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The distribution of 4-full numbers

by

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1. Introduction. A positive integer n is called 4-*full* whenever $p\mid n$ implies that $p^4\mid n$, where p denotes a prime number. Let $Q_4(x)$ be the number of 4-full numbers not exceeding x, for x sufficiently large. The problem of finding an asymptotic formula for $Q_4(x)$ with a good error term has a long and distinguished history, beginning with a famous paper of Erdős and Szekeres [3]. Elementary (Abel summation, Euler–Maclaurin summation), analytic (Perron formula, residue theorem), and exponential sum methods have subsequently been used to attack the problem. Let

$$F(s)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}\quad \text{for } \operatorname{Re}(s)>1,$$

where $\chi(n)$ is the character function of 4-full integers; then

$$F(s)=\frac{\zeta(4s)\zeta(5s)\zeta(6s)\zeta(7s)}{\zeta(10s)}\sum_{n=1}^{\infty}\frac{c(n)}{n^s},$$

the Dirichlet series $\sum c(n)n^{-s}$ is absolutely convergent for $\operatorname{Re}(s)>1/11$, so that by the residue theorem we can write

$$Q_4(x)=\sum_{4\le i\le 7}W_i x^{1/i}+\Delta(x),$$

with $\Delta(x)$ an error term. Let

$$\lambda=\inf\{\varrho:\Delta(x)\ll x^{\varrho}\}.$$

The following list of upper bounds of λ can be found in the literature:

$1/5=0.2$, Erdős and Szekeres [3] (1935),
$1/6=0.1666\ldots$, Bateman and Grosswald [2] (1958),
$169/1360=0.1242\ldots$, Krätzel [7] (1972),
$257/2072=0.1240\ldots$, Ivić [4] (1978),
$3187/25852=0.1232\ldots$, Ivić [5] (1981),
$3091/25981=0.1189\ldots$, Ivić and Shiu [6] (1982),

$5/44 = 0.1136\ldots$, Krätzel [8] (1983),
$21/187 = 0.1122\ldots$, Krätzel [10] (1989);

in particular, the last result of Krätzel was obtained by using the three-dimensional lattice point results of his paper [9]. Professor Krätzel informed the author in June 1993 that Dr. Menzer (Jena) already got a further improvement on his result.

The purpose of this paper is to give a better upper bound for λ. We will show the following

THEOREM 1. $\lambda \le 6/59 = 0.1016\ldots$

Our result is near but still falls short of the expected bound, namely, $\lambda \le 0.1$. In light of the argument involved in Krätzel [10], it suffices to deduce the following

THEOREM 2. *For any* $\varepsilon > 0$,

$$\sum_{n_1^4 n_2^5 n_3^6 n_4^7 \le x} 1 = Ax^{1/4} + Bx^{1/5} + Cx^{1/6} + Dx^{1/7} + O(x^{6/59+\varepsilon})$$

with some absolute constants A, B, C *and* D.

Following the approach of Krätzel, Theorem 2 can be reduced to 4-dimensional exponential sums, which can be estimated by a combination of Kolesnik's method and a refined version of the Bombieri–Fouvry–Iwaniec method (cf. [11], [12]).

The author is grateful to Professor Ekkehard Krätzel for providing a lot of reprints of his papers on divisors problems, without which the present work would not exist.

2. Proof of Theorem 2 (reduction). From (4), (5) and (7) of [10],

$$\sum_{n_1^4 n_2^5 n_3^6 n_4^7 \le x} 1 = Ax^{1/4} + Bx^{1/5} + Cx^{1/6} + Dx^{1/7} + E(x),$$

where

$$E(x) = -\sum_{(a,b,c,d)} S(a,b,c,d;x) + O(x^{1/11}),$$

(a,b,c,d) runs through all permutations of $(4,5,6,7)$, and

$$S(a,b,c,d;x) = \sum\nolimits_1 \psi\bigg(\bigg(\frac{x}{n_1^a n_2^b n_3^c}\bigg)^{1/d}\bigg),$$

$\psi(t) = t - [t] - \frac{1}{2}$ ($[t]$ is the integral part of t), with $\sum_1$ denoting summation over all lattice points (n_1, n_2, n_3) with

$$n_1^a n_2^b n_3^{c+d} \le x, \qquad 1 \le n_1\,(\le)\,n_2 \le n_3\,;$$

here $n_1\,(\le)\,n_2$ means $n_1 \le n_2$ if $(a,b) = (a_i, a_j)$ for $i < j$, and $n_1 < n_2$ otherwise, and we have set $(4,5,6,7) = (a_1,a_2,a_3,a_4)$. We can write $S(a,b,c,d;x)$ as $O((\ln x)^3)$ subsums of the type $S(a,b,c,d;\boldsymbol{N})$, together with a permissible error, where $\boldsymbol{N} = (N_1,N_2,N_3)$, N_v's are positive integers, and

$$S(a,b,c,d;\boldsymbol{N}) = \sum\nolimits_2 \psi\biggl(\biggl(\frac{x}{n_1^a n_2^b n_3^c}\biggr)^{1/d}\biggr)$$

with $\sum_2$ denoting summation over lattice points (n_1,n_2,n_3) with

$$(0)\quad n_1^a n_2^b n_3^{c+d} \le x, \quad 1 \le n_1\,(\le)\,n_2 \le n_3, \quad N_v \le n_v < 2N_v \quad (v=1,2,3).$$

By means of the Fourier series treatment of the function $\psi(t)$ (cf. (18) and (19) of [11]), for a parameter $K \in [10, x^{1/2}]$ and some number $H \in [1, K^2]$ (H depends on K), we have the estimate

$$(1)\qquad x^{-\varepsilon} S(a,b,c,d;\boldsymbol{N}) \ll N_1N_2N_3K^{-1} + \min(1, K/H)(\Phi(H;\boldsymbol{N}) + \Psi(H;\boldsymbol{N})),$$

where

$$(2)\qquad \begin{aligned} \Phi(H;\boldsymbol{N}) &= H^{-1}\sum_{h\sim H}\Bigl|\sum\nolimits_3 e(f(h,n_1,n_2,n_3))\Bigr|,\\ f(h,n_1,n_2,n_3) &= h\biggl(\frac{x}{n_1^a n_2^b n_3^c}\biggr)^{1/d} \end{aligned}$$

with $\sum_3$ denoting summation over lattice points (n_1,n_2,n_3) with

$$(*)\quad n_1^a n_2^b n_3^{c+d} \le x, \quad 1 \le n_1 < n_2 \le n_3, \quad N_v \le n_v < 2N_v \quad (v=1,2,3),$$

and

$$\Psi(H;\boldsymbol{N}) = H^{-1}\sum_{h\sim H}\Bigl|\sum\nolimits_4 e(f_1(h,n_2,n_3))\Bigr|,$$
$$f_1(h,n_2,n_3) = h(xn_2^{-a-b}n_3^{-c})^{1/d}$$

with $\sum_4$ denoting summation over lattice points (n_2,n_3) with

$$(\#)\qquad n_2^{a+b} n_3^{c+d} \le x, \quad 1 \le n_2 \le n_3, \quad N_v \le n_v < 2N_v \quad (v=2,3)$$

(that is, (#) is obtained from (0) by taking $n_1 = n_2$).

Throughout this paper we use the notations $r \sim R$ and $r \cong R$ to mean $1 \le r/R < 2$ and $C_1 \le r/R \le C_2$, respectively; C_i $(i = 1,2,3,\ldots)$ will be some absolute constants. As usual, $e(\xi) = \exp(2\pi i\xi)$ for a real number ξ.

As the contribution of $\Psi(H;\boldsymbol{N})$ is always negligible when compared with that of $\Phi(H;\boldsymbol{N})$, we will omit $\Psi(H;\boldsymbol{N})$ from our argument throughout. For convenience we can assume that $x=\sqrt{5}\cdot Z$, where Z is an integer, that is, x is a quadratic irrational (otherwise we can replace x by $5^{1/2}[x5^{-1/2}]$ and add a permissible error in (1)). To deal with $\Phi(H;\boldsymbol{N})$ we first transform summation over n_3 to summation over u via the following lemma.

LEMMA 1. *Let $f(x)$ and $g(x)$ be algebraic functions for $x\in[a,b]$, satisfying*

$$|f''(x)|\cong R^{-1},\quad f'''(x)\ll (RU)^{-1},$$
$$|g(x)|\le H,\quad g'(x)\ll HU_1^{-1},\quad U,U_1\ge 1.$$

Then

$$\sum_{a\le n\le b} g(n)e(f(n))$$
$$=\sum_{\alpha\le u\le\beta} b_u\frac{g(n(u))}{\sqrt{f''(n(u))}}e(f(n(u))-un(u)+1/8)$$
$$+O(H\ln(\beta-\alpha+2)+H(b-a+R)(U^{-1}+U_1^{-1}))$$
$$+O\biggl(H\min\biggl(R^{1/2},\max\biggl(\frac{1}{\langle\alpha\rangle},\frac{1}{\langle\beta\rangle}\biggr)\biggr)\biggr),$$

where $[\alpha,\beta]$ is the image of $[a,b]$ under the mapping $y=f'(x)$, $n(u)$ is determined by the equation $f'(n(u))=u$, and

$$b_u=\begin{cases}1 & \text{for } \alpha<u<\beta,\\ 1/2 & \text{for } u=\alpha=\text{integer or } u=\beta=\text{integer};\end{cases}$$

the function $\langle x\rangle$ is defined as follows:

$$\langle x\rangle=\begin{cases}\|x\| & \text{if } x \text{ is not an integer},\\ \beta-\alpha & \text{otherwise},\end{cases}$$

where $\|x\|=\min_{n\in\mathbb{Z}}|x-n|$; and $\sqrt{f''}>0$ if $f''>0$, and $\sqrt{f''}=i\sqrt{|f''|}$ if $f''<0$.

Proof. This is Lemma 1.4 of [13].

Now put

$$X=xn_1^{-a},$$
$$M_1=\max(N_3,n_2),\quad M_2=\min((Xn_2^{-b})^{1/(c+d)},2N_3),$$
$$U_1=\frac{hc}{d}(Xn_2^{-b}M_2^{-c-d})^{1/d},\quad U_2=\frac{hc}{d}(Xn_2^{-b}M_1^{-c-d})^{1/d}.$$

Lemma 1 yields

$$(3)\qquad \sum_{M_1\le n_3\le M_2} e(g)$$

$$= \sum_{U_1<u<U_2} C_1(X^{-1}h^{-d}n_2^b u^{2d+c})^{1/(2(d+c))}e(g)$$

$$+ O\biggl(N_3(HF)^{-1} + \ln x + \min\biggl((N_3^2H^{-1}F^{-1})^{1/2}, \frac{1}{U_2-hc/d}\biggr)\biggr)$$

$$+ O\biggl(\sum_{1\le i\le 2}\min\biggl((N_3^2H^{-1}F^{-1})^{1/2}, \frac{1}{\|T(n_2,X_i)\|}\biggr)\biggr)$$

$$+ R(h,n_1,n_2),$$

where

$$g = C_2(Xh^du^cn_2^{-b})^{1/(c+d)},\quad X_1=\max(n_2,N_3),\quad X_2=2N_3,$$

$$F=(XN_3^{-c}N_2^{-b})^{1/d},$$

$$R(h,n_1,n_2)=\begin{cases}\frac12 C_1(X^{-1}h^{-d}n_2^bU_1^{2d+c})^{-1/(2(c+d))}e(g)\\ \qquad\qquad\text{if } M_2=(Xn_2^{-b})^{1/(d+c)} \text{ and } U_1 \text{ integer},\\ 0\quad \text{otherwise},\end{cases}$$

$$T(n_2,w)=\frac{hc}{d}(Xn_2^{-b}w^{-d-c})^{1/d}.$$

We find that (as $F\gg N_3$)

$$(4)\qquad \sum_{n_1}\sum_{n_2}\min\biggl((N_3^2H^{-1}F^{-1})^{1/2}, \frac{1}{U_2-hc/d}\biggr)\ll x^{1/11}.$$

Let $G=(xN_1^{-a}N_2^{-b}N_3^{-c})^{1/d}$ ($\cong F$). By Hilfssatz 4 of [9] we get

$$(5)\qquad \sum_{n_1}\sum_{n_2}\min\biggl((N_3^2H^{-1}F^{-1})^{1/2}, \frac{1}{\|T(n_2,X_i)\|}\biggr)$$

$$\ll N_1(1+HGN_3^{-1})((N_3^2H^{-1}G^{-1})^{1/2}+H^{-1}G^{-1}N_2N_3)\ln x$$

$$\ll N_1(HG)^{1/2}\ln x + x^{1/11}.$$

Finally, an application of the exponent pair $(1/6,4/6)$ gives

$$(6)\qquad \sum_{n_1}\sum_{n_2} R(h,n_1,n_2)$$

$$\ll N_1(N_3^2(HG)^{-1})^{1/2}(N_2^{4/6}(GHN_2^{-1})^{1/6}+N_2H^{-1}G^{-1})$$

$$\ll N_1N_2^{1/2}N_3G^{-1/3}+N_1N_2N_3G^{-3/2}\ll N_1N_2^{1/2}N_3^{2/3}+N_1N_2$$

$$\ll (N_1^7N_2^6N_3^9)^{13/132}+N_1N_2\ll x^{13/132}.$$

From (2)–(6) we get

$$\Phi(H;\boldsymbol{N}) \ll H^{-1}(N_3^2H^{-1}G^{-1})^{1/2}\sum_{h\sim H}\Big|\sum\nolimits_5 g_1(n_1)g_2(n_2)g_3(u)e(g)\Big| + N_1(HG)^{1/2}\ln x + x^{13/132}, \tag{7}$$

where $\sum_5$ denotes summation over lattice points (n_1,n_2,u) with

$$1\le n_1<n_2,\quad N_v\le n_v<2N_v\quad (v=1,2),\quad U_1<u<U_2,$$

and $g_i(\cdot)$ $(i=1,2,3)$ are monomials with $|g_i(\cdot)|\le 1$.

3. Three estimates for $S(a,b,c,d;\boldsymbol{N})$

LEMMA 2. *Let $f(x,y)=Ax^\alpha y^\beta$, $g(x,y)=Bx^\gamma y^\delta$, where α, β, γ and δ are rationals with $\alpha\beta(\alpha+\beta-1)(\alpha+\beta-2)\ne 0$, $A>0$, and suppose that $g(x,y)\cong G$ holds for (x,y) with $x\sim X$ and $y\sim Y$. Moreover, suppose that $D\subset\{(x,y)\mid x\sim X, y\sim Y\}$, D is embraced by $O(1)$ algebraic curves, and $X\gg Y$. Let $N=XY$, $F=AX^\alpha Y^\beta$. Then*

$$\sum_{(x,y)\in D} g(x,y)e(f(x,y)) \ll (N+AN)^\varepsilon G(\sqrt[6]{F^2N^3}+N^{5/6} + \sqrt[8]{N^8F^{-1}X^{-1}}+NF^{-1/4}+NY^{-1/2}).$$

Proof. This is a "weighted" version of Lemma 9 of [11], and can be obtained similarly by Kolesnik's original method.

Applying Lemma 2 to the variables n_2 and u of the multiple sum of (7), we get

$$x^{-\varepsilon}\Phi(H;\boldsymbol{N}) \ll \sqrt[6]{(HG)^2N_1^6N_2^3N_3^3}+\sqrt[8]{(HG)^3N_1^8N_2^7} + (HG)^{1/2}N_1N_2^{1/2}+N_1N_2N_3^{1/2}+x^\varphi,$$

where $\varphi=13/132$; thus from (1) we obtain

$$x^{-2\varepsilon}S(a,b,c,d;\boldsymbol{N}) \ll N_1N_2N_3K^{-1}+\sqrt[6]{(KG)^2N_1^6N_2^3N_3^3} + \sqrt[8]{(KG)^3N_1^8N_2^7}+(KG)^{1/2}N_1N_2^{1/2} + N_1N_2N_3^{1/2}+x^\varphi. \tag{8}$$

To choose K optimally we need the following

LEMMA 3. *Let M, N, u_m, v_n, A_m, B_n be positive $(1\le n\le N, 1\le m \le M)$, and Q_1 and Q_2 be given non-negative numbers with $Q_1<Q_2$. Then*

there is a number Q such that $Q_1 \le Q \le Q_2$ and

$$\sum_{1\le m\le M} A_m Q^{u_m} + \sum_{1\le n\le N} B_n Q^{-v_n} \ll \sum_{1\le n\le N}\sum_{1\le m\le M} (A_m^{v_n} B_n^{u_m})^{1/(u_m+v_n)} + \sum_{1\le m\le M} A_m Q_1^{u_m} + \sum_{1\le n\le N} B_n Q_2^{-v_n}.$$

Proof. This is Lemma 2 of [11].

By Lemma 3 we can choose a $K \in [0, x^{1/2}]$ optimally in (8) so that

$$x^{-2\varepsilon} S(a,b,c,d;\boldsymbol{N}) \ll \sqrt[8]{G^2 N_1^8 N_2^5 N_3^5} + \sqrt[11]{G^3 N_1^{11} N_2^{10} N_3^3} + \sqrt[3]{G N_1^3 N_2^2 N_3} + N_1 N_2 N_3^{1/2} + x^{\varphi}. \tag{9}$$

This is our first estimate. To get the second, we apply the next

LEMMA 4. *Let $H \ge 1$, $X \ge 1$, $Y \ge 1000$, α, β and γ be real numbers with $\alpha\gamma(\gamma-1)(\beta-1) \ne 0$, and let $A > C(\alpha,\beta,\gamma) > 0$ and $f(h,x,y) = Ah^\alpha x^\beta y^\gamma$. Define*

$$S(H,X,Y) = \sum_{(h,x,y)\in D} C_1(h,x) C_2(y) e(f(h,x,y)),$$

where D is a region contained in the rectangle $\{(h,x,y) \mid h \sim H, x \sim X, y \sim Y\}$ such that for any fixed pair (h_0,x_0), the intersection $D \cap \{(h_0,x_0,y) \mid y \sim Y\}$ has at most $O(1)$ segments. Also, suppose that $|C_1(h,x)| \le 1$, $|C_2(y)| \le 1$ and $F = AH^\alpha X^\beta Y^\gamma \gg Y$. Then, for $L = \ln((A+1)HXY+2)$ and $M = \max(1, FY^{-2})$,

$$L^{-3} S(H,X,Y) \ll \sqrt[22]{(HX)^{19} Y^{13} F^3} + HXY^{5/8}(1 + Y^7 F^{-4})^{1/16} + \sqrt[32]{(HX)^{29} Y^{28} F^{-2} M^5} + \sqrt[4]{(HX)^3 Y^4 M}.$$

Proof. This is Theorem 3 of [12].

We apply Lemma 4 to the triple (h,u,n_2) of (7), with the choice $(h,x,y) = (h,u,n_2)$, and we estimate the sum over n_1 trivially in (7), thus obtaining (note that $u \cong HGN_3^{-1}$):

$$x^{-\varepsilon} \Phi(H;\boldsymbol{N}) \ll \sqrt[22]{H^8 G^{11} N_1^{22} N_2^{13} N_3^3} + (HG)^{1/2} N_1 N_2^{5/8} + \sqrt[16]{(HG)^4 N_1^{16} N_2^{17}} + \sqrt[32]{H^8 G^{11} N_1^{32} N_2^{28} N_3^3} + \sqrt[32]{H^{13} G^{16} N_1^{32} N_2^{18} N_3^3} + \sqrt[4]{G N_1^4 N_2^4 N_3} + \sqrt[4]{(HG)^2 N_1^4 N_2^2} + x^{\varphi}. \tag{10}$$

From (1) and (10) we find that

$$x^{-2\varepsilon}S(a,b,c,d;\boldsymbol{N}) \ll N_1N_2N_3K^{-1} + \sqrt[22]{K^8G^{11}N_1^{22}N_2^{13}N_3^3} + (KG)^{1/2}N_1N_2^{5/8} + \sqrt[16]{(KG)^4N_1^{16}N_2^{17}} + \sqrt[32]{K^8G^{11}N_1^{32}N_2^{28}N_3^3} + \sqrt[32]{K^{13}G^{16}N_1^{32}N_2^{18}N_3^3} + \sqrt[4]{GN_1^4N_2^4N_3} + \sqrt[4]{(KG)^2N_1^4N_2^2} + x^{\varphi}. \tag{11}$$

Choosing $K \in [0, x^{1/2}]$ optimally via Lemma 3, we obtain

$$x^{-2\varepsilon}S(a,b,c,d;\boldsymbol{N}) \ll \sqrt[30]{G^{11}N_1^{30}N_2^{21}N_3^{11}} + \sqrt[24]{(GN_3)^8N_2^{18}N_1^{24}} + \sqrt[20]{(GN_3)^4N_2^{21}N_1^{20}} + \sqrt[40]{(GN_3)^{11}N_2^{36}N_1^{40}} + \sqrt[45]{(GN_3)^{16}N_2^{31}N_1^{45}} + \sqrt[5]{(GN_3)^2N_2^3N_1^5} + \sqrt[4]{GN_3N_2^4N_1^4} + x^{\varphi}. \tag{12}$$

This is the second estimate. To deduce the third, we first relax the severe constraint $n_1 < n_2$ and $U_1 < u < U_2$ by the next familiar lemma.

LEMMA 5. *Let T, α and β be real, $T > 0$, $\beta > 0$. Then*

$$\frac{1}{\pi}\int_{-T}^{T} e^{it\alpha}\frac{\sin t\beta}{t}\,dt = \begin{cases} 1 + O(1/(T(\beta-|\alpha|))) & \text{if } |\alpha| \le \beta, \\ O(1/(T(|\alpha|-\beta))) & \text{if } |\alpha| > \beta. \end{cases}$$

We can apply Lemma 5 as follows. For instance, we want to remove the inequality

$$u > U_1 = \frac{hc}{d}(xn_1^{-a}n_2^{-b}(2N_3)^{-c-d})^{1/d}$$

when $M_2 = 2N_3$; then we choose $T = x^{1000}$, $\alpha = \ln(x(hc)^d)$ and $\beta = \ln((ud)^d n_1^a n_2^b (2N_3)^{c+d})$ in Lemma 5. As $x = \sqrt{5}Z$ with Z an integer, the contribution of $O(T(||\alpha|-\beta|)^{-1})$ of Lemma 5 can be estimated satisfactorily by using the fact that $|x - p/q| \gg q^{-2}x^{-1}$ for all rationals p/q with $q > 0$ (which is implied by a special case of Liouville's theorem, cf. §5 of Chapter 6 in Baker [1]). In this fashion we can remove all the relationships between the lattice points (h, n_2) and (n_1, u) consecutively. Thus from (7) we get

$$x^{-\varepsilon}\Phi(H;\boldsymbol{N}) \ll (H^{-3}G^{-1}N_3^2)^{1/2}\sum_{h\sim H}\sum_{n_2\sim N_2}\Big|\sum_{(n_1,u)\in D_1} C(n_1,u)e(C_2(xh^du^cn_1^{-a}n_2^{-b})^{\delta})\Big| + N_1(HG)^{1/2} + x^{\varphi}, \tag{13}$$

where, for brevity, $\delta = 1/(c+d)$, and D_1 is a region contained in $\{(n_1,u) \mid n_1 \sim N_1, u \cong U = HGN_3^{-1}\}$, independent of h and n_2; $|C(n_1,u)| \le 1$. By

Lemma 4 of [11] we get from (13) the estimate

$$x^{-2\varepsilon}\Phi(H;\boldsymbol{N})^2 \ll H^{-2}N_3^2B_1B_2 + HGN_1^2 + x^{2\varphi}, \tag{14}$$

where B_1 is the number of lattice points $(h, n_2, \widetilde{h}, \widetilde{n}_2)$ such that

$$h, \widetilde{h} \sim H, \quad n_2, \widetilde{n}_2 \sim N_2, \quad |(h^d n_2^{-b})^{1/(c+d)} - (\widetilde{h}^d \widetilde{n}_2^{-b})^{1/(c+d)}| \ll \Delta R_1,$$

with $\Delta = (HG)^{-1}$ and $R_1 = (H^dN_2^{-b})^{1/(c+d)}$, and where B_2 is the number of lattice points $(n_1, u, \widetilde{n}_1, \widetilde{u})$ such that

$$n_1, \widetilde{n}_1 \sim N_1, \quad u, \widetilde{u} \cong HGN_3^{-1},$$
$$|(u^c n_1^{-a})^{1/(c+d)} - (\widetilde{u}^c \widetilde{n}_1^{-a})^{1/(c+d)}| \ll \Delta R_2,$$

with $R_2 = ((GHN_3^{-1})^cN_1^{-a})^{1/(c+d)}$. By Lemma 5 of [11],

$$B_1 \ll (HN_2 + HN_2N_2G^{-1})(\ln x)^2 \ll HN_2 \ln^2 x, \tag{15}$$
$$B_2 \ll (N_1HGN_3^{-1} + HGN_1^2N_3^{-2})(\ln x)^2 \ll N_1HGN_3^{-1}\ln^2 x. \tag{16}$$

From (14)–(16) we get

$$x^{-2\varepsilon}\Phi(H;\boldsymbol{N}) \ll (GN_1N_2N_3)^{1/2} + N_1(HG)^{1/2} + x^{\varphi},$$

which, in conjunction with (1) and the choice $K = (G^{-1}N_2^2N_3^3)^{1/3}$, gives

$$\begin{aligned} x^{-3\varepsilon}S(a,b,c,d;\boldsymbol{N}) &\ll (GN_1N_2N_3)^{1/2} + \sqrt[3]{GN_1^3N_2N_3} + x^{\varphi} \\ &\ll (GN_1N_2N_3)^{1/2} + x^{\varphi}. \end{aligned} \tag{17}$$

This is our third estimate.

4. Proof of Theorem 2 (completion). Recall that $N_1 \ll N_2 \ll N_3$; thus

$$(N_3 \ll)\, G := (xN_1^{-a}N_2^{-b}N_3^{-c})^{1/d} \ll (xN_1^{-7}N_2^{-6}N_3^{-5})^{1/4} \tag{18}$$

for any permutation (a,b,c,d) of $(4,5,6,7)$. By our three estimates (9), (12), (17), and the fact (18), we have, with $\eta = 4\varepsilon$,

$$\begin{aligned} x^{-\eta}S(a,b,c,d;\boldsymbol{N}) \ll\ & \sqrt[16]{xN_1^9N_2^4N_3^5} + \sqrt[44]{x^3N_1^{23}N_2^{22}N_3^{-3}} \\ & + \sqrt[12]{xN_1^5N_2^2N_3^{-1}} + N_1N_2N_3^{1/2} + x^{\varphi} \\ \ll\ & \sqrt[16]{xN_1^9N_2^4N_3^5} + \sqrt[44]{x^3N_1^{23}N_2^{19}} \\ & + \sqrt[12]{xN_1^5N_2} + N_1N_2N_3^{1/2} + x^{\varphi}, \end{aligned} \tag{19}$$

$$x^{-\eta}S(a,b,c,d;\boldsymbol{N}) \ll \sqrt[120]{x^{11}N_1^{43}N_2^{18}N_3^{-11}} + \sqrt[24]{x^2N_1^{10}N_2^6N_3^{-2}} + \sqrt[20]{xN_1^{13}N_2^{15}N_3^{-1}} + \sqrt[160]{x^{11}N_1^{83}N_2^{78}N_3^{-11}} + \sqrt[45]{x^4N_1^{17}N_2^7N_3^{-4}} + \sqrt[10]{xN_1^3N_3^{-1}} + \sqrt[16]{xN_1^9N_2^{10}N_3^{-1}} + x^{\varphi}, \tag{20}$$

and

$$x^{-\eta}S(a,b,c,d;\boldsymbol{N}) \ll \sqrt[8]{xN_1^{-3}N_2^{-2}N_3^{-1}} + x^{\varphi}. \tag{21}$$

It remains to deduce the required estimate from (19)–(21).

From (19) and (21) it is seen that

$$x^{-\eta}S(a,b,c,d;\boldsymbol{N}) \ll \sqrt[16]{xN_1^9N_2^4N_3^5} + E_1 + E_2 + E_3 + x^{\varphi},$$

where

$$E_1 = \min(\sqrt[44]{x^3(N_1N_2)^{21}}, \sqrt[8]{x(N_1N_2)^{-3}}) \le x^{0.1},$$
$$E_2 = \min(\sqrt[12]{x(N_1N_2)^3}, \sqrt[8]{x(N_1N_2)^{-3}}) \le x^{0.1},$$
$$E_3 = \min(N_1N_2N_3^{1/2}, \sqrt[8]{xN_1^{-3}N_2^{-2}N_3^{-1}}) \ll (xN_1^{-1})^{0.1} \le x^{0.1}.$$

Thus

$$x^{-\eta}S(a,b,c,d;\boldsymbol{N}) \ll \sqrt[16]{xN_1^9N_2^4N_3^5} + x^{0.1}. \tag{22}$$

From (20) and (22) we obtain

$$x^{-\eta}S(a,b,c,d;\boldsymbol{N}) \ll \sum_{4\le i\le 10} E_i + x^{0.1}, \tag{23}$$

where

$$E_4 = \min(\sqrt[10]{xN_1^3N_3^{-1}}, \sqrt[16]{xN_1^9N_2^4N_3^5}) \ll \sqrt[33]{x^3N_1^{12}N_2^2}, \tag{24}$$

$$E_5 = \min(\sqrt[120]{x^{11}N_1^{43}N_2^{18}N_3^{-18}}, \sqrt[16]{xN_1^9N_2^4N_3^5}) \ll \sqrt[888]{x^{73}N_1^{377}N_2^{162}}, \tag{25}$$

$$E_6 = \min(\sqrt[12]{xN_1^5N_2^3N_3^{-1}}, \sqrt[16]{xN_1^9N_2^4N_3^5}) \ll \sqrt[76]{x^6N_1^{34}N_2^{19}}, \tag{26}$$

$$E_7 = \min(\sqrt[20]{xN_1^{13}N_2^{15}N_3^{-1}}, \sqrt[16]{xN_1^9N_2^4N_3^5}) \ll \sqrt[116]{x^6N_1^{74}N_2^{79}}, \tag{27}$$

$$E_8 = \min(\sqrt[160]{x^{11}N_1^{83}N_2^{78}N_3^{-11}}, \sqrt[16]{xN_1^9N_2^4N_3^5}) \ll \sqrt[488]{x^{33}N_1^{257}N_2^{217}}, \tag{28}$$

$$E_9 = \min(\sqrt[45]{x^4N_1^{17}N_2^7N_3^{-4}}, \sqrt[16]{xN_1^9N_2^4N_3^5}) \ll \sqrt[289]{x^{24}N_1^{121}N_2^{51}}, \tag{29}$$

$$E_{10} = \min(\sqrt[16]{xN_1^9N_2^{10}N_3^{-1}}, \sqrt[16]{xN_1^9N_2^4N_3^5}) \ll \sqrt[96]{x^6(N_1N_2)^{54}}. \tag{30}$$

By (21) and (22) we get

$$(31)\quad x^{-\eta}S(a,b,c,d;\boldsymbol{N}) \ll \min\big(\sqrt[16]{xN_1^9N_2^4N_3^5},\ \sqrt[8]{xN_1^{-3}N_2^{-2}N_3^{-1}}\big) + x^{0.1} \ll \sqrt[28]{(xN_1^{-1}N_2^{-1})^3} + x^{0.1}.$$

For brevity we set $J = N_1N_2$. From (23) to (31) we find that

$$x^{-\eta}S(a,b,c,d;\boldsymbol{N}) \ll \sum_{11\le i\le 17} E_i + x^{0.1},$$

where

$$\begin{aligned}
E_{11} &= \min\big(\sqrt[28]{(xJ^{-1})^3},\ \sqrt[33]{x^3J^7}\big) \ll x^{6/59},\\
E_{12} &= \min\big(\sqrt[28]{(xJ^{-1})^3},\ \sqrt[888]{x^{73}J^{269.5}}\big) \ll x^{0.1007},\\
E_{13} &= \min\big(\sqrt[28]{(xJ^{-1})^3},\ \sqrt[76]{x^6J^{26.5}}\big) \ll x^{0.1006},\\
E_{14} &= \min\big(\sqrt[28]{(xJ^{-1})^3},\ \sqrt[116]{x^6J^{79}}\big) \ll x^{0.1},\\
E_{15} &= \min\big(\sqrt[28]{(xJ^{-1})^3},\ \sqrt[488]{x^{33}J^{237}}\big) \ll x^{0.1},\\
E_{16} &= \min\big(\sqrt[28]{(xJ^{-1})^3},\ \sqrt[289]{x^{24}J^{86}}\big) \ll x^{0.1008},\\
E_{17} &= \min\big(\sqrt[28]{(xJ^{-1})^3},\ \sqrt[48]{x^3J^{27}}\big) \ll x^{0.1},
\end{aligned}$$

as required.

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