ACTA ARITHMETICA
LXVII.2 (1994)

On special values of generalized p-adic hypergeometric functions

by

Kaori Ota (Tokyo)

We generalize results for ratios of generalized hypergeometric functions obtained by P. T. Young [9] to the case of a power of p. More precisely, we generalize Theorems 3.2 and 3.4 in [9]. In order to do that, introducing a generalization of the p-adic Gamma function to a power of p seems to simplify expressions of the results, and this is done in Section 1. There we prove properties of this generalized Gamma function. In Section 2, after proving some preliminary results for certain ratios of binomial coefficients and giving a "truncated" version of Dwork's theorem, we prove the theorems. We follow very closely the proofs of the corresponding theorems in [9], and our methods also apply to the well-poised series (Theorems 3.1 and 3.3 in [9]) treated in [10]. Some applications are also given in [10].

Throughout this paper, p is a prime, ν_p denotes the p-adic valuation normalized by $\nu_p(p) = 1$ and by the p-adic expansion we mean the standard p-adic expansion whose coefficients are all in $\{0, 1, \ldots, p-1\}$.

1. The p-adic Gamma function is defined by

$$\Gamma_p(n+1) = (-1)^{n+1} \prod_{\substack{j=1 \\ p \nmid j}}^{n} j$$

for a positive integer n and is extended by continuity to all of $\mathbb{Z}_p$:

$$\Gamma_p : \mathbb{Z}_p \to \mathbb{Z}_p^*$$

(cf. [6]). Here $\mathbb{Z}_p$ is the ring of p-adic integers.

For a non-negative integer l, we define a map h_l from $\mathbb{Z}_p$ to $\mathbb{Z}_p$ by

$$h_l(x) = \sum_{j \geq l} x_j p^{j-l}$$

for $x \in \mathbb{Z}_p$ with the p-adic expansion $\sum_{j=0}^{\infty} x_j p^j$. Then h_l is obviously continuous. Let t be a positive integer and $q = p^t$.

DEFINITION. For $x \in \mathbb{Z}_p$, we define a map Γ_q from $\mathbb{Z}_p$ to $\mathbb{Z}_p^*$ by

$$\Gamma_q(x+1) = \prod_{l=0}^{t-1} \Gamma_p(h_l(x)+1).$$

Then Γ_q is continuous and, for $t=1$, Γ_q coincides with Γ_p. N. Koblitz constructed and studied an extension of the p-adic Gamma function (cf. [4], [5]). There he uses Γ_q for a different function, but this will cause no confusion.

Γ_q has the following properties analogous to Γ_p.

PROPOSITION 1. (1) *If* $x \equiv y \pmod{p^n}$ *for* $n \geq t-1$, *then*

$$\Gamma_q(x) \equiv \Gamma_q(y) \pmod{p^{n-(t-1)}}.$$

(2) *For a positive integer* n,

$$\Gamma_q(n+1) = (-1)^A p^{-B} \prod_{\substack{j=1 \\ q \nmid j}}^{n} j,$$

where

$$A = t + \sum_{i=0}^{t-1} \left[\frac{n}{p^i}\right] \quad \text{and} \quad B = \sum_{i=1}^{t} \left[\frac{n}{p^i}\right] - t\left[\frac{n}{p^t}\right],$$

and $[x]$ *denotes the largest integer not exceeding* x.

(3) *We have*

$$\Gamma_q(x+1) = \begin{cases} (-1)^{k+1} h_k(x) \Gamma_q(x) & \text{if } x \notin q\mathbb{Z}_p \text{ and } k = \nu_p(x), \\ (-1)^t \Gamma_q(x) & \text{if } x \in q\mathbb{Z}_p. \end{cases}$$

(4) *For* p *odd*,

$$\Gamma_q(x)\Gamma_q(1-x) = (-1)^{t-1+R_t(x)},$$

where $R_t(x)$ *is the integer in* $\{1, 2, \ldots, q\}$ *satisfying* $R_t(x) \equiv x \pmod q$. *For* $p = 2$,

$$\Gamma_q(x)\Gamma_q(1-x) = \begin{cases} (-1)^{t+1+x_t} & \text{if } x \notin q\mathbb{Z}_p \text{ and } x \notin \mathbb{Z}_p^*, \\ (-1)^{t+x_t} & \text{if } x \in q\mathbb{Z}_p \text{ or } x \in \mathbb{Z}_p^*, \end{cases}$$

where x_t *is the coefficient of* p^t *in the* p*-adic expansion of* x.

(5) *Let* N *be an integer* ≥ 2, *prime to* p. *Then*

$$\prod_{i=0}^{N-1} \Gamma_q\left(\frac{x+i}{N}\right) = \Gamma_q(x) \prod_{i=1}^{N-1} \Gamma_q\left(\frac{i}{N}\right) g_N(x)^{-1},$$

where $g_N(x) = N^{R_t(x)-1} N^{(q-1)xR_t'(x)}$ *with* $R_t'(x) = (x - R_t(x))/q$.

(6) *For Gauss sums, we have the following. For p odd and j in $\{1,2,\ldots,q-2\}$,*

$$\Gamma_q\left(\frac{j}{1-q}+1\right)=(-1)^t q^{-1}\pi^{s(j)}\tau_q(\omega_q^j,\psi_{\pi,q}).$$

Here τ_q is the Gauss sum defined by

$$\tau_q(\chi,\psi)=-\sum_{x\in\mathbb{F}_q}\chi(x)\psi(x),$$

$\pi^{p-1}=-p$, $\psi_{\pi,q}(x)=\zeta_\pi^{tr(x)}$ *is the additive character of $\mathbb{F}_q$ with the p-th root of unity ζ_π satisfying $\zeta_\pi\equiv 1+\pi \pmod{\pi^2}$, ω_q is the Teichmüller character, and $s(j)=\sum_{i=0}^{t-1} j_i$ for the p-adic expansion $j=\sum_{i=0}^{t-1} j_i p^i$.*

Proof. We use the properties of Γ_p (cf. [6], Chap. 14).

(1) If $x\equiv y \pmod{p^n}$, then $\Gamma_p(x)\equiv\Gamma_p(y) \pmod{p^n}$, so this is obvious.

(2) We have

$$\begin{aligned}\prod_{\substack{j=1\\ q\nmid j}}^{n} j &= \prod_{\substack{j=1\\ p\nmid j}}^{n} j \prod_{\substack{j=1\\ p\nmid j}}^{[n/p]}(pj)\ldots\prod_{\substack{j=1\\ p\nmid j}}^{[n/p^{t-1}]}(p^{t-1}j)\\ &=(-1)^{n+1}\Gamma_p(n+1)(-1)^{[n/p]+1}p^{[n/p]-[n/p^2]}\Gamma_p([n/p]+1)\ldots\\ &\quad\ldots(-1)^{[n/p^{t-1}]+1}(p^{t-1})^{[n/p^{t-1}]-[n/p^t]}\Gamma_p([n/p^{t-1}]+1).\end{aligned}$$

Since

$$\prod_{i=0}^{t-1}\Gamma_p([n/p^i]+1)=\prod_{i=0}^{t-1}\Gamma_p(h_i(n)+1)=\Gamma_q(n+1),$$

the result follows.

(3) Γ_p satisfies the following functional equation:

$$\Gamma_p(x+1)=\begin{cases}-x\Gamma_p(x) & \text{if } x\in\mathbb{Z}_p^*,\\ -\Gamma_p(x) & \text{if } x\in p\mathbb{Z}_p.\end{cases}$$

Set $f(x)$ to be $\Gamma_q(x+1)$. When $k=\nu_p(x+1)<\infty$, let $x+1=\sum_{j\ge k} y_j p^j$ be the p-adic expansion of $x+1$. Then for $l\le k$, we have

$$\begin{aligned}h_l(x)&=(p-1)+\ldots+(p-1)p^{k-1-l}+(y_k-1)p^{k-l}+y_{k+1}p^{k+1-l}+\ldots\\ &=-1+h_l(x+1).\end{aligned}$$

For $l>k$, we have $h_l(x)=h_l(x+1)$.

1. If $k\ge t$, then

$$f(x+1)=(-1)^t\prod_{l=0}^{t-1}\Gamma_p(h_l(x)+1)=(-1)^t f(x).$$

This also holds for $k=\infty$.

2. If $k < t$, then

$$f(x+1) = (-1)^{k+1} h_k(x+1) \prod_{l=0}^{t-1} \Gamma_p(h_l(x)+1) = (-1)^{k+1} h_k(x+1) f(x).$$

By substituting $x-1$ for x, the result follows.

(4) Γ_p satisfies the following:

$$\Gamma_p(x)\Gamma_p(1-x) = \begin{cases} (-1)^{R_1(x)} & \text{if } p > 2, \\ \varepsilon(x) & \text{if } p = 2, \end{cases}$$

where

$$\varepsilon(x) = \begin{cases} -1 & \text{if } x \equiv 0, 1 \pmod 4, \\ 1 & \text{if } x \equiv 2, 3 \pmod 4. \end{cases}$$

Assume that $k = \nu_p(x) < \infty$ and let $x = \sum_{l=k}^{\infty} x_l p^l$ be the p-adic expansion of x. Then for $l \le k$, we have

$$h_l(-x) = (p - x_k)p^{k-l} + \sum_{j \ge k+1} (p - x_j - 1)p^{j-l} = -h_l(x).$$

For $l > k$, we have

$$h_l(-x) = -1 - h_l(x).$$

1. If $k \ge t$, then

$$\begin{aligned} \Gamma_q(x+1)\Gamma_q(1-x) &= \prod_{l=0}^{t-1} \{\Gamma_p(h_l(x)+1)\Gamma_p(-h_l(x)+1)\} \\ &= (-1)^t \prod_{l=0}^{t-1} \{\Gamma_p(h_l(x))\Gamma_p(1-h_l(x))\} \\ &= \begin{cases} (-1)^t \prod_{l=0}^{t-1} (-1)^p = 1 & \text{for } p > 2, \\ (-1)^t \prod_{l=0}^{t-1} \varepsilon(h_l(x)) & \text{for } p = 2. \end{cases} \end{aligned}$$

Since $\Gamma_q(x+1) = (-1)^t \Gamma_q(x)$, we have

$$\Gamma_q(x)\Gamma_q(1-x) = \begin{cases} (-1)^t & \text{if } p > 2, \\ (-1)^{t+x_t} & \text{if } p = 2. \end{cases}$$

This also holds for $k = \infty$.

2. If $k < t$, then

$$\begin{aligned} &\Gamma_q(x+1)\Gamma_q(1-x) \\ &\quad = \prod_{l=0}^{t-1} \Gamma_p(h_l(x)+1) \prod_{l=0}^{k} \Gamma_p(-h_l(x)+1) \prod_{l=k+1}^{t-1} \Gamma_p(-h_l(x)) \end{aligned}$$

$$= \Big\{\prod_{l=0}^{t-1} \Gamma_p(h_l(x)+1)\Big\}(-1)^{k+1}(-h_k(x))\prod_{l=0}^{t-1}\Gamma_p(-h_l(x))$$

$$= \begin{cases} (-1)^k h_k(x)(-1)^{k+(p-x_k)+\ldots+(p-x_{t-1})} & \text{if } p>2,\\ (-1)^k h_k(x)\prod_{l=0}^{t-1}\varepsilon(-h_l(x)) & \text{if } p=2.\end{cases}$$

Since $\Gamma_q(x+1) = (-1)^{k+1}h_k(x)\Gamma_q(x)$, we have

$$\Gamma_q(x)\Gamma_q(1-x) = \begin{cases} (-1)^{t-1+R_t(x)} & \text{if } p>2,\\ (-1)^{t+x_t} & \text{if } p=2 \text{ and } x\in\mathbb{Z}_p^*,\\ (-1)^{t-1+x_t} & \text{if } p=2 \text{ and } x\notin\mathbb{Z}_p^*.\end{cases}$$

(5) Set

$$g_N(x) = \prod_{i=1}^{N-1}\Gamma_q\Big(\frac{i}{N}\Big)\prod_{i=0}^{N-1}\Gamma_q\Big(\frac{x+i}{N}\Big)^{-1}\Gamma_q(x).$$

Then for non-zero x,

$$\frac{g_N(x+1)}{g_N(x)} = \frac{\Gamma_q(x+1)\Gamma_q(x/N)}{\Gamma_q(x)\Gamma_q((x+N)/N)}$$
$$= \begin{cases} h_k(x)/h_k(x/N) & \text{if } x\notin q\mathbb{Z}_p \text{ and } \nu_p(x)=k,\\ 1 & \text{if } x\in q\mathbb{Z}_p.\end{cases}$$

If $\nu_p(x)=k$, then $h_k(x) = N\cdot h_k(x/N)$, so

$$\frac{g_N(x+1)}{g_N(x)} = \begin{cases} N & \text{if } x\notin q\mathbb{Z}_p,\\ 1 & \text{if } x\in q\mathbb{Z}_p.\end{cases}$$

By continuity in x, we only need to consider positive integers. Hence

$$g_N(n) = \begin{cases} N\cdot g_N(n-1) & \text{if } n-1\notin q\mathbb{Z}_p,\\ g_N(n-1) & \text{if } n-1\in q\mathbb{Z}_p\end{cases}$$
$$= N^{n-1-[(n-1)/q]}g_N(1).$$

As $g_N(1)=1$ and $[(n-1)/q] = R_t'(n)$, we get the result.

(6) By the Gross–Koblitz formula (cf. [6], Chap. 15),

$$(-1)^t q^{-1}\pi^{s(j)}\tau_q(\omega_q^j,\psi_{\pi,q}) = \prod_{l=0}^{t-1}\Gamma_p\Big(1-\Big\langle\frac{p^l j}{q-1}\Big\rangle\Big).$$

But

$$1-\Big\langle\frac{p^{t-l}j}{q-1}\Big\rangle = 1+h_l\Big(\frac{j}{1-q}\Big),$$

so the result follows. ∎

Remarks. 1. Just like Γ_p,

$$\prod_{i=1}^{N-1} \Gamma_q\Big(\frac{i}{N}\Big) = \begin{cases} \pm 1 & \text{if } N \text{ is odd,} \\ \pm 1 \text{ or } \pm i & \text{if } N \text{ is even.} \end{cases}$$

2. For $0 < a, b < a + b < q - 1$,

$$J(\omega_q^{-a}, \omega_q^{-b}) = \frac{\tau_q(\omega_q^{-a}, \psi_{\pi,q})\tau_q(\omega_q^{-b}, \psi_{\pi,q})}{\tau_q(\omega_q^{-(a+b)}, \psi_{\pi,q})}$$
$$= (-p)^{\nu_p(\binom{a+b}{b})} \frac{\Gamma_q\big(\frac{a}{q-1}\big)\Gamma_q\big(\frac{b}{q-1}\big)}{\Gamma_q\big(\frac{a+b}{q-1}\big)},$$

where $J(\chi_1, \chi_2)$ denotes the Jacobi sum. This follows from (6).

2. We generalize theorems in [9] to a power of p. Let p be an odd prime and let q denote p^t as before.

PROPOSITION 2. *Let a and b be integers satisfying $0 < a < b < q - 1$. For $r \geq 0$, set*

$$n_r = b\frac{q^r - 1}{q - 1} \quad \text{and} \quad m_r = a\frac{q^r - 1}{q - 1}.$$

Then for $r > 0$,

$$\frac{\binom{n_r}{m_r}}{\binom{n_{r-1}}{m_{r-1}}} = (-1)^t(-p)^{\nu_p(\binom{b}{a})} \frac{\Gamma_q(1 + n_r)}{\Gamma_q(1 + m_r)\Gamma_q(1 + n_r - m_r)}$$
$$= (-p)^{\nu_p(\binom{b}{a})} \frac{\Gamma_q(-m_r)\Gamma_q(m_r - n_r)}{\Gamma_q(-n_r)}.$$

Proof. From Proposition 1(2), we have

$$\Gamma_q(1 + n_r) = (-1)^{t+\Sigma_{i=0}^{t-1}[n_r/p^i]} p^{-(\Sigma_{i=1}^{t}[n_r/p^i] - t[n_r/p^t])} \prod_{\substack{j=1 \\ q\nmid j}}^{n_r} j,$$

and from

$$\prod_{j=1}^{n_r} j = \Big(\prod_{\substack{j=1 \\ q\nmid j}}^{n_r} j\Big)\Big(\prod_{j=1}^{n_{r-1}} (qj)\Big),$$

we have

$$\frac{n_r!}{n_{r-1}!} = q^{n_{r-1}} \prod_{\substack{j=1 \\ q\nmid j}}^{n_r} j.$$

Hence

$$\frac{\binom{n_r}{m_r}}{\binom{n_{r-1}}{m_{r-1}}} = \Big(q^{n_{r-1}} \prod_{\substack{j=1 \\ q\nmid j}}^{n_r} j\Big)\Big(q^{m_{r-1}} \prod_{\substack{j=1 \\ q\nmid j}}^{m_r} j\Big)^{-1}\Big(q^{n_{r-1}-m_{r-1}} \prod_{\substack{j=1 \\ q\nmid j}}^{n_r-m_r} j\Big)^{-1}$$

$$= (-1)^{e_1} p^{e_2} \frac{\Gamma_q(1+n_r)}{\Gamma_q(1+m_r)\Gamma_q(1+n_r-m_r)},$$

where

$$e_1 = \sum_{i=0}^{t-1}\left[\frac{n_r}{p^i}\right] + \sum_{i=0}^{t-1}\left[\frac{m_r}{p^i}\right] + \sum_{i=0}^{t-1}\left[\frac{n_r-m_r}{p^i}\right] + 3t,$$

$$e_2 = \sum_{i=1}^{t}\left[\frac{n_r}{p^i}\right] - \sum_{i=1}^{t}\left[\frac{m_r}{p^i}\right] - \sum_{i=1}^{t}\left[\frac{n_r-m_r}{p^i}\right] - t\left[\frac{n_r}{p^t}\right] + t\left[\frac{m_r}{p^t}\right] + t\left[\frac{n_r-m_r}{p^t}\right].$$

From simple calculations, we can show that $e_1 \equiv \nu_p\big(\binom{b}{a}\big) + t \pmod 2$, and $e_2 = \nu_p\big(\binom{b}{a}\big)$.

Also from Proposition 1(4),

$$\Gamma_q(1+n_r) = (-1)^{t-1+R_t(-n_r)}\Gamma_q(-n_r)^{-1} = (-1)^{t-1+q-b}\Gamma_q(-n_r)^{-1}$$

and similar identities hold, so we get the result. ■

The following corollary has been given in [10] in a more general form.

COROLLARY 3 ([10], Theorem 2.2 and Corollary 2.3). *With the same notations and assumptions as in Proposition* 2, *for* $r > 0$,

$$\frac{\binom{n_r}{m_r}}{\binom{n_{r-1}}{m_{r-1}}} \equiv J(\omega_q^{-a}, \omega_q^{a-b}) \pmod{p^{tr-(t-1)+\nu_p(\binom{b}{a})}}.$$

P r o o f. Since

$$-n_r \equiv \frac{b}{q-1} \pmod{q^r}, \qquad -m_r \equiv \frac{a}{q-1} \pmod{q^r}$$

and

$$m_r - n_r \equiv \frac{b-a}{q-1} \pmod{q^r},$$

from Proposition 1(1) we get

$$\Gamma_q(-n_r) \equiv \Gamma_q\Big(\frac{b}{q-1}\Big) \pmod{p^{tr-(t-1)}}, \quad \text{and so on.}$$

Hence the result follows from Remark 2 after Proposition 1. ■

Remark. From Proposition 2, for
$$n_r = b\frac{q^r-1}{q-1} \quad \text{and} \quad m_r = a\frac{q^r-1}{q-1}$$
with $0 < a < b < q-1$,
$$\nu_p\left(\binom{n_r}{m_r}\right) = r\cdot\nu_p\left(\binom{b}{a}\right).$$
Note also that from $\nu_p(n!) = (n - s(n))/(p-1)$,
$$\nu_p\left(\binom{b}{a}\right) = 0 \ \Leftrightarrow\ s(b) = s(a) + s(b-a).$$

As in [1] and [9], for α in $\mathbb{Z}_p$ we define α' and μ_α as follows:
$$p\alpha' - \alpha = \mu_\alpha \in \{0,1,\ldots,p-1\}.$$
Also for $i \ge 1$, we define
$$\alpha^{(i)} = (\alpha^{(i-1)})' \quad \text{and} \quad \mu_\alpha^{(i)} = \mu_{\alpha^{(i)}}.$$
Then the following properties hold:

1. If $-\alpha = \sum_{i=0}^{\infty} a_i p^i$ is the p-adic expansion of $-\alpha$, then
$$\mu_\alpha^{(i)} = a_i \quad \text{and} \quad \alpha^{(i)} = -\sum_{j=i}^{\infty} a_j p^{j-i} = -h_i(-\alpha).$$

2. For $\alpha = a/(q-1)$ with $0 \le a = \sum_{i=0}^{t-1} a_i p^i \le q-1$,
$$\mu_\alpha^{(i)} = a_j, \qquad \alpha^{(i)} = \frac{a_j + a_{j+1}p + \ldots + a_{j-1}p^{t-1}}{q-1} \qquad \text{if } i \equiv j \pmod t$$
with $j \in \{0,1,\ldots,t-1\}$, and $\alpha^{(t)} = \alpha$.

3. For $n_r = a(q^r-1)/(q-1)$ with $0 \le a \le q-1$, $(-n_r)^{(t)} = -n_{r-1}$.

The generalized hypergeometric function ${}_kF_l(\alpha_1,\alpha_2,\ldots,\alpha_k;\beta_1,\beta_2,\ldots\ldots,\beta_l;X)$ is defined by
$${}_kF_l\left(\begin{matrix}\alpha_1,\alpha_2,\ldots,\alpha_k \\ \beta_1,\beta_2,\ldots,\beta_l\end{matrix}\ ;X\right) = \sum_{s=0}^{\infty}\frac{(\alpha_1)_s(\alpha_2)_s\ldots(\alpha_k)_s}{(\beta_1)_s(\beta_2)_s\ldots(\beta_l)_s\cdot s!}X^s,$$
where $(\alpha)_n = \alpha(\alpha+1)\ldots(\alpha+n-1)$ for $n > 0$ and $(\alpha)_0 = 1$.

For $i \ge 0$, define
$${}_kF_l^{(i)}\left(\begin{matrix}\alpha_1,\alpha_2,\ldots,\alpha_k \\ \beta_1,\beta_2,\ldots,\beta_l\end{matrix}\ ;X\right) = {}_kF_l\left(\begin{matrix}\alpha_1^{(i)},\alpha_2^{(i)},\ldots,\alpha_k^{(i)} \\ \beta_1^{(i)},\beta_2^{(i)},\ldots,\beta_l^{(i)}\end{matrix}\ ;X\right).$$
Also as in [9], we denote by
$${}_k\mathcal{F}_l^{(t)}\left(\begin{matrix}\alpha_1,\alpha_2,\ldots,\alpha_k \\ \beta_1,\beta_2,\ldots,\beta_l\end{matrix}\ ;X\right)$$

the analytic element which extends the ratio

$$\frac{{}_kF_l\left(\begin{matrix}\alpha_1,\alpha_2,\ldots,\alpha_k\\ \beta_1,\beta_2,\ldots,\beta_l\end{matrix};X\right)}{{}_kF_l^{(t)}\left(\begin{matrix}\alpha_1,\alpha_2,\ldots,\alpha_k\\ \beta_1,\beta_2,\ldots,\beta_l\end{matrix};X^q\right)}.$$

For a series $F(X)=\sum_{s=0}^{\infty}A(s)X^s$, set $F_n(X)=\sum_{s=0}^{p^n-1}A(s)X^s$.

PROPOSITION 4. *Let α and β be elements of $\mathbb{Z}_p$ such that if $-\alpha=\sum_{i=0}^{\infty}a_ip^i$ and $-\beta=\sum_{i=0}^{\infty}b_ip^i$ are p-adic expansions, then*

$$a_i+b_i<p \quad \textit{for any } i. \tag{4.1}$$

Then

$${}_2\mathcal{F}_1^{(t)}\left(\begin{matrix}\alpha,\beta\\ 1\end{matrix};1\right)=\frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)}.$$

Proof. Let

$$S=\{(\sigma,\tau)\in\mathbb{Z}_p\times\mathbb{Z}_p\mid p\text{-adic expansions of }-\sigma\text{ and }-\tau\text{ satisfy (4.1)}\}.$$

Note that if $(\sigma,\tau)\in S$, then $(\sigma',\tau')\in S$, so $(\sigma^{(i)},\tau^{(i)})\in S$ for any i.

From Theorem 2 in [3],

$$\Theta(\alpha^{(i)},\beta^{(i)};1)=\frac{\Gamma_p(\alpha^{(i)})\Gamma_p(\beta^{(i)})}{\Gamma_p(\alpha^{(i)}+\beta^{(i)})},$$

where $\Theta(\sigma,\tau;X)$ is the analytic element which extends the ratio

$$\frac{{}_2F_1\left(\begin{matrix}\sigma,\tau\\ 1\end{matrix};X\right)}{{}_2F_1^{(1)}\left(\begin{matrix}\sigma,\tau\\ 1\end{matrix};X^p\right)}.$$

Then we have

$$\alpha^{(i)}+\beta^{(i)}=(\alpha+\beta)^{(i)} \quad \text{for any } i,\quad \prod_{i=0}^{t-1}\Gamma_p(\alpha^{(i)})=\Gamma_q(\alpha),$$

$$\prod_{i=0}^{t-1}\Gamma_p(\beta^{(i)})=\Gamma_q(\beta) \quad\text{and}\quad \prod_{i=0}^{t-1}\Gamma_p((\alpha+\beta)^{(i)})=\Gamma_q(\alpha+\beta).$$

As

$${}_2\mathcal{F}_1^{(t)}\left(\begin{matrix}\alpha,\beta\\ 1\end{matrix};1\right)=\prod_{i=0}^{t-1}\Theta(\alpha^{(i)},\beta^{(i)};1),$$

the result follows. ∎

Remark. Let $a=\sum_{i=0}^{t-1} a_i p^i$ and $b=\sum_{i=0}^{t-1} b_i p^i$ be p-adic expansions of a and b satisfying $0<a,b<q-1$ and $a+b<q-1$. If $a_i+b_i<p$ for any i with $0\le i\le t-1$, then

$$ {}_2\mathcal{F}_1^{(t)}\left(\begin{matrix}\frac{a}{q-1},\frac{b}{q-1}\\ 1\end{matrix};1\right)=J(\omega_q^{-a},\omega_q^{-b}) $$

(cf. (3.3) in [9]).

Set $\alpha=a/(q-1)$ and $\beta=b/(q-1)$. Then $(\alpha,\beta)\in S$ and $\nu_p\big(\binom{a+b}{b}\big)=0$, so the statement holds from Remark 2 after Proposition 1 and Proposition 4.

In order to use a uniform convergence argument as in [9], we seem to need not only Dwork's theorem (Theorem 1.1 in [2], Theorem 2.3 in [9]) but also a "truncated" version of it, which is the following:

THEOREM 5. *For an integer* $r\ge 0$, *let* $A^{(r)}(m)$ *and* $g_r(m)$ *take values in* $\mathbb{Z}_p$ *for non-negative integers* m. *Moreover, suppose that* $A^{(r)}$ *and* g_r *satisfy the following*:

1. $|A^{(r)}(0)|=1$;
2. $A^{(r)}(m)\in g_r(m)\mathbb{Z}_p$ *for* $m\ge 0$;
3. *For a positive integer* c *with* $p^L\le c<p^{L+1}$, $L>0$,

$$ A^{(r)}(m)\ne 0 \quad \text{and} \quad g_r(m)\ne 0 \quad \text{for } m\le h_r(c); $$

4. *For integers* s *and* r *with* $0\le r\le L-1$ *and* $0\le s\le L-r-1$, *and for integers* a,μ *and* m *with* $0\le a<p$, $0\le\mu<p^s$ *and* $m\ge 0$, *we have*

$$ \frac{A^{(r)}(a+p\mu+mp^{s+1})}{A^{(r)}(a+p\mu)}-\frac{A^{(r+1)}(\mu+mp^s)}{A^{(r+1)}(\mu)}\in p^{s+1}\frac{g_{s+r+1}(m)}{g_r(a+p\mu)}\mathbb{Z}_p. $$

Then for $0\le s\le L-1$ *and* $m\ge 0$, *we have*

$$ F(X)G_{m,s}(X^p)\equiv G(X^p)F_{m,s+1}(X) \pmod{g_{s+1}(m)p^{s+1}[[X]]}, $$

where

$$ F(X)=\sum_{n=0}^{\infty}A^{(0)}(n)X^n,\qquad G(X)=\sum_{n=0}^{\infty}A^{(1)}(n)X^n, $$

$$ F_{m,s}(X)=\sum_{n=mp^s}^{(m+1)p^s-1}A^{(0)}(n)X^n \quad \text{and} \quad G_{m,s}(X)=\sum_{n=mp^s}^{(m+1)p^s-1}A^{(1)}(n)X^n. $$

Notice that under our assumptions on the $A^{(r)}$, the denominators in hypothesis 4 are non-zero. The proof of Theorem 1.1 in [2] works as well in this case.

THEOREM 6. *Let* $\theta_1,\dots,\theta_n$ *and* $\sigma_1,\dots,\sigma_{n-1}$ *be elements of* $\mathbb{Q}\cap\mathbb{Z}_p$, *and suppose that none of the* σ_i *are zero or negative integers*, $\sigma_i\ne 1$ *for* $1\le i$

$\le q'$ and $\sigma_i = 1$ for $i > q'$. Set

$$F(X) = {}_nF_{n-1}\left(\begin{matrix} \theta_1, \ldots, \theta_n \\ \sigma_1, \ldots, \sigma_{n-1} \end{matrix}\ ; X\right) = \sum_{m=0}^{\infty} A(m) X^m.$$

Suppose further that

(1) *$|\sigma_j^{(\nu)}| = 1$ for $1 \le j \le q'$ and $\nu \ge 0$;*

(2) *for a positive integer c with $p^L \le c < p^{L+1}, L > 0$,*

$$A(m) \begin{cases} = 0 & \text{for } m > c, \\ \ne 0 & \text{for } m \le c, \end{cases} \tag{6.1}$$

i.e., for some i we have $\theta_i = -c$;

(3) *for each fixed i with $0 \le i \le L$, supposing that the indices are rearranged so that $\mu_{\theta_1}^{(i)} \le \ldots \le \mu_{\theta_n}^{(i)}$ and $\mu_{\sigma_1}^{(i)} \le \ldots \le \mu_{\sigma_{q'}}^{(i)}$, we have $\mu_{\sigma_j}^{(i)} > \mu_{\theta_{j+1}}^{(i)}$ for $j = 1, \ldots, q'$;*

(4) *for i with $\theta_i = -c$, $\mu_{\sigma_l}^{(r)} \ne \mu_{\theta_i}^{(r)}$ for l and r with $1 \le l \le q'$ and $1 \le r \le L$.*

Set

$$A^{(s)}(m) = \frac{\prod_{i=1}^{n} (\theta_i^{(s)})_m}{m! \prod_{j=1}^{n-1} (\sigma_j^{(s)})_m} \quad \text{and} \quad g_s(m) = \frac{A^{(s)}(m)}{\prod_{j=1}^{q'} (m + \sigma_j^{(s)})}. \tag{6.2}$$

Then conditions of Theorem 5 *are satisfied and we get*, *for s and m with $0 \le s \le L-1$ and $m \ge 0$,*

$$F(X) F'_{m,s}(X^p) \equiv F'(X^p) F_{m,s+1}(X) \pmod{g_{s+1}(m) p^{s+1}[[X]]},$$

where

$$F'(X) = {}_nF_{n-1}^{(1)}\left(\begin{matrix} \theta_1, \ldots, \theta_n \\ \sigma_1, \ldots, \sigma_{n-1} \end{matrix}\ ; X\right).$$

Notice that from assumption (2) we have

$$A^{(r)}(m) \begin{cases} = 0 & \text{for } m > h_r(c), \\ \ne 0 & \text{for } m \le h_r(c). \end{cases} \tag{6.1'}$$

P r o o f o f T h e o r e m 6. This is a truncated version of Theorem 3.1 of [2] which was proved by using Lemma 1 of [1], and Lemmas 2.1, 2.2, 3.1 and 3.2 of [2]. Here we only give corresponding statements without proofs except for Lemma A.

For an integer a in $\{0, 1, \ldots, p-1\}$ and θ in $\mathbb{Z}_p$, we define a function ϱ by

$$\varrho(a, \theta) = \begin{cases} 1 & \text{if } a > \mu_\theta, \\ 0 & \text{otherwise.} \end{cases}$$

Also we use $\varrho(a+, x)$ to denote the limit of $\varrho(b, x)$ as b approaches a from the right (cf. [2]).

Lemma A ([1], Lemma 1). *Let θ be an element of $\mathbb{Q}\cap\mathbb{Z}_p$ and let a, μ, m and s be integers such that $0\le a<p$ and $\mu,m,s\ge 0$.*

(1) *If* $(\theta)_{a+p\mu+mp^{s+1}}\ne 0$, *then*

$$\frac{(\theta)_{a+p\mu+mp^{s+1}}}{(\theta')_{\mu+mp^s}}\equiv\frac{(\theta)_{mp^{s+1}}}{(\theta')_{mp^s}}\cdot\frac{(\theta)_{a+p\mu}}{(\theta')_{\mu}}\left(1+\frac{mp^s}{\theta'+\mu}\right)^{\varrho(a,\theta)}\pmod{1+p^{s+1}},$$

where we define

$$\left(1+\frac{mp^s}{\theta'+\mu}\right)^{\varrho(a,\theta)}=1\quad\text{for the case } \theta'+\mu=0,$$

and

$$\frac{(\theta)_{mp^{s+1}}}{(\theta')_{mp^s}}\equiv(-p)^{mp^s}\pmod{1+p^{s+1}}.$$

(2) *If* $(\theta)_{a+p\mu}\ne 0$, *then*

$$\nu_p\left(\frac{(\theta)_{a+p\mu}}{(\theta')_{\mu}}\right)=\mu+(1+\nu_p(\mu+\theta'))\varrho(a,\theta),$$

where we define $(1+\nu_p(\mu+\theta'))\varrho(a,\theta)=0$ *for* μ *with* $\mu+\theta'=0$. (We show in the proof that when $\mu+\theta'=0$, $\varrho(a,\theta)=0$.)

Proof. When θ is not zero nor a negative integer, this is Lemma 1 in [1]. For $\theta=0$, we have $a=\mu=m=\theta'=0$, so the lemma holds.

For a negative integer θ, the proof of (1) goes exactly in the same way as that of Lemma 1, so we only prove (2). Let $\theta'-1=\sum_{s=0}^{\infty}\beta_s p^s$ be the p-adic expansion and α be such that $\theta-1\equiv\alpha\pmod p$, $\alpha\in\{0,1,\ldots,p-1\}$. Then

$$\theta=1+\alpha+\sum_{s=0}^{\infty}\beta_s p^{s+1}.$$

For an integer $r\ge 2$, set

$$\theta_r=1+\alpha+\sum_{s=0}^{r-2}\beta_s p^{s+1}\quad\text{and}\quad\theta'_r=1+\sum_{s=0}^{r-1}\beta_s p^s.$$

As θ_r and θ'_r approach θ and θ' respectively as $r\to\infty$, and as $(\theta)_{a+p\mu}\ne 0$, there exists N such that for $r\ge N$,

$$\nu_p((\theta_r)_{a+p\mu})=\nu_p((\theta)_{a+p\mu})\quad\text{and}\quad\nu_p((\theta'_r)_{\mu})=\nu_p((\theta')_{\mu}).$$

We take N so large that for $r\ge N$, $\beta_s=p-1$ for any $s\ge r-1$. As in the proof of Lemma 1 of [1], the left side of (2) equals

$$(*)=\frac{1}{p-1}\{a+\alpha+(p-1)\mu-\beta_{r-1}-s(\theta_r+a+p\mu-1)+s(\theta'_r+\mu-1)\},$$

where for a non-negative integer n with the p-adic expansion $n=\sum_{i=0}^{k}n_i p^i$, $s(n)=\sum_{i=0}^{k}n_i$.

1. If $a+\alpha < p$, then $\varrho(a,\theta)=0$, and

$$\theta_r + a + p\mu - 1 = a + \alpha + pA, \quad \theta'_r + \mu - 1 = A + \beta_{r-1}p^{r-1},$$

where $A = \sum_{s=0}^{r-2}\beta_s p^s + \mu$.

Suppose that $A \ge p^{r-1}$. Then for $r \ge N$,

$$\theta = 1 + \alpha + \sum_{s=0}^{r-2}\beta_s p^{s+1} - p^r$$

and

$$\theta + (a + p\mu - 1) = a + \alpha + pA - p^r \ge a + \alpha \ge 0.$$

This means that $(\theta)_{a+p\mu} = 0$, which contradicts our assumption. Hence $A < p^{r-1}$ and we have

$$s(\theta_r + a + p\mu - 1) = a + \alpha + s(A) \quad \text{and} \quad s(\theta'_r + \mu - 1) = s(A) + \beta_{r-1},$$

so $(*) = \mu$ as desired.

2. If $a+\alpha \ge p$ then $\varrho(a,\theta)=1$, and

$$\theta_r + a + p\mu - 1 = (a + \alpha - p) + pB, \quad \theta'_r + \mu = B + \beta_{r-1}p^{r-1},$$

where $B = \mu + 1 + \sum_{s=0}^{r-2}\beta_s p^s = A + 1$.

Suppose that $B \ge p^{r-1}$. Then as before we get a contradiction. So $B < p^{r-1}$ and

$$s(\theta_r + a + p\mu - 1) = a + \alpha - p + s(B) \quad \text{and} \quad s(\theta'_r + \mu) = s(B) + \beta_{r-1}.$$

If $\mu + \theta' \ne 0$, then the same proof as for Lemma 1 of [1] works. If $\mu + \theta' = 0$, then

$$\theta + a + p\mu - 1 = p\theta' - \mu_\theta + a + p\mu - 1 = a - \mu_\theta - 1 < 0,$$

since $(\theta)_{a+p\mu} \ne 0$, hence $\varrho(a,\theta) = 0$. ∎

For Lemmas B through E, let $\theta_1,\ldots,\theta_n$ and $\sigma_1,\ldots,\sigma_{n-1}$ be elements of $\mathbb{Q}\cap\mathbb{Z}_p$, and suppose that none of the σ_i are zero or negative integers, $\sigma_i \ne 1$ for $1 \le i \le q'$ and $\sigma_i = 1$ for $i > q'$. Let g_s be defined by (6.2). We put for $\nu \ge 0$,

$$N_{\theta^{(\nu)}}(a) = \sum_{i=1}^{n}\varrho(a,\theta_i^{(\nu)}), \quad N_{\theta^{(\nu)}}(a+) = \sum_{i=1}^{n}\varrho(a+,\theta_i^{(\nu)}),$$
$$N_{\sigma^{(\nu)}}(a) = \sum_{j=1}^{q'}\varrho(a,\sigma_j^{(\nu)}), \quad N_{\sigma^{(\nu)}}(a+) = \sum_{j=1}^{q'}\varrho(a+,\sigma_j^{(\nu)}).$$

In particular, we put

$$N_\theta(a) = N_{\theta^{(0)}}(a) \quad \text{and} \quad N_\sigma(a+) = N_{\sigma^{(0)}}(a+).$$

Lemma B ([2], Lemma 2.1). *For integers a and μ with $0 \le a < p$ and $\mu \ge 0$, if $g_0(a+p\mu) \ne 0$, then*

$$\nu_p\biggl(\frac{g_0(a+p\mu)}{g_1(\mu)}\biggr)$$
$$= N_\theta(a) - N_\sigma(a+) + \sum_{i=1}^{n} \varrho(a,\theta_i)\nu_p(\mu+\theta_i') + \sum_{j=1}^{q'}(1-\varrho(a+,\sigma_j))\nu_p(\mu+\sigma_j'),$$

where we define $\varrho(a,\theta_i)\nu_p(\mu+\theta_i') = 0$ if $\mu+\theta_i' = 0$. (Note that in this case $\varrho(a,\theta_i) = 0$.)

Lemma C ([2], Lemma 2.2). *Suppose that the following conditions are satisfied*:

(1) $|\sigma_j^{(\nu)}| = 1$ *for* $1 \le j \le q'$, $0 \le \nu \le L+1$;
(2) $N_{\theta^{(\nu)}}(a) \ge N_{\sigma^{(\nu)}}(a+)$ *for* $0 \le \nu \le L$ *and* $0 \le a < p$.

Then for ν with $0 \le \nu \le L+1$ we have

$$g_\nu(n) \in \mathbb{Z}_p \quad \text{for } 0 \le n < p^{L-\nu+1}.$$

Remark. Suppose that $g_0(n) = 0$ for $n > c$ with c such that $p^L \le c < p^{L+1}$. Then $g_\nu(n) = 0$ for $n > h_\nu(c)$ for $0 \le \nu \le L$, and also for $\nu \ge L+1$ we have $g_\nu(n) = 0$ for $n > 0$ and $g_\nu(0) = (\prod_{j=1}^{q'} \sigma_j^{(\nu)})^{-1}$. Hence if the condition (1) is satisfied for any $\nu \ge 0$, then

$$g_\nu(n) \in \mathbb{Z}_p \quad \text{for any } \nu \ge 0 \text{ and } n \ge 0.$$

Lemma D ([2], Lemma 3.1). *Suppose that $|\sigma_j^{(s)}| = 1$ for $1 \le j \le q'$. For an integer μ with $0 \le \mu < p^s$, we have $\nu_p(\sigma_j+\mu) \le s$ for $1 \le j \le q'$.*

Lemma E ([2], Lemma 3.2). *Let s be a non-negative integer. Suppose that the following conditions are satisfied*:

(1) *for integers i and ν with $0 \le i < p$ and $0 \le \nu \le s$,*

$$N_{\theta^{(\nu)}}(i) \ge N_{\sigma^{(\nu)}}(i+) + \chi(N_{\sigma^{(\nu)}}(i+)),$$

where χ denotes the characteristic function of the set of strictly positive real numbers;

(2) *for* $1 \le j \le q'$, $|\sigma_j^{(s+1)}| = 1$.

Then for non-negative integers a, μ, m with $0 \le a < p$ and $\mu < p^s$, if $g_0(a+p\mu+mp^{s+1}) \ne 0$, then

(i) $u := \dfrac{g_0(a+p\mu+mp^{s+1})}{g_{s+1}(m)} \in \mathbb{Z}_p$,

(ii) $\dfrac{u}{mp^s+\mu+\theta_i'}\varrho(a,\theta_i) \equiv 0 \pmod p$ *for* $1 \le i \le n$, *where we define the left side to be zero if* $mp^s+\mu+\theta_i'=0$ (*in this case* $\varrho(a,\theta_i)=0$),

(iii) $\dfrac{u}{\mu+\sigma_j'}\varrho(a,\sigma_j)\equiv 0 \pmod p$ *for* $1\le j\le q'$.

From the above lemmas we can prove that the hypotheses of Theorem 5 are satisfied except for hypothesis 4 when $A^{(r)}(a+p\mu+mp^{s+1})=0$ and $A^{(r+1)}(\mu+mp^s)\ne 0$.

Suppose that $A^{(r)}(a+p\mu+mp^{s+1})=0$ and $A^{(r+1)}(\mu+mp^s)\ne 0$. Then from (6.1′),

$$a+p\mu+mp^{s+1}>h_r(c) \quad \text{and} \quad \mu+mp^s\le h_{r+1}(c).$$

If $c=\sum_{i=0}c_ip^i$ is the p-adic expansion of c, then

$$(*) \qquad c_r-a<0 \quad \text{and} \quad \mu+mp^s=h_{r+1}(c).$$

For i with $\theta_i=-c$, we have $\mu^{(r)}_{\theta_i}=c_r$, so $\varrho(a,\theta^{(r)}_i)=1$ and

$$N_{\theta^{(r)}}(a)\ge N_{\sigma^{(r)}}(a+)+1.$$

Also from $(*)$, $\mu=c_{r+1}+c_{r+2}p+\ldots+c_{s+r}p^{s-1}$, hence from Lemma B we have

$$\nu_p\left(\frac{g_r(a+p\mu)}{g_{r+1}(\mu)}\right)\ge s+1.$$

From assumption (4), $\mu+\sigma^{(r+1)}_l\in\mathbb{Z}_p^*$, so

$$\frac{g_{r+1}(\mu)}{A^{(r+1)}(\mu)}\in\mathbb{Z}_p.$$

As $A^{(r+1)}(\mu+mp^s)/g_{r+1}(\mu+mp^s)$ and $g_{r+1}(\mu+mp^s)/g_{s+r+1}(m)$ are elements of $\mathbb{Z}_p$ by the definition and Lemma B, we have

$$\frac{A^{(r+1)}(\mu+mp^s)}{A^{(r+1)}(\mu)}$$
$$=\frac{A^{(r+1)}(\mu+mp^s)}{g_{r+1}(\mu+mp^s)}\cdot\frac{g_{r+1}(\mu+mp^s)}{g_{s+r+1}(m)}\cdot\frac{g_r(a+p\mu)}{g_{r+1}(\mu)}\cdot\frac{g_{r+1}(\mu)}{A^{(r+1)}(\mu)}\cdot\frac{g_{s+r+1}(m)}{g_r(a+p\mu)}$$
$$\in p^{s+1}\frac{g_{s+r+1}(m)}{g_r(a+p\mu)}\mathbb{Z}_p. \ ■$$

Remark. For $m=0$, we get

$$F(X)F_s'(X^p)\equiv F'(X^p)F_{s+1}(X) \pmod{p^{s+1}[[X]]} \quad \text{for } 0\le s\le L-1.$$

But from conditions on the $A(m)$,

$$F_r(X)=F(X) \quad \text{for } r\ge L+1 \quad \text{and} \quad F'_{r'}(X)=F'(X) \quad \text{for } r'\ge L.$$

Hence for $s\ge L$,

$$F(X)F_s'(X^p)=F(X)F'(X^p)=F_{s+1}(X)F'(X^p),$$

so for $s \geq 0$ we have

$$F(X)F_s'(X^p) \equiv F'(X^p)F_{s+1}(X) \pmod{p^{s+1}[[X]]}.$$

Similarly, for $i \geq 0$, we have

$$F^{(i)}(X)F_s^{(i+1)}(X^p) \equiv F^{(i+1)}(X^p)F_{s+1}^{(i)}(X) \pmod{p^{s+1}[[X]]}.$$

Hence from these congruences we can show just as in Theorem 3 of [1] that $F(X)/F'(X^p)$ can be prolonged to an analytic element f of support $D = \{x \in \mathbb{Z}_p : |F_1^{(i)}(x)| = 1 \text{ for } i \geq 0\}$, where

$$f(x) \equiv F_{s+1}(x)/F_s'(x^p) \pmod{p^{s+1}\mathbb{Z}_p}. \tag{r.1}$$

Suppose further that sequences $\lambda_j(r)$ and $\nu_j(r)$ of elements of $\mathbb{Q} \cap \mathbb{Z}_p$ converge respectively to θ_j and σ_j. Consider

$${}_rH(X) = {}_nF_{n-1}\left(\begin{matrix}\lambda_1(r), \ldots, \lambda_n(r) \\ \nu_1(r), \ldots, \nu_{n-1}(r)\end{matrix}\ ;\ X\right) = \sum_{m=0}^{\infty} {}_rB(m)X^m.$$

If the $\lambda_j(r)$ and the $\nu_j(r)$ satisfy the conditions of Theorem 6, then ${}_rH(X)/{}_rH'(X^p)$ can be prolonged to ${}_rh$ of support $D_r = \{s \in \mathbb{Z}_p : |{}_rH_1^{(i)}(x)| = 1$ for $i \geq 0\}$. Then for $x \in D$,

$$f(x) = \lim_{r \to \infty} {}_rh(x).$$

This follows from $(r.1)$ and a continuity argument as in [3] and [9].

Theorem 7. *Let* $a = \sum_{i=0}^{t-1} a_i p^i$, $b = \sum_{i=0}^{t-1} b_i p^i$, $c = \sum_{i=0}^{t-1} c_i p^i$, $d = \sum_{i=0}^{t-1} d_i p^i$ *and* $e = \sum_{i=0}^{t-1} e_i p^i$ *be* p*-adic expansions of* a, b, c, d *and* e, *respectively. Assume that for each* i,

$$a_i + b_i + c_i = d_i + e_i - (p-1) \quad \text{and} \quad a_i, b_i, c_i < d_i, e_i.$$

Then

$${}_3\mathcal{F}_2^{(t)}\left(\begin{matrix}\frac{a}{q-1}, \frac{b}{q-1}, \frac{c}{q-1} \\ \frac{d}{q-1}, \quad \frac{e}{q-1}\end{matrix}\ ;\ 1\right) = \frac{\Gamma_q\left(\frac{d}{q-1}\right)\Gamma_q\left(\frac{d-a-b}{q-1}\right)\Gamma_q\left(\frac{d-a-c}{q-1}\right)\Gamma_q\left(\frac{d-b-c}{q-1}\right)}{\Gamma_q\left(\frac{d-a}{q-1}\right)\Gamma_q\left(\frac{d-b}{q-1}\right)\Gamma_q\left(\frac{d-c}{q-1}\right)\Gamma_q\left(\frac{d-a-b-c}{q-1}\right)}.$$

Proof. For $r \geq 1$, let

$$m_r = (d-b)\frac{q^r-1}{q-1},$$
$$n_r = (q-1-b)\frac{q^r-1}{q-1},$$
$$s_r = a\frac{q^r-1}{q-1},$$
$$u_r = (q-1+c-d)\frac{q^r-1}{q-1}.$$

Note that $d-b$, $q-1-b$ and $q-1+c-d$ are all positive.

The following identity holds (cf. [8] and Proof of Theorem 3.2 in [9]):

$$ {}_3F_2\left(\begin{matrix} n_r+1,\ -s_r,\ n_r-m_r-u_r \\ n_r+1-s_r-u_r,\ n_r+1-m_r \end{matrix}\ ;1\right) = \frac{\binom{m_r}{s_r}\binom{n_r}{u_r}}{\binom{n_r}{s_r+u_r}\binom{n_r+s_r-m_r}{s_r}}. $$

Under our assumptions on a, b, c, d and e, we have

$$ \nu_p\left(\binom{n_r}{s_r+u_r}\right) = \nu_p\left(\binom{n_r+s_r-m_r}{s_r}\right) = \nu_p\left(\binom{m_r}{s_r}\right) = \nu_p\left(\binom{n_r}{u_r}\right) = 0. $$

Hence

$$ {}_3F_2\left(\begin{matrix} n_r+1,\ -s_r,\ n_r-m_r-u_r \\ n_r+1-s_r-u_r,\ n_r+1-m_r \end{matrix}\ ;1\right) \in \mathbb{Z}_p^*. $$

Let $\alpha = n_r+1$, $\beta=-s_r$, $\gamma = n_r-m_r-u_r$, $\sigma = n_r+1-s_r-u_r$ and $\tau = n_r+1-m_r$. Then

$$ (\mu_\alpha^{(i)}, \mu_\beta^{(i)}, \mu_\gamma^{(i)}, \mu_\sigma^{(i)}, \mu_\tau^{(i)}) = (b_j, a_j, c_j, e_j, d_j) $$

for $i \le tr-1$ and $i \equiv j \pmod t$ with $j \in \{0,1,\ldots,t-1\}$, and also for $i \le tr-1$,

$$ \alpha^{(i)} = h_i(n_r)+1, \quad \beta^{(i)} = -h_i(s_r), \quad \gamma^{(i)} = h_i(n_r)-h_i(m_r)-h_i(u_r), $$
$$ \sigma^{(i)} = h_i(n_r)+1-h_i(s_r)-h_i(u_r), \quad \tau^{(i)} = h_i(n_r)+1-h_i(m_r). $$

Hence

$$ {}_3F_2\left(\begin{matrix} \alpha^{(i)}, \beta^{(i)}, \gamma^{(i)} \\ \sigma^{(i)}, \tau^{(i)} \end{matrix}\ ;1\right) = \frac{\binom{h_i(m_r)}{h_i(s_r)}\binom{h_i(n_r)}{h_i(u_r)}}{\binom{h_i(n_r)}{h_i(s_r)+h_i(u_r)}\binom{h_i(n_r)+h_i(s_r)-h_i(m_r)}{h_i(s_r)}}. $$

Under our assumptions on a, b, c, d and e, we have

$$ \left| {}_3F_2^{(i)}\left(\begin{matrix} n_r+1, -s_r, n_r-m_r-u_r \\ n_r+1-s_r-u_r, n_r+1-m_r \end{matrix}\ ;1\right)\right| = \left| {}_3F_2\left(\begin{matrix} \alpha^{(i)}, \beta^{(i)}, \gamma^{(i)} \\ \sigma^{(i)}, \tau^{(i)} \end{matrix}\ ;1\right)\right| = 1 \tag{7.1} $$

for any i.

Set

$$ F(X) = {}_3F_2\left(\begin{matrix} \frac{a}{q-1}, \frac{b}{q-1}, \frac{c}{q-1} \\ \frac{d}{q-1}, \frac{e}{q-1} \end{matrix}\ ;X\right), $$
$$ G(X) = {}_3F_2\left(\begin{matrix} n_r+1,\ -s_r,\ n_r-m_r-u_r \\ n_r+1-s_r-u_r,\ n_r+1-m_r \end{matrix}\ ;X\right). $$

Then the conditions in Theorem 6 are satisfied for $G^{(j)}(X)$ with $c = \min\{h_j(s_r), -h_j(n_r)+h_j(m_r)+h_j(u_r)\}$ and $L = tr-1-j$ for $0 \le j \le t-1$, and we get, in particular,

$$G^{(i)}(X) \equiv G^{(i+1)}(X^p)G_1^{(i)}(X) \pmod{p[X]}.$$

Hence from (7.1), $|G_1^{(i)}(1)| = 1$ for any i. So by continuity, $|F_1^{(i)}(1)| = 1$ for any i.

As for $F(X)$, Dwork's theorem (Theorem 2.3 of [9]) holds and the ratio $F(X)/F(X^q)$ can be prolonged to $X = 1$. From Remark after Theorem 6, its value is

$$\begin{aligned}
&{}_3\mathcal{F}_2^{(t)}\left(\begin{matrix}\frac{a}{q-1}, \frac{b}{q-1}, \frac{c}{q-1} \\ \frac{d}{q-1}, \quad \frac{e}{q-1}\end{matrix}\ ; 1\right) \\
&= \lim_{r\to\infty} {}_3\mathcal{F}_2^{(t)}\left(\begin{matrix}n_r+1, -s_r, n_r-m_r-u_r \\ n_r+1-s_r-u_r, n_r+1-m_r\end{matrix}\ ; 1\right) \\
&= \lim_{r\to\infty} \frac{{}_3F_2\left(\begin{matrix}n_r+1,\ -s_r,\ n_r-m_r-u_r \\ n_r+1-s_r-u_r,\ n_r+1-m_r\end{matrix}\ ; 1\right)}{{}_3F_2\left(\begin{matrix}(n_r+1)^{(t)},\ (-s_r)^{(t)},\ (n_r-m_r-u_r)^{(t)} \\ (n_r+1-s_r-u_r)^{(t)},\ (n_r+1-m_r)^{(t)}\end{matrix}\ ; 1\right)} \\
&= \lim_{r\to\infty} \frac{\binom{m_r}{s_r}\binom{n_r}{u_r}\binom{n_{r-1}}{s_{r-1}+u_{r-1}}\binom{n_{r-1}+s_{r-1}-m_{r-1}}{s_{r-1}}}{\binom{n_r}{s_r+u_r}\binom{n_r+s_r-m_r}{s_r}\binom{m_{r-1}}{s_{r-1}}\binom{n_{r-1}}{u_{r-1}}} \\
&= \frac{\Gamma_q\left(\frac{d-a-b}{q-1}\right)\Gamma_q\left(\frac{d-b-c}{q-1}\right)\Gamma_q\left(\frac{d-a-c}{q-1}\right)\Gamma_q\left(\frac{d}{q-1}\right)}{\Gamma_q\left(\frac{d-a}{q-1}\right)\Gamma_q\left(\frac{d-b}{q-1}\right)\Gamma_q\left(\frac{d-c}{q-1}\right)\Gamma_q\left(\frac{d-a-b-c}{q-1}\right)}. \ \blacksquare
\end{aligned}$$

Remark. The condition "$a_i, b_i, c_i < d_i, e_i$" in Theorem 7 can be slightly weakened so as to satisfy the hypotheses of Theorem 6 for $G^{(j)}(X)$ with $0 \le j \le t-1$.

THEOREM 8. *Let* $a = \sum_{i=0}^{t-1} a_i p^i$, $b = \sum_{i=0}^{t-1} b_i p^i$ *and* $c = \sum_{i=0}^{t-1} c_i p^i$ *be* p*-adic expansions of* a, b *and* c*, respectively. Assume that for each* i,

(1) $a_i + b_i \le (p-1)/2$, $c_i \le (p-1)/2$ *and* $a_i \ne c_i$,

(2) *if* $f_i = \max\{2a_i, 2b_i, c_i\}$ *and* $g_i = \max\{\{2a_i, 2b_i, c_i\} - \{f_i\}\}$, *then*

$$f_i < \max\left\{\frac{p-1}{2} + a_i + b_i, 2c_i\right\} \quad \textit{and} \quad g_i < \min\left\{\frac{p-1}{2} + a_i + b_i, 2c_i\right\}.$$

Then

$${}_3\mathcal{F}_2^{(t)}\left(\begin{matrix}\frac{2a}{q-1}, \frac{2b}{q-1}, \frac{c}{q-1} \\ \frac{1}{2} + \frac{a+b}{q-1}, \frac{2c}{q-1}\end{matrix}\ ; 1\right)$$

$$= \frac{\Gamma_q\left(\frac{1}{2}\right)\Gamma_q\left(\frac{1}{2}+\frac{a+b}{q-1}\right)\Gamma_q\left(\frac{1}{2}+\frac{c}{q-1}\right)\Gamma_q\left(\frac{1}{2}+\frac{c-a-b}{q-1}\right)}{\Gamma_q\left(\frac{1}{2}+\frac{a}{q-1}\right)\Gamma_q\left(\frac{1}{2}+\frac{b}{q-1}\right)\Gamma_q\left(\frac{1}{2}+\frac{c-a}{q-1}\right)\Gamma_q\left(\frac{1}{2}+\frac{c-b}{q-1}\right)}.$$

P r o o f. For $r \geq 0$, let

$$n_r = a\frac{q^r-1}{q-1}, \qquad m_r = \frac{1}{2} + d\frac{q^r-1}{q-1}$$

with $d = (q-1)/2 - b$ and $\gamma = c/(q-1)$. From Watson's theorem (cf. [7] and Proof of Theorem 3.4 in [9]),

$$(8.1) \quad {}_3F_2\left(\begin{matrix} -2n_r,\, 2m_r,\, \gamma \\ \frac{1}{2}+m_r-n_r,\, 2\gamma \end{matrix}\,;1\right) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+\gamma)\Gamma(\frac{1}{2}+m_r-n_r)\Gamma(\frac{1}{2}+\gamma+n_r-m_r)}{\Gamma(\frac{1}{2}-n_r)\Gamma(\frac{1}{2}+m_r)\Gamma(\frac{1}{2}+\gamma+n_r)\Gamma(\frac{1}{2}+\gamma-m_r)}.$$

Here

$$(8.2) \quad \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-n_r)} = \prod_{\substack{l=1 \\ q\nmid 1/2-l}}^{n_r}\left(\frac{1}{2}-l\right)\prod_{\substack{l=1 \\ q\mid 1/2-l}}^{n_r}\left(\frac{1}{2}-l\right) = \prod_{\substack{l=1 \\ q\nmid 1/2-l}}^{n_r}\left(\frac{1}{2}-l\right)q^{n_{r-1}}\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-n_{r-1})}.$$

From Proposition 1(3), for any v in $\mathbb{Z}_p$,

$$\Gamma_q\left(\frac{1}{2}-v+1\right) = \begin{cases} (-1)^{k+1}h_k(1/2-v)\Gamma_q(1/2-v) & \text{if } \nu_p(1/2-v)=k<t, \\ (-1)^t\Gamma_q(1/2-v) & \text{if } \nu_p(1/2-v)\geq t. \end{cases}$$

Hence

$$\Gamma_q\left(\frac{1}{2}\right) = (-1)^{tx_t}\Bigg\{(-1)^{tx_{t-1}}\prod_{\substack{j=1 \\ \nu_p(1/2-j)=t-1}}^{n_r} h_{t-1}\left(\frac{1}{2}-j\right)\Bigg\}\ldots$$
$$\ldots\Bigg\{(-1)^{x_0}\prod_{\substack{j=1 \\ \nu_p(1/2-j)=0}}^{n_r} h_0\left(\frac{1}{2}-j\right)\Bigg\}\Gamma_q\left(\frac{1}{2}-n_r\right),$$

where

$$x_j = \begin{cases} \#\{l \mid 1\leq l\leq n_r,\ \nu_p(1/2-l)=j\} & \text{for } 0\leq j\leq t-1, \\ \#\{l \mid 1\leq l\leq n_r,\ \nu_p(1/2-l)\geq t\} & \text{for } j=t. \end{cases}$$

If $\nu_p(1/2-j)\geq k$, then $h_k(1/2-j) = p^{-k}(1/2-j)$. So

$$\Gamma_q\left(\frac{1}{2}\right) = (-1)^{\Sigma_{j=0}^{t-1}(j+1)x_j+tx_t}p^{-\Sigma_{j=1}^{t-1}jx_j}\Bigg\{\prod_{\substack{l=1 \\ q\nmid 1/2-l}}^{n_r}\left(\frac{1}{2}-l\right)\Bigg\}\Gamma_q\left(\frac{1}{2}-n_r\right).$$

Hence from (8.2) we have

$$\frac{\Gamma(\frac{1}{2})/\Gamma(\frac{1}{2}-n_r)}{\Gamma(\frac{1}{2})/\Gamma(\frac{1}{2}-n_{r-1})} = (-1)^{\Sigma_{j=0}^{t-1}(j+1)x_j+tx_t} q^{n_{r-1}} p^{\Sigma_{j=1}^{t-1} jx_j} \frac{\Gamma_q(\frac{1}{2})}{\Gamma_q(\frac{1}{2}-n_r)}. \tag{8.3}$$

Similarly,

$$\frac{\Gamma(\frac{1}{2}+m_r)/\Gamma(\frac{1}{2}+m_r-n_r)}{\Gamma(\frac{1}{2}+m_{r-1})/\Gamma(\frac{1}{2}+m_{r-1}-n_{r-1})} = (-1)^{\Sigma_{j=0}^{t-1}(j+1)y_j+ty_t} q^{n_{r-1}} p^{\Sigma_{j=1}^{t-1} jy_j} \frac{\Gamma_q(\frac{1}{2}+m_r)}{\Gamma_q(\frac{1}{2}+m_r-n_r)}, \tag{8.4}$$

$$\frac{\Gamma(\frac{1}{2}+\gamma+n_r)/\Gamma(\frac{1}{2}+\gamma)}{\Gamma(\frac{1}{2}+\gamma+n_{r-1})/\Gamma(\frac{1}{2}+\gamma)} = (-1)^{\Sigma_{j=0}^{t-1}(j+1)z_j+tz_t} q^{n_{r-1}} p^{\Sigma_{j=1}^{t-1} jz_j} \frac{\Gamma_q(\frac{1}{2}+\gamma+n_r)}{\Gamma_q(\frac{1}{2}+\gamma)} \tag{8.5}$$

and

$$\frac{\Gamma(\frac{1}{2}+\gamma+n_r-m_r)/\Gamma(\frac{1}{2}+\gamma-m_r)}{\Gamma(\frac{1}{2}+\gamma+n_{r-1}-m_{r-1})/\Gamma(\frac{1}{2}+\gamma-m_{r-1})} = (-1)^{\Sigma_{j=0}^{t-1}(j+1)u_j+tu_t} q^{n_{r-1}} p^{\Sigma_{j=1}^{t-1} ju_j} \frac{\Gamma_q(\frac{1}{2}+\gamma+n_r-m_r)}{\Gamma_q(\frac{1}{2}+\gamma-m_r)}, \tag{8.6}$$

where

$$y_j = \begin{cases} \#\{l \mid 1 \le l \le n_r,\ \nu_p(1/2+m_r-l)=j\} & \text{for } 0 \le j \le t-1, \\ \#\{l \mid 1 \le l \le n_r,\ \nu_p(1/2+m_r-l)\ge t\} & \text{for } j=t, \end{cases}$$

$$z_j = \begin{cases} \#\{l \mid 1 \le l \le n_r,\ \nu_p(1/2+\gamma+n_r-l)=j\} & \text{for } 0 \le j \le t-1, \\ \#\{l \mid 1 \le l \le n_r,\ \nu_p(1/2+\gamma+n_r-l)\ge t\} & \text{for } j=t, \end{cases}$$

$$u_j = \begin{cases} \#\{l \mid 1 \le l \le n_r,\ \nu_p(1/2+\gamma+n_r-m_r-l)=j\} & \\ \qquad\qquad\qquad\qquad \text{for } 0 \le j \le t-1, & \\ \#\{l \mid 1 \le l \le n_r,\ \nu_p(1/2+\gamma+n_r-m_r-l)\ge t\} & \text{for } j=t. \end{cases}$$

As for the computation of x_j,

$$\frac{1}{2} - l \equiv 0 \pmod{p^j} \ \Leftrightarrow\ l \equiv \frac{1+p^j}{2} \pmod{p^j}.$$

So

$$x_j = \begin{cases} n_{r-1} & \text{for } j=t, \\ h_j(n_r)-h_{j+1}(n_r) & \text{for } 0 \le j \le t-1. \end{cases} \tag{8.7}$$

Similarly,

$$y_j = z_j = u_j = \begin{cases} n_{r-1} & \text{for } j=t, \\ h_j(n_r)-h_{j+1}(n_r) & \text{for } 0 \le j \le t-1. \end{cases} \tag{8.8}$$

Since

$$(8.9)\quad {}_3F_2\left(\begin{matrix} -2n_r, 2m_r, \gamma \\ \frac{1}{2}+m_r-n_r, 2\gamma \end{matrix}; 1\right)$$

$$= \prod_{s=1}^{r}\left\{\frac{\Gamma(\frac{1}{2})/\Gamma(\frac{1}{2}-n_s)}{\Gamma(\frac{1}{2})/\Gamma(\frac{1}{2}-n_{s-1})}\left(\frac{\Gamma(\frac{1}{2}+\gamma+n_s)/\Gamma(\frac{1}{2}+\gamma)}{\Gamma(\frac{1}{2}+\gamma+n_{s-1})/\Gamma(\frac{1}{2}+\gamma)}\right)^{-1}\right.$$

$$\times\left(\frac{\Gamma(\frac{1}{2}+m_s)/\Gamma(\frac{1}{2}+m_s-n_s)}{\Gamma(\frac{1}{2}+m_{s-1})/\Gamma(\frac{1}{2}+m_{s-1}-n_{s-1})}\right)^{-1}$$

$$\left.\times\frac{\Gamma(\frac{1}{2}+\gamma+n_s-m_s)/\Gamma(\frac{1}{2}+\gamma-m_s)}{\Gamma(\frac{1}{2}+\gamma+n_{s-1}-m_{s-1})/\Gamma(\frac{1}{2}+\gamma-m_{s-1})}\right\},$$

we see from (8.3)–(8.8) that the exponent of p in the right hand side of (8.9) is zero, so

$${}_3F_2\left(\begin{matrix} -2n_r, 2m_r, \gamma \\ \frac{1}{2}+m_r-n_r, 2\gamma \end{matrix}; 1\right) \in \mathbb{Z}_p^*.$$

Also for $i \le tr-1$, we have

$$(\tfrac{1}{2}+m_r-n_r)^{(i)} = \tfrac{1}{2}+\tfrac{1}{2}(2m_r)^{(i)}+\tfrac{1}{2}(-2n_r)^{(i)}, \qquad (2\gamma)^{(i)} = 2\gamma^{(i)}.$$

Hence we can show similarly

$${}_3F_2^{(i)}\left(\begin{matrix} -2n_r, 2m_r, \gamma \\ \frac{1}{2}+m_r-n_r, 2\gamma \end{matrix}; 1\right) \in \mathbb{Z}_p^* \quad \text{for any } i \ge 0.$$

Since for $i \le tr-1$ we have

$$(\mu_{-2n_r}^{(i)}, \mu_{2m_r}^{(i)}, \mu_{\gamma}^{(i)}, \mu_{1/2+m_r-n_r}^{(i)}, \mu_{2\gamma}^{(i)}) = \left(2a_j, 2b_j, c_j, \frac{p-1}{2}+a_j+b_j, 2c_j\right)$$

where $i \equiv j \pmod t$ with $j \in \{0, 1, \ldots, t-1\}$, the conditions in Theorem 6 are satisfied for ${}_3F_2^{(i)}\left(\begin{smallmatrix} -2n_r, 2m_r, \gamma \\ \frac{1}{2}+m_r-n_r, 2\gamma \end{smallmatrix}; X\right)$ with $c = h_i(2n_r)$ and $L = tr-1-i$ for $0 \le i \le t-1$.

So as in Theorem 7, for

$$F(X) = {}_3F_2\left(\begin{matrix} \frac{2a}{q-1}, \frac{2b}{q-1}, \frac{c}{q-1} \\ \frac{1}{2}+\frac{a+b}{q-1}, \frac{2c}{q-1} \end{matrix}; X\right)$$

the ratio $F(X)/F(X^q)$ can be prolonged to $X = 1$, and its value is

$$
{}_3\mathcal{F}_2^{(t)}\left(\begin{matrix}\frac{2a}{q-1},\ \frac{2b}{q-1},\ \frac{c}{q-1}\\ \frac{1}{2}+\frac{a+b}{q-1},\ \frac{2c}{q-1}\end{matrix};1\right)
= \lim_{r\to\infty}\frac{{}_3F_2\left(\begin{matrix}-2n_r,\ 2m_r,\ \frac{c}{q-1}\\ \frac{1}{2}+m_r-n_r,\ \frac{2c}{q-1}\end{matrix};1\right)}{{}_3F_2\left(\begin{matrix}(-2n_r)^{(t)},\ (2m_r)^{(t)},\ \left(\frac{c}{q-1}\right)^{(t)}\\ \left(\frac{1}{2}+m_r-n_r\right)^{(t)},\ \left(\frac{2c}{q-1}\right)^{(t)}\end{matrix};1\right)}
$$

$$
= \frac{\Gamma_q\left(\frac{1}{2}\right)\Gamma_q\left(\frac{1}{2}+\frac{a+b}{q-1}\right)\Gamma_q\left(\frac{1}{2}+\frac{c}{q-1}\right)\Gamma_q\left(\frac{1}{2}+\frac{c-a-b}{q-1}\right)}{\Gamma_q\left(\frac{1}{2}+\frac{a}{q-1}\right)\Gamma_q\left(\frac{1}{2}+\frac{b}{q-1}\right)\Gamma_q\left(\frac{1}{2}+\frac{c-a}{q-1}\right)\Gamma_q\left(\frac{1}{2}+\frac{c-b}{q-1}\right)}. \ \blacksquare
$$

Remarks. 1. Conditions on a, b and c in Theorem 8 are satisfied if for each i,

$$
a_i + b_i \le \frac{p-1}{2}, \quad c_i \le \frac{p-1}{2},
$$

and

$$
\begin{cases} c_i > a_i > 0 & \text{if } a_i \le b_i,\\ c_i > b_i > 0 \text{ and } c_i \ne a_i & \text{if } a_i \ge b_i.\end{cases}
$$

2. From the condition (1), c_i cannot be equal to a_i for any i. If $c_i = a_i$ for some i and $c_j \ne b_j$ for any j, then by interchanging a and b, we still get the same result.

As in [9], from the above theorems we obtain some identities among $\mathcal{F}$'s.

(1) For $a = \sum_{i=0}^{t-1} a_i p^i$, $b = \sum_{i=0}^{t-1} b_i p^i$ and $c = \sum_{i=0}^{t-1} c_i p^i$ with $a_i, b_i, c_i < p-1$ and $a_i + b_i + c_i = p-1$ for $0 \le i \le t-1$,

$$
{}_3\mathcal{F}_2^{(t)}\left(\begin{matrix}\frac{a}{q-1},\ \frac{b}{q-1},\ \frac{c}{q-1}\\ 1, 1\end{matrix};1\right) = {}_2\mathcal{F}_1^{(t)}\left(\begin{matrix}\frac{a}{q-1},\ \frac{b}{q-1}\\ 1\end{matrix};1\right)^2.
$$

This follows from Proposition 4 and Theorem 7 (cf. (3.27) in [9]).

(2) For $a = \sum_{i=0}^{t-1} a_i p^i$ and $b = \sum_{i=0}^{t-1} b_i p^i$ with $0 < a_i + b_i \le (p-1)/2$ and $\max\{a_i, b_i\} < (p-1)/2$ for $0 \le i \le t-1$, let $d_i = a_i + b_i + (p-1)/2$ and $d = \sum_{i=0}^{t-1} d_i p^i$. Then

$$
{}_3\mathcal{F}_2^{(t)}\left(\begin{matrix}\frac{a}{q-1},\ \frac{b}{q-1},\ \frac{1}{2}\\ \frac{d}{q-1}, 1\end{matrix};1\right) = {}_3\mathcal{F}_2^{(t)}\left(\begin{matrix}\frac{2a}{q-1},\ \frac{2b}{q-1},\ \frac{1}{2}\\ \frac{d}{q-1}, 1\end{matrix};1\right).
$$

This follows from Theorems 7 and 8 (cf. (3.28) in [9]).

(3) From Proposition 1(1), Proposition 2 and Theorem 3.2 of [10], we get

for $a=\sum_{i=0}^{t-1} a_i p^i$ with $0<a_i\le (p-1)/3$ and for $m_r=a(q^r-1)/(q-1)$,

$$\frac{\binom{3m_r}{2m_r}\binom{2m_r}{m_r}}{\binom{3m_{r-1}}{2m_{r-1}}\binom{2m_{r-1}}{m_{r-1}}} \equiv (-1)^a {}_3\mathcal{F}_2^{(t)}\left(\begin{matrix}\frac{2a}{q-1}, \frac{2a}{q-1}, \frac{2a}{q-1} \\ 1,1\end{matrix}\ ;1\right) \pmod{pq^{r-1}}$$

(cf. (3.24) in [9]).

References

[1] B. Dwork, *p-adic cycles*, Inst. Hautes Études Sci. Publ. Math. 37 (1969), 27–115.
[2] —, *On p-adic differential equations IV*, Ann. Sci. École Norm. Sup. 6 (1973), 295–315.
[3] N. Koblitz, *The hypergeometric function with p-adic parameters*, in: Proceedings, Queens Number Theory Conference, 1979, Queen's University, 1980, 319–328.
[4] —, *q-extension of the p-adic Gamma function*, Trans. Amer. Math. Soc. 260 (1980), 449–457.
[5] —, *q-extension of the p-adic Gamma function II*, ibid. 273 (1982), 111–129.
[6] S. Lang, *Cyclotomic Fields II*, Graduate Texts in Math. 69, Springer, 1980.
[7] L. J. Slater, *Generalized Hypergeometric Functions*, Cambridge Univ. Press, London, 1966.
[8] J. Riordan, *Combinatorial Identities*, Wiley, New York, 1968.
[9] P. T. Young, *Apéry numbers, Jacobi sums, and special values of generalized p-adic hypergeometric functions*, J. Number Theory 41 (1992), 231–255.
[10] —, *On Jacobi sums, multinomial coefficients, and p-adic hypergeometric functions*, ibid., to appear.

DEPARTMENT OF MATHEMATICS
TSUDA COLLEGE
KODAIRA, TOKYO 187, JAPAN
E-mail: OTA@TSUDA.AC.JP

Received on 30.4.1993
and in revised form on 31.12.1993 (2427)