{p}$ remains prime in M. Hence, by class field theory, A_{K^+} is a cyclic group which is generated by the ideal class in A_{K^+} containing the $(h_{K^+}/|A_{K^+}|)$-th power of $\mathfrak{p}$. Therefore n is even. This conclusion shows that $\mathfrak{p}$ divides 2, because $\mathfrak{p}$ must be unramified in K if $\mathfrak{p}$ is prime to 2.

As in the introduction, let k be a totally real number field and let Γ denote the set of all CM-fields K with $K^+ = k$. We fix k from now on.

THEOREM 2. *The following three statements are equivalent*:

(2-i) $2 \nmid h_k$ *and* $r(E_k/E_k^*) = [k : \mathbb{Q}]$,

(2-ii) *there exist infinitely many CM-fields in* Γ *of odd relative class number*,

(2-iii) *there exist infinitely many CM-fields in* Γ *of odd class number.*

Proof. Clearly, (2-i) and (2-ii) imply (2-iii) while (2-iii) implies (2-ii). It therefore suffices to prove that (2-i) is equivalent with (2-ii). Now, assuming (2-i), we let $\mathfrak{r}$ denote the product of distinct infinite primes of k. Let c be the ideal class containing the principal ideal (3) in the ray class group of k modulo $(4)\mathfrak{r}$. Let ξ be any algebraic integer in k such that (ξ) is a prime ideal of k in c. It follows that, for some $\varepsilon \in E_k$, $\varepsilon\xi$ is totally positive in k and congruent to 3 modulo (4). Let $K = k(\sqrt{-\varepsilon\xi})$. Then K is a CM-field contained in Γ and (ξ) is the unique prime ideal ramified in K. Hence Theorem 1 shows that (2-i) implies $2 \nmid h_K^-$. Since there exist infinitely many prime ideals of k in c, we can take infinitely many such CM-fields as K.

We next assume (2-ii), so that, by class field theory, there exists a CM-field in Γ with odd relative class number in which a prime ideal of k dividing an odd prime is ramified. We then have $2 \nmid h_k$ by Theorem 1 and Lemma 2. Furthermore, by Theorem 1, we also have $r(E_k/E_k^*) = [k : \mathbb{Q}]$. Thus (2-i) follows from (2-ii).

Remark. As readily seen, (2-i) of Theorem 2 is equivalent to the condition that h_k^* is odd.

For instance, suppose k to be a real quadratic number field. Then, by Theorem 2, there exist infinitely many CM-fields in Γ with odd relative class number if and only if $k = \mathbb{Q}(\sqrt{p})$ for some prime number $p \not\equiv -1 \pmod 4$.

The following is an immediate consequence of Theorem 1.

PROPOSITION 1. *Let* L *be the Hilbert* 2*-class field over* k. *Assume that* L *is not cyclic over* k *or* $r(E_L/E_L^*) \le [L : \mathbb{Q}] - 2$. *Then any CM-field in* Γ *has even relative class number.*

2. Let F be any number field. Then we denote by F_∞ the basic $\mathbb{Z}_2$-extension over F; namely, we denote by $\mathbb{Q}_\infty$ the unique $\mathbb{Z}_2$-extension over $\mathbb{Q}$ in $\mathbb{C}$ and let $F_\infty = F\mathbb{Q}_\infty$ unless $F = \mathbb{Q}$. We write μ_F and λ_F respectively for the Iwasawa μ-invariant and the Iwasawa λ-invariant associated with the $\mathbb{Z}_2$-extension F_∞/F. We further let $\tau(F) = 1$ or 0 according as the ramification indices for $F_\infty/\mathbb{Q}_\infty$ of all primes of F_∞ lying above 2 are even or not. Note that a prime of F_∞ is ramified for F_∞/F if and only if the prime lies above 2. By the *Hilbert* 2-*class field* over F *in the narrow sense*, we mean the maximal unramified abelian 2-extension over F in which no prime ideal of F is ramified. Now, for any CM-field K, let

$$\mu_K^- = \mu_K - \mu_{K^+}, \qquad \lambda_K^- = \lambda_K - \lambda_{K^+},$$

and denote by s_K the number of finite primes of $K_\infty^+ = K^+\mathbb{Q}_\infty$ ramified in K_∞ but not lying above 2. It is easy to see $s_K < \infty$.

In this section, we shall prove:

THEOREM 3. *Let $k = k_0 \subseteq \ldots \subseteq k_n \subseteq k_{n+1} \subseteq \ldots \subseteq k_\infty$ be the tower of intermediate fields of k_∞/k such that $[k_n : k] = 2^n$ for each integer $n \geq 0$, let m denote the maximal integer ≥ 0 such that k_m/k is unramified, and let L^* denote the Hilbert* 2-*class field over k in the narrow sense. Then there exist infinitely many CM-fields K in Γ with $\mu_K^- = \lambda_K^- = 0$ if and only if the following conditions are satisfied*:

(3-i) h_{k_m} *is odd*,
(3-ii) $r(E_{k_m}/E_{k_m}^*) = [k_m : \mathbb{Q}] - \tau(k)$, *i.e.*, $h_{k_m}^* = 2^{\tau(k)} h_{k_m}$,
(3-iii) *just one prime ideal of L^* divides* 2.

For the proof, we prepare two lemmas.

LEMMA 3. *Let F'/F be a cyclic extension of number fields with* 2-*power degree such that just one prime ideal of F is ramified in F'. If h_F^* is odd, then so is $h_{F'}^*$.*

Proof. Modifying the proof of II in [6], one can get a simple proof of this fact.

LEMMA 4 (Corollary of Theorems 1 and 4 of [7]). *Let K be a CM-field such that $\mu_K^- = \lambda_K^- = 0$. Then $s_K = 1$ or 0 and*, *in the case $s_K = 1$*, *just one prime of K_∞^+ lies above* 2.

Proof. Let n be any integer ≥ 0. Writing K_n^+ for the intermediate field of K_∞^+/K^+ with degree 2^n over K^+, let A_n^* denote the Sylow 2-subgroup of the ideal class group of K_n^+ in the narrow sense. Let ϱ be the number of primes of K_∞^+ lying above 2. As $\mu_K^- = 0$ implies $\mu_K = 0$ (see, e.g., Proposition 13.24 of [12]), it then follows from Theorem 4 of [7] that

$$r(A_n^*) \geq \varrho - 1 + \tau(K^+) \quad \text{if } n \text{ is sufficiently large.}$$

Furthermore, by Theorem 1 of [7],

$$r(A_n^*) - \tau(K^+) - 1 + s_K \le \lambda_K^- = 0 \qquad \text{if } n \text{ is sufficiently large.}$$

Hence we have $\varrho - 1 \le 1 - s_K$, so that s_K equals 1 or 0 and, in the case $s_K = 1$, ϱ equals 1.

Proof of Theorem 3. We denote by Ω the set of CM-fields K in Γ satisfying $\mu_K^- = \lambda_K^- = 0$.

Let us first assume (3-i)–(3-iii). Since $L^* \supseteq k_m = k_\infty \cap L^*$, we see from (3-iii) that only a prime of $k_\infty L^*$ is ramified for $k_\infty L^*/L^*$ and it is totally ramified for $k_\infty L^*/L^*$. Furthermore, by (3-iii), L^* is cyclic over k so that $h_{L^*}^*$ is odd. Therefore, for any integer $n \ge m$, $h_{k_n L^*}^*$ is odd by Lemma 3 and hence $k_n L^*$ is the Hilbert 2-class field over k_n in the narrow sense. We then also have

$$[k_n L^* : k_n] = 2^{\tau(k)} \tag{2.1}$$

because (3-i) and (3-ii) imply $[L^* : k_m] = 2^{\tau(k)}$ by class field theory. Let $\mathfrak{p}$ be any prime ideal of k which does not divide 2 but remains prime in k_1. It follows that $\mathfrak{p}$ remains prime in k_n for every integer $n \ge 0$. Let $\mathfrak{l}$ be the prime ideal of k dividing 2. As $\mathfrak{l}$ remains prime in L^*, we deduce from class field theory that $\mathfrak{p}^a \mathfrak{l}^b = (\omega)$ holds with an odd integer a, an integer b, and a totally positive element ω of k. Let

$$J = k(\sqrt{-\omega}).$$

Obviously, J is a CM-field in Γ such that $s_J = 1$ and hence $J_\infty \not\ni \sqrt{-1}$. Therefore, by Theorem 1 of [7] and by (2.1),

$$\lambda_J^- = \tau(k) - \tau(k) = 0.$$

Thus J belongs to Ω. It is now clear that Ω is an infinite set.

Assume next that Ω is infinite. Then Ω contains a CM-field K in which a prime ideal of k not dividing 2 is ramified, so that, by Lemma 4,

$$s_K = 1, \qquad K_\infty \not\ni \sqrt{-1}. \tag{2.2}$$

Let j be any integer ≥ 0, and let L_j denote the Hilbert 2-class field over k_j. Note not only that L_j is totally real but also that $L_j K$ is a CM-field. We find $s_{L_j K} = [L_j k_\infty : k_\infty]$ since the unique finite prime of k_∞ ramified in K_∞ and not lying above 2 is completely decomposed in $L_j k_\infty$. However, by Theorem 3 of [7], we have $\mu_{L_j K}^- = \lambda_{L_j K}^- = 0$. Hence $L_j k_\infty = k_\infty$ follows from Lemma 4. Consequently,

$$L_j = k_j \qquad \text{whenever } j \ge m. \tag{2.3}$$

In particular, h_{k_m} is odd. As just one prime of k_∞ lies above 2 by Lemma 4, there always exists a unique prime ideal of L_j dividing 2. Now, for each integer $n \ge 0$, let L_n^* denote the Hilbert 2-class field over k_n in the narrow sense. As the restriction map $\mathrm{Gal}(L^*_{j+1}/k_{j+1}) \to \mathrm{Gal}(L^*_j/k_j)$ is surjective in case $j \geq m$, we see from $\mu_K = 0$ that

$$r(\mathrm{Gal}(L^*_{j+1}/k_{j+1})) = r(\mathrm{Gal}(L^*_j/k_j)) \quad \text{if } j \text{ is sufficiently large.}$$

Furthermore, by Theorem 1 of [7] and (2.2), we have

$$r(\mathrm{Gal}(L^*_j/k_j)) = \tau(k) \quad \text{if } j \text{ is sufficiently large.}$$

It therefore follows from (2.3) and Lemma 3 that

$$[L^*_j : k_j] = 2^{\tau(k)} \quad \text{whenever } j \geq m.$$

In particular, $h^*_{k_m} = 2^{\tau(k)} h_{k_m}$. On the other hand, by Theorem 4 of [7], the unique prime ideal of L_j dividing 2 remains prime in L^*_j if $\tau(k) = 1$ and j is sufficiently large. Hence the prime ideal of k dividing 2 remains prime in $L^* = L^*_m$. The conditions (3-i)–(3-iii) are thus satisfied and the proof is completed.

COROLLARY. *There exist infinitely many CM-fields K in Γ with $\mu^-_K = \lambda^-_K = 0$ if and only if there exist infinitely many CM-fields K in Γ with $\mu_K = \lambda_K = 0$.*

Proof. In fact, if there exist infinitely many CM-fields K in Γ with $\mu^-_K = \lambda^-_K = 0$, then we have $\mu_k = \lambda_k = 0$ by Theorem 3 (cf. [6]).

Suppose $[k : \mathbb{Q}] = 2$ in Theorem 3. Then $m = 0$ or 1 by genus theory. Furthermore, by Theorem 3, infinitely many CM-fields K with $K^+ = k$ satisfy $\mu^-_K = \lambda^-_K = 0$ if and only if either $k = \mathbb{Q}(\sqrt{p})$ for some prime number $p \not\equiv 1 \pmod 8$ or $k = \mathbb{Q}(\sqrt{2p})$ for some prime number $p \equiv 5 \pmod 8$ (cf. Example after Theorem 1). Here, in the first case, $m = 0$ and

$$\tau(k) = \begin{cases} 0 & \text{if } p = 2 \text{ or } p \equiv 5 \pmod 8, \\ 1 & \text{if } p \equiv 3 \pmod 4; \end{cases}$$

in the second case, $m = 1$ and $\tau(k) = 0$.

The following are almost immediate consequences of Theorem 1 of [7].

PROPOSITION 2. *Let K be a CM-field. For each integer $n \geq 0$, let K^+_n denote the intermediate field of K^+_∞/K^+ with degree 2^n over K^+ and let M^*_n denote the Hilbert 2-class field over K^+_n in the narrow sense. Then $\mu^-_K = \lambda^-_K = 0$ if and only if*

$$r(\mathrm{Gal}(M^*_n/K^+_n)) + s_K + \delta = \tau(K^+) + 1$$

for every sufficiently large integer $n \geq 0$, where $\delta = 1$ or 0 according as K_∞ contains $\sqrt{-1}$ or not.

PROPOSITION 3. *Let m and k_m be the same as in Theorem 3. Assume that $r(A_{k_m}) \geq \tau(k) + 2$ or $r(E_{k_m}/E^*_{k_m}) \leq [k_m : \mathbb{Q}] - \tau(k) - 2$. Then either $\mu^-_K > 0$ or $\lambda^-_K > 0$ for any CM-field K with $K^+ = k$.*

3. We shall finally make a simple remark, omitting the details. Let l be any odd prime and let $\mathbb{Z}_l$ denote the additive group of the l-adic integer ring. For each number field F, let $\mu_l(F)$ and $\lambda_l(F)$ denote respectively the Iwasawa μ- and λ-invariants associated with the basic $\mathbb{Z}_l$-extension over F. Let, for any CM-field K,

$$\mu_l^-(K) = \mu_l(K) - \mu_l(K^+), \quad \lambda_l^-(K) = \lambda_l(K) - \lambda_l(K^+).$$

Then, as is well known, K satisfies $\mu_l^-(K) = \lambda_l^-(K) = 0$ if and only if l neither divides h_K^- nor is divisible by any prime ideal of K^+ decomposed in K (see, e.g., Criterion 1.0 of [3]). We denote by Ω_l the set of CM-fields K' in Γ for which $\mu_l^-(K') = \lambda_l^-(K') = 0$.

Now, it seems quite likely that Ω_l is always an infinite set and hence there exist infinitely many CM-fields in Γ with relative class number prime to l. This certainly holds in the case $l = 3$; indeed, as Theorem 3 of [2] is refined by Theorem 1 of [10], so Theorem I.3 of [1] can be refined enough to show that a subset of Ω_3 has a positive "density" in Γ (cf. Proposition 2 and Theorem 3 of [10]). On the other hand, the main result of [9] states that Ω_l is an infinite set unless l divides the non-zero integer $w\zeta_k(-1)$. Here ζ_k denotes the Dedekind zeta function of k and

$$w = 2\prod_p p^{n(p)},$$

with p ranging over the prime numbers ramified in $k(\sqrt{-1}, \sqrt{-3})$ and $n(p)$ denoting for each p the maximal integer ≥ 0 such that the $p^{n(p)}$-th roots of unity are contained in some quadratic extension over k. (For the special case where $k = \mathbb{Q}$, see also [5].)

References

[1] B. Datskovsky and D. J. Wright, *Density of discriminants of cubic extensions*, J. Reine Angew. Math. 386 (1988), 116–138.

[2] H. Davenport and H. Heilbronn, *On the density of discriminants of cubic fields, II*, Proc. Roy. Soc. London Ser. A 322 (1971), 405–420.

[3] E. Friedman, *Iwasawa invariants*, Math. Ann. 271 (1985), 13–30.

[4] H. Hasse, *Bericht über neuere Untersuchungen und Probleme aus der Theorie der algebraischen Zahlkörper*, Phisica-Verlag, Würzburg–Wien, 1970.

[5] K. Horie, *A note on basic Iwasawa λ-invariants of imaginary quadratic fields*, Invent. Math. 88 (1987), 31–38.

[6] K. Iwasawa, *A note on class numbers of algebraic number fields*, Abh. Math. Sem. Univ. Hamburg 20 (1956), 257–258.

[7] Y. Kida, *Cyclotomic $\mathbb{Z}_2$-extensions of J-fields*, J. Number Theory 14 (1982), 340–352.

[8] T. Kubota, *Über den bizyklischen biquadratischen Zahlkörper*, Nagoya Math. J. 10 (1956), 65–85.

[9] H. Naito, *Indivisibility of class numbers of totally imaginary quadratic extensions and their Iwasawa invariants*, J. Math. Soc. Japan 43 (1991), 185–194.

[10] J. Nakagawa and K. Horie, *Elliptic curves with no rational points*, Proc. Amer. Math. Soc. 104 (1988), 20–24.

[11] L. Rédei and H. Reichardt, *Die Anzahl der durch* 4 *teilbaren Invarianten der Klassengruppe eines beliebigen quadratischen Zahlkörpers*, J. Reine Angew. Math. 170 (1933), 69–74.

[12] L. C. Washington, *Introduction to Cyclotomic Fields*, Springer, New York, 1982.

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Received on 2.6.1993
and in revised form on 28.2.1994 (2439)