ACTA ARITHMETICA
LXVII.1 (1994)

# The number of cube-full numbers in an interval

by

HONG-QUAN LIU (Harbin)

**1. Introduction.** A positive integer $n$ is called *cube-full* if $p \mid n$ implies that also $p^3 \mid n$; here $p$ denotes a prime number. Let $Q_3(x)$ denote the number of cube-full numbers in the interval $[1, x]$, where $x$ is a sufficiently large number. On the one hand, an asymptotic formula for $Q_3(x)$ can be given with an error term $o(x^{1/8})$, and this order of magnitude cannot be reduced to $O(x^{\mu})$ with some $\mu < 1/8$ without any hypothesis. On the other hand, an asymptotic formula does hold for $Q_3(x + x^{2/3+\mu}) - Q_3(x)$ with a $\mu < 1/8$ by using results for exponential sums. In fact, Shiu [7] proved that

$$(*) \qquad Q_3(x + x^{2/3+\mu}) - Q_3(x) \to Cx^{\mu}, \qquad \text{for any } \frac{140}{1123} < \mu < \frac{1}{3};$$

here $140/1123 = 0.1246607\ldots$ $(< 1/8)$. The aim of this paper is to give an improvement on his result. We will prove

THEOREM 1. $(*)$ *holds for*

$$\frac{1}{3} > \mu > \frac{11}{92} = 0.1195652\ldots$$

Taking the idea from the author's paper [3] dealing with squarefull integers, we first give a reduction of our problem, which connects $(*)$ with some exponential sums, but this time (in contrast to [3]) triple exponential sums are inevitable, and, to obtain our result, it is crucial to apply the method developed in [2] and [4]. Actually, we also give an improvement of Krätzel's result about $\Delta(x; 3, 4, 5)$, where $\Delta(x; 3, 4, 5)$ is the error term in the asymptotic formula

$$\sum_{x \geq r^3 s^4 t^5} 1 = A_3^* x^{1/3} + A_4^* x^{1/4} + A_5^* x^{1/5} + \Delta(x; 3, 4, 5),$$

$$A_i^* = \prod_{\substack{3 \leq n \leq 5 \\ n \neq i}} \zeta\left(\frac{n}{i}\right), \qquad i = 3, 4, 5.$$

Our estimates then imply $\Delta(x;3,4,5) \ll x^{11/92+\varepsilon}$, where $\varepsilon$ is an arbitrarily small positive number, while [1] finds $\Delta(x;3,4,5) \ll x^{22/177+\varepsilon}$, with $22/177 = 0.1242937\ldots$

**2. Reduction.** We prove

THEOREM 2. *Let*

$$S_{a,b,c}(x) = \sum_{(m,n)\in D} \psi\bigg(\bigg(\frac{x}{m^b n^c}\bigg)^{1/a}\bigg), \qquad \psi(\xi) = \xi - [\xi] - \frac{1}{2},$$

*where* $(a,b,c)$ *is any permutation of* $(3,4,5)$, $D$ *is the range* $\{(m,n) \mid m^{a+b}n^c \le x, m \ge n\}$ *or* $\{(m,n) \mid m^{a+b}n^c \le x, m \ge x^{1/12}\}$ *or* $\{(m,n) \mid n \le m \le x^{1/12}\}$. *Then*

$$S_{a,b,c}(x) \ll x^{11/92+\varepsilon}$$

*implies the assertion of Theorem* 1.

Proof. Put $B = x^{\varepsilon}$, and suppose that $11/92 + \varepsilon < \mu < 1/3$. It is well known that

$$Q_3(x) = \sum_{\substack{a^3b^4c^5\in I\\ b,c\le B}} |\mu(bc)| + \sum_{\substack{a^3b^4c^5\in I\\ b>B \text{ or } c>B}} |\mu(bc)| = \sum\nolimits_1 + \sum\nolimits_2, \quad \text{say},$$

where $I$ is the interval $(x, x + x^{2/3+\mu}]$. We have

$$\sum\nolimits_1 = ((x + x^{2/3+\mu})^{1/3} - x^{1/3}) \sum_{b\le B} \frac{|\mu(b)|}{b^{4/3}} \sum_{\substack{(c,b)=1\\ c\le B}} \frac{|\mu(c)|}{c^{5/3}} + O(B^2),$$

$$\sum_{\substack{(c,b)=1\\ c\le B}} \frac{|\mu(c)|}{c^{5/3}} = \sum_{\substack{c=1\\ (c,b)=1}}^{\infty} \frac{|\mu(c)|}{c^{5/3}} + O(B^{-2/3})$$

$$= \frac{\zeta(5/3)}{\zeta(10/3)} \prod_{p\mid b} \bigg(1 + \frac{1}{p^{5/3}}\bigg)^{-1} + O(B^{-2/3}),$$

$$\sum_{b\le B} \frac{|\mu(b)|}{b^{4/3}} \prod_{p\mid b} \bigg(1 + \frac{1}{p^{5/3}}\bigg)^{-1} = \prod_p \bigg(1 + \frac{1}{p^{4/3}}\bigg(1 + \frac{1}{p^{5/3}}\bigg)^{-1}\bigg) + O(B^{-1/3}),$$

$$(x + x^{2/3+\mu})^{1/3} - x^{1/3} = \frac{1}{3} x^{\mu}(1 + O(x^{2\mu-2/3})),$$

and thus

$$\sum\nolimits_1 = Cx^{\mu}(1 + o(1)), \qquad C = \frac{1}{3} \cdot \frac{\zeta(5/3)}{\zeta(10/3)} \prod_p \bigg(1 + \frac{1}{p^{4/3}}\bigg(1 + \frac{1}{p^{5/3}}\bigg)^{-1}\bigg).$$

It remains to estimate $\sum_2$. First we consider the portion of $\sum_2$ with $b > B$, which we denote by $\sum_{21}$. We have, with $x_1 = x + x^{2/3+\mu}$ and $a(m) = \sum_{m=a^3c^5} 1$,

$$(**) \qquad \sum_{21} \le \sum_{B<b\le x_1^{1/12}} \sum_{(xb^{-4})^{1/3}<ac^{5/3}\le(x_1b^{-4})^{1/3}} 1 + \sum_{m\le x_1^{2/3}} a(m) \sum_{(xm^{-1})^{1/4}<b\le(x_1m^{-1})^{1/4}} 1.$$

At this stage, to treat the sums of $(**)$ we need to cite the next two lemmata.

Lemma 2.1. *For $y \ge 1$, $\varrho > 0$, $1 \le \mu \le y^{1/(\varrho+1)} \le v$, $\mu^\varrho v = y$,*

$$\sum_{y^{1/(\varrho+1)}<n\le v} \psi\left(\sqrt[\varrho]{\frac{y}{n}}\right) = \sum_{\mu<n\le y^{1/(\varrho+1)}} \psi\left(\frac{y}{n^\varrho}\right) - \varrho\psi_2(\mu)y\mu^{-1-\varrho} + O(y\mu^{-2-\varrho}+1),$$

*where $\psi_2(z) := \frac{1}{2}\psi(z)^2 - 1/24$.*

Lemma 2.2. *With the same assumptions as in Lemma* 2.1,

$$\sum_{m^\varrho n\le y} 1 = \zeta(\varrho)y + \zeta\left(\frac{1}{\varrho}\right)y^{1/\varrho} - \sum_{n\le\mu} \psi\left(\frac{y}{n^\varrho}\right) - \sum_{n\le v} \psi\left(\sqrt[\varrho]{\frac{y}{n}}\right) - \varrho\psi_2(\mu)y\mu^{-1-\varrho} + O(y\mu^{-2-\varrho}) + 1/4 - \psi(\mu)\psi(v) + O(\mu v^{-1}).$$

Lemma 2.1 is Hilfssatz 4 of P. G. Schmidt [6], and Lemma 2.2 is (8) on p. 37 of [6]. Now by Lemma 2.2 we have, with $\mu = (x_1b^{-4})^{1/8}$,

$$\sum_{ac^{5/3}\le(x_1b^{-4})^{1/3}} 1 = \zeta\left(\frac{5}{3}\right)(x_1b^{-4})^{1/3} + \zeta\left(\frac{3}{5}\right)(x_1b^{-4})^{1/5} - \sum_{n\le\mu} \psi((x_1b^{-4}n^{-5})^{1/3}) - \sum_{n\le\mu} \psi((x_1b^{-4}n^{-3})^{1/5}) + O(1)$$

and a similar formula holds for $x_1$ being replaced by $x$. Thus the first sum of $(**)$ is (note that we can assume, for example, $\mu < 0.125$, in light of [7])

$$\le O(B^{-1/3}x^\mu) + S_1,$$

where $S_1$ is a linear combination of $S_{a,b,c}(\Omega)$, $\Omega = x$ or $x_1$, and $(a,b,c)$ belongs to the set $\{(3,4,5),(3,5,4),(5,4,3),(5,3,4)\}$. To deal with the second sum in $(**)$ we apply the technique in Schmidt [6]. First it is easy to observe that

$$\sum_{m\le x_1^{2/3}} a(m) \sum_{(xm^{-1})^{1/4}<b\le(x_1m^{-1})^{1/4}} 1 \le I_1 + I_2,$$

where

$$I_1 = \sum_{a^3c^5\le x_1^{2/3},a\le x_1^{1/12}} \sum_{(xa^{-3}c^{-5})^{1/4}<b\le(x_1a^{-3}c^{-5})^{1/4}} 1,$$

$$I_2 = \sum_{a^3c^5\le x_1^{2/3},c\le x_1^{1/12}} \sum_{(xa^{-3}c^{-5})^{1/4}<b\le(x_1a^{-3}c^{-5})^{1/4}} 1.$$

It is easy to see that

$$I_1 = O(x^{\mu-\varepsilon}) + S(x_1) - S(x),$$

$$S(x_1) = \sum_{a\le x_1^{1/12}} \sum_{c\le(x_1^{2/3}a^{-3})^{1/5}} \psi\Big(\Big(\frac{x_1}{a^3c^5}\Big)^{1/4}\Big)$$

and $S(x)$ is defined similarly. We have

$$\sum_{c\le(x_1^{2/3}a^{-3})^{1/5}} \psi\Big(\Big(\frac{x_1}{a^3c^5}\Big)^{1/4}\Big) = \Big(\sum_{c\le(x_1a^{-3})^{1/9}} + \sum_{(x_1a^{-3})^{1/9}<c\le(x_1^{2/3}a^{-3})^{1/5}}\Big)\psi(\cdot) = I_3 + I_4, \quad \text{say.}$$

From $I_3$ we can get sums of the type $S_{a,b,c}(x_1)$ easily, and for $I_4$ we choose $y=(x_1a^{-3})^{1/5}$, $\varrho=4/5$, $\mu=x_1^{1/12}$, $v=(x_1^{2/3}a^{-3})^{1/5}$ in Lemma 2.1, to obtain

$$I_4 = \sum_{x_1^{1/12}<n\le(x_1a^{-3})^{1/9}} \psi\Big(\Big(\frac{x_1}{a^3n^4}\Big)^{1/5}\Big) + O(y\mu^{-1-\varrho}+1).$$

Thus from $I_4$ we also get sums of the type $S_{a,b,c}(\Omega)$, with a permissible error. Similarly we can treat $I_2$. The other portion of $\sum_2$ with the condition $c>B$ can be treated along the same lines by using Lemmata 2.1 and 2.2. Hence our problem is reduced to treating a linear combination of sums of the type $S_{a,b,c}(\Omega)$. The proof of Theorem 2 can thus been finished.

**3. Three general estimates for $S_{a,b,c}(x)$.** Clearly we can assume that

$$D = \{(m,n) \mid m^{a+b}n^c \le x,\ m\ge n\};$$

the other two cases can be treated similarly and more easily. We need

LEMMA 3.1. *Let $H \geq 1$, $X \geq 1$, $Y \geq 1000$, let $\alpha$, $\beta$ and $\gamma$ be real numbers such that $\alpha\gamma(\gamma-1)(\beta-1) \neq 0$, and for $A > C(\alpha,\beta,\gamma) > 0$, let $f(h,x,y) = Ah^\alpha x^\beta y^\gamma$. Define*

$$S(H,X,Y) = \sum_{(h,x,y)\in D} C_1(h,x)C_2(y)e(f(h,x,y)),$$

*where $D$ is a region contained in the rectangle*

$$\{(h,x,y) \mid h \sim H, x \sim X, y \sim Y\} \qquad (h \sim H \text{ means that } H \leq h < 2H, \text{ etc.})$$

*such that for any fixed pair $(h_0,x_0)$, the intersection $D \cap \{(h_0,x_0,y) \mid y \sim Y\}$ has at most $O(1)$ segments. Also, suppose $|C_1(h,x)| \leq 1$, $|C_2(y)| \leq 1$, $F = AH^\alpha X^\beta Y^\gamma \gg Y$. Then*

$$L^{-3}S(H,X,Y) \ll \sqrt[22]{(HX)^{19}Y^{13}F^3} + HXY^{5/8}(1+Y^7F^{-4})^{1/16} + \sqrt[32]{(HX)^{29}Y^{28}F^{-2}M^5} + \sqrt[4]{(HX)^3Y^4M},$$

*where $L = \ln(AHXY+2)$, $M = \max(1, FY^{-2})$.*

LEMMA 3.2. *Let $f(x,y)$ be an algebraic function in the rectangle $D_0 = \{(x,y) \mid x \sim X,\ y \sim Y\}$, $f(x,y) = Ax^\alpha y^\beta$ for $(x,y) \in D_0$, $D$ be a subdomain of $D_0$ bounded by $O(1)$ algebraic curves. Suppose that $X \gg Y$, $N = XY$, $A > 0$, $F = AX^\alpha Y^\beta$, and $\alpha\beta(\alpha+\beta-1)(\alpha+\beta-2) \neq 0$. Then*

$$S := (NF)^{-\varepsilon} \sum_{(x,y)\in D} e(f(x,y)) \ll \sqrt[6]{F^2N^3} + N^{5/6} + \sqrt[8]{N^8F^{-1}X^{-1}} + NF^{-1/4} + NY^{-1/2}.$$

Lemma 3.1 is Theorem 3 of [4], and Lemma 3.2 is Lemma 9 of [2]. We also need

LEMMA 3.3. *Let $f(x)$ and $g(x)$ be algebraic functions in the interval $[a,b]$, and*

$$|f''(x)| \cong R^{-1}, \qquad |f'''(x)| \ll (RU)^{-1},$$
$$|g(x)| \leq H, \qquad |g'(x)| \ll HU_1^{-1}, \qquad U, U_1 \geq 1.$$

*Then*

$$\sum_{a\leq n\leq b} g(n)e(f(n)) = \sum_{\alpha\leq u\leq\beta} b_u \frac{g(n(u))}{\sqrt{f''(n(u))}} e(f(n(u)) - un(u) + 1/8) + O(H\ln(\beta-\alpha+2) + H(b-a+R)(U^{-1}+U_1^{-1})) + O(H\min(R^{1/2}, \max(1/\langle\alpha\rangle, 1/\langle\beta\rangle))),$$

*where $[\alpha,\beta]$ is the image of $[a,b]$ under the mapping $y = f'(x)$, $n(u)$ is determined by the equation $f'(n(u)) = u$, $b_u = 1/2$ or $1$ according as $u$ is* *one of* $\alpha$, $\beta$ *or not*, $\langle x\rangle$ *is defined by*

$$\langle x\rangle = \begin{cases} \|x\| & \text{if } x \text{ is not an integer}, \ \|x\| = \min_n |x-n|, \ n\in\mathbb{Z}, \\ \beta-\alpha & \text{if } x \text{ is an integer}, \end{cases}$$

*and* $\sqrt{f''} > 0$ *if* $f'' > 0$, $\sqrt{f''} = i\sqrt{|f''|}$ *if* $f'' < 0$.

Proof. This is Theorem 2.2 of [5].

Now we proceed to deal with $S_{a,b,c}(x)$. It suffices to estimate $S(M,N)$, where

$$S(M,N) = \sum_{(m,n)\in D} \psi((xm^{-b}n^{-c})^{1/a}),$$

$$D = D(M,N) = \{(m,n) \mid m\sim M, n\sim N, m^{a+b}n^c \le x, m\ge n\},$$

and $M$, $N$ are any positive integers such that

$$MN > x^{11/92}, \quad 2M\ge N, \quad M^{a+b}N^c \le x.$$

Then, using the familiar reduction (cf. [2]), for a parameter $K\in[100, MN]$, we get, with $\eta = \varepsilon^2$, and some $H\le K^2$,

$$x^{-\eta}S(M,N) \ll MNK^{-1} + \min(1, K/H)\Phi(H,M,N), \tag{0}$$

where

$$\Phi(H,M,N) = H^{-1}\sum_{h\sim H}\Big|\sum_{(m,n)\in D} e(f(h,m,n))\Big|,$$

$$f(h,m,n) = h\left(\frac{x}{m^b n^c}\right)^{1/a}.$$

At this stage, we can assume that $x$ is irrational. We apply Lemma 3.3 to the summation over $m$, and get, with $M_1 = \max(M,n)$, $M_2 = \min((xn^{-c})^{1/(a+b)}, 2M)$,

$$\sum_{M_1\le m\le M_2} e(f(h,m,n))$$

$$= \sum_{U_1<u<U_2} C_1(x^{-1}h^{-a}n^c u^{2a+b})^{-1/(2(a+b))} e(C_2(xh^a u^b n^{-c})^{1/(a+b)})$$

$$+ O\left(\frac{M}{HF} + \ln x\right) + O\left(\min\left(\left(\frac{M^2}{HF}\right)^{1/2}, \frac{1}{U_2 - hb/a}\right)\right.$$

$$\left. + \sum_{1\le i\le 2}\min\left(\left(\frac{M^2}{HF}\right)^{1/2}, \frac{1}{\|g(n,X_i)\|}\right)\right) + R(h,n),$$

where $C_1, C_2, C_3, \ldots$ denote certain constants,

$$U_1 = \frac{hb}{a}(xn^{-c}M_2^{-a-b})^{1/a}, \quad U_2 = \frac{hb}{a}(xn^{-c}M_1^{-a-b})^{1/a},$$

$$g(n,X)=\frac{hb}{a}(xn^{-c}X^{-a-b})^{1/a},$$
$$X_1=\max(n,M),\qquad X_2=2M,\qquad F=(xM^{-b}N^{-c})^{1/a},$$
$$R(h,n)=\begin{cases}\frac{1}{2}C_1(x^{-1}h^{-a}n^cu^{2a+b})^{-1/(2(a+b))}e(C_2(xh^au^bn^{-c})^{1/(a+b)}) \\ \qquad\qquad \text{for } M_2=(xn^{-c})^{1/(a+b)}, u=U_1=hb/a; \\ 0 \quad \text{otherwise.}\end{cases}$$

It is easy to see that

$$\sum_{n\sim N}\min\left(\left(\frac{M^2}{HF}\right)^{1/2},\frac{1}{U_2-hb/a}\right)\ll x^{1/12}\ln x,$$
$$\sum_{n\in I}R(n,h)\ll\left(\frac{M^2}{HF}\right)^{1/2}\left(N^{4/6}(FHN^{-1})^{1/6}+\frac{N}{FH}\right)$$
$$\ll x^{1/12}\qquad(\text{note that } F\gg M,\ldots),$$

where $I$ is some subinterval of $[N,2N]$. We consider

$$\Omega=\sum_{n\sim N}\min\left(\left(\frac{M^2}{HF}\right)^{1/2},\frac{1}{\|g(n,X_1)\|}\right)$$
$$=\Big(\sum_{n\le M,n\sim N}+\sum_{n>M,n\sim N}\Big)\min(A,B)=\Omega_1+\Omega_2,$$

where

$$\Omega_1=\sum_{n\le M,n\sim N}\min\left(\left(\frac{M^2}{HF}\right)^{1/2},\frac{1}{\|g_1(n)\|}\right),$$
$$g_1(n)=\frac{hb}{a}(xn^{-c}M^{-a-b})^{1/a},$$
$$\Omega_2=\sum_{n>M,n\sim N}\min\left(\left(\frac{M^2}{HF}\right)^{1/2},\frac{1}{\|g_2(n)\|}\right),$$
$$g_2(n)=\frac{hb}{a}(xn^{-c-a-b})^{1/a}=\frac{hb}{a}(xn^{-12})^{1/a}.$$

As $g_1(y)$ is monotonic, and $g_1'(y)\cong H(xN^{-c-a}M^{-b-a})^{1/a}$ for $y\sim N$, by Hilfssatz 4 of Krätzel [1] we have

$$\Omega_1\ll(1+H(xN^{-c}M^{-c-a})^{1/a})$$
$$\times((M^2H^{-1}F^{-1})^{1/2}+H^{-1}(x^{-1}N^{c+a}M^{b+a})^{1/a}\ln x)$$
$$\ll(HF)^{1/2}+x^{1/12}\ln x$$

and similarly, $\Omega_2\ll(HF)^{1/2}+x^{1/12}\ln x$ (note that $\Omega_2\ne0$ implies $N\gg M$), and

$$\sum_{n\sim N}\min\left(\left(\frac{M^2}{HF}\right)^{1/2},\frac{1}{\|g(n,X_2)\|}\right)\ll (HF)^{1/2}+x^{1/12}\ln x.$$

From the above observations, we achieve, after a double Abelian summation, the estimate

$$\Phi(H,M,N)\ll H^{-1}\left(\frac{M^2}{HF}\right)^{1/2}\sum_{h\sim H}\Big|\sum_{(u,n)\in D'}e(C_2(xh^a u^b n^{-c})^{1/(a+b)})\Big| + (HF)^{1/2}+x^{1/12}\ln x, \tag{1}$$

where $D'$ is a range contained in $\{(u,n)\mid n\sim N, C_3\le HF/(uM)\le C_4\}$, bounded by $O(1)$ algebraic curves. We will apply the next lemma to choose parameters optimally.

Lemma 3.4. *Let $M>0$, $N>0$, $u_m>0$, $v_n>0$, $A_m>0$, $B_n>0$ $(1\le m\le M,\ 1\le n\le N)$, and let $Q_1$ and $Q_2$ be given nonnegative numbers, $Q_1\le Q_2$. Then there exists an $Q\in[Q_1,Q_2]$ with*

$$\sum_{1\le m\le M}A_mQ^{u_m}+\sum_{1\le n\le N}B_nQ^{-v_n}\ll\sum_{1\le m\le M}\sum_{1\le n\le N}(A_m^{v_n}B_n^{u_m})^{1/(u_m+v_n)} + \sum_{1\le m\le M}A_mQ_1^{u_m}+\sum_{1\le n\le N}B_nQ_2^{-v_n}.$$

This is Lemma 2 of [2]. Now if we apply Lemma 3.2 to the inner double sum in (1), put our estimate into (0), and choose $K$ optimally via Lemma 3.4, we get

Lemma 1.

$$x^{-\varepsilon}S(M,N)\ll\sqrt[8a]{x^2M^{5a-2b}N^{5a-2c}}+\sqrt[11a]{x^3M^{3a-3b}N^{10a-3c}}+x^{11/92}.$$

If we choose $(h,x,y)=(h,u,n)$ in Lemma 3.1, then we get an estimate for the triple exponential sum in (1). Putting it into (0), and choosing $K$ optimally via Lemma 3.4, we immediately get

Lemma 2.

$$\begin{aligned}x^{-\varepsilon}S(M,N)\ll{}&\sqrt[30]{F^{11}M^{11}N^{21}}+\sqrt[24]{F^8M^8N^{18}}+\sqrt[20]{F^4M^4N^{21}}\\&+\sqrt[40]{F^{11}M^{11}N^{36}}+\sqrt[45]{F^{16}M^{16}N^{31}}\\&+\sqrt[5]{F^2M^2N^3}+\sqrt[4]{FMN^4}+x^{11/92}.\end{aligned}$$

To obtain our last general estimate, we apply Lemma 3.3 to the summation over $n$ in (1), to get

$$\Phi(H,M,N)\ll\frac{MN}{H^2F}\sum_{h\sim H}\Big|\sum_{(u,v)\in D''}P(u)Q(v)e(G(u,v,h))\Big| + (HF)^{1/2}+x^{1/12}\ln x, \tag{2}$$

where $|P(u)| \le 1$, $|Q(v)| \le 1$, and $D''$ is a suitable domain contained in the rectangle $\{(u,v) \mid HF/(Mu) \in [C_3, C_4],\, HF/(Nv) \in [C_5, C_6]\}$. We then apply Lemma 3.1 to the sum of (2) by choosing $(h,x,y) = (h,u,v)$, and taking $K$ optimally via Lemma 3.4, we get

LEMMA 3.

$$\begin{aligned} x^{-\varepsilon} S(M,N) \ll\ & \sqrt[32]{F^{13}M^{13}N^{19}} + \sqrt[13]{F^5M^5N^8} + \sqrt[29]{F^{13}M^{13}N^{12}} \\ & + \sqrt[52]{F^{23}M^{23}N^{24}} + \sqrt[47]{F^{18}M^{18}N^{29}} \\ & + \sqrt[6]{F^3M^3N^2} + \sqrt[5]{F^2M^2N^3} + x^{11/92}. \end{aligned}$$

**4. Proof of Theorem 1.** As $M \gg N$, we easily observe that

$$F \ll (xM^{-4}N^{-5})^{1/3},$$

thus by Lemmata 1 and 3 we obtain

$$x^{-\varepsilon} S(M,N) \ll \sqrt[24]{x^2M^7N^5} + \sqrt[33]{x^3N^{12}} + x^{11/92}, \tag{3}$$

$$\begin{aligned} x^{-\varepsilon} S(M,N) \ll\ & \sqrt[96]{x^{13}M^{-13}N^{-8}} + \sqrt[39]{x^5M^{-5}N^{-1}} + \sqrt[87]{x^{13}M^{-13}N^{-29}} \\ & + \sqrt[156]{x^{23}M^{-23}N^{-43}} + \sqrt[141]{x^{18}M^{-18}N^{-3}} \\ & + \sqrt[18]{x^3M^{-3}N^{-9}} + \sqrt[15]{x^2M^{-2}N^{-1}} + x^{11/92}, \end{aligned} \tag{4}$$

and thus

$$\begin{aligned} x^{-\varepsilon} S(M,N) \ll\ & \sum_{1 \le i \le 11} R_i + \sqrt[87]{x^{13}M^{-13}N^{-29}} + \sqrt[156]{x^{23}M^{-23}N^{-43}} \\ & + \sqrt[6]{xM^{-1}N^{-3}} + x^{11/92}, \end{aligned}$$

where

$$\begin{aligned} R_1 &= \min(\sqrt[24]{x^2M^7N^5},\ \sqrt[96]{x^{13}M^{-13}N^{-8}}) \\ &\ll (\sqrt[24]{x^2(M^{13}N^8)^{4/7}})^{7/23} (\sqrt[96]{x^{13}M^{-13}N^{-8}})^{16/23} = x^{11/92}, \end{aligned}$$

$$\begin{aligned} R_2 &= \min(\sqrt[24]{x^2M^7N^5},\ \sqrt[39]{x^5M^{-5}N^{-1}}) \\ &\ll (\sqrt[12]{xM^5N})^{4/17} (\sqrt[39]{x^5M^{-5}N^{-1}})^{13/17} = x^{2/17} < x^{0.119}, \end{aligned}$$

$$\begin{aligned} R_3 &= \min(\sqrt[24]{x^2M^7N^5},\ \sqrt[141]{x^{18}M^{-18}N^{-3}}) \\ &\ll (\sqrt[12]{x(M^6N)^{6/7}})^{14/61} (\sqrt[47]{x^6M^{-6}N^{-1}})^{47/61} = x^{43/366} < x^{0.118}, \end{aligned}$$

$$\begin{aligned} R_4 &= \min(\sqrt[24]{x^2M^7N^5},\ \sqrt[15]{x^2M^{-2}N^{-1}}) \\ &\ll (\sqrt[12]{x(M^2N)^2})^{2/7} (\sqrt[15]{x^2M^{-2}N^{-1}})^{5/7} = x^{5/42} < x^{0.119}, \end{aligned}$$

$$R_5 = \min(\sqrt[33]{x^3N^{12}},\ \sqrt[96]{x^{13}M^{-13}N^{-8}})$$
$$\ll (\sqrt[11]{xN^4})^{77/205}(\sqrt[96]{x^{13}N^{-21}})^{128/205} = x^{73/615} < x^{0.119},$$

$$R_6 = \min(\sqrt[11]{xN^4},\ \sqrt[39]{x^5M^{-5}N^{-1}})$$
$$\ll (\sqrt[11]{xN^4})^{11/37}(\sqrt[39]{x^5N^{-6}})^{26/37} = x^{13/111} < x^{0.118},$$

$$R_7 = \min(\sqrt[11]{xN^4},\ \sqrt[87]{x^{13}M^{-13}N^{-29}})$$
$$\ll (\sqrt[11]{xN^4})^{77/135}(\sqrt[87]{x^{13}N^{-42}})^{58/135} = x^{47/405} < x^{0.117},$$

$$R_8 = \min(\sqrt[11]{xN^4},\ \sqrt[156]{x^{23}M^{-23}N^{-43}})$$
$$\ll (\sqrt[11]{xN^4})^{363/675}(\sqrt[156]{x^{23}N^{-66}})^{46/675} = x^{56/675} < x^{0.1},$$

$$R_9 = \min(\sqrt[11]{xN^4},\ \sqrt[47]{x^6M^{-6}N^{-1}})$$
$$\ll (\sqrt[11]{xN^4})^{77/265}(\sqrt[47]{x^6N^{-7}})^{188/265} = x^{31/265} < x^{0.117},$$

$$R_{10} = \min(\sqrt[11]{xN^4}, \sqrt[6]{xM^{-1}N^{-3}})$$
$$\ll (\sqrt[11]{xN^4})^{11/17}(\sqrt[6]{xN^{-4}})^{6/17} = x^{2/17},$$

$$R_{11} = \min(\sqrt[11]{xN^4},\ \sqrt[15]{x^2M^{-2}N^{-1}})$$
$$\ll (\sqrt[11]{xN^4})^{11/31}(\sqrt[15]{x^2N^{-3}})^{20/31} = x^{11/93}.$$

Hence we have, with $\theta = 11/92$,

$$x^{-\varepsilon}S(M,N) \ll \sqrt[87]{x^{13}M^{-13}N^{-29}} + \sqrt[156]{x^{23}M^{-23}N^{-43}} + \sqrt[6]{xM^{-1}N^{-3}} + x^{\theta}, \tag{5}$$

which, in conjunction with (3), gives

$$x^{-\varepsilon}S(M,N) \ll \sqrt[24]{x^2M^7N^5} + R_7 + R_8 + R_{10} + x^{\theta} \ll \sqrt[24]{x^2M^7N^5} + x^{\theta}. \tag{6}$$

By Lemma 2 we have

$$x^{-\varepsilon}S(M,N) \ll \sqrt[90]{x^{11}M^{-11}N^8} + \sqrt[15]{x^2M^{-2}N^{-1}} + \sqrt[60]{x^4M^{-4}N^{43}} + \sqrt[120]{x^{11}M^{-11}N^{53}} + \sqrt[12]{xM^{-1}N^7} + x^{\theta}. \tag{7}$$

Note that actually $\sqrt[87]{x^{13}M^{-13}N^{-29}} \ll \sqrt[6]{xM^{-1}N^{-3}}$, thus from (5) we get

$$x^{-\varepsilon}S(M,N) \ll \sqrt[156]{x^{23}M^{-23}N^{-43}} + \sqrt[6]{xM^{-1}N^{-3}} + x^{\theta}. \tag{8}$$

From (7) and (8) we get

$$x^{-\varepsilon}S(M,N) \ll \sqrt[90]{x^{11}M^{-11}N^{8}} + \sqrt[15]{x^{2}M^{-2}N^{-1}} + \sum_{1\le i\le 6} S_i + x^{\theta},$$

where

$$\begin{aligned} S_1 &= \min(\sqrt[60]{x^4N^{39}},\ \sqrt[156]{x^{23}N^{-66}}) \\ &\le (\sqrt[60]{x^4N^{39}})^{110/279}(\sqrt[156]{x^{23}N^{-66}})^{169/279} = x^{43/372} < x^{0.116}, \end{aligned}$$

$$\begin{aligned} S_2 &= \min(\sqrt[60]{x^4N^{39}}, \sqrt[6]{xN^{-4}}) \\ &\le (\sqrt[60]{x^4N^{39}})^{40/79}(\sqrt[6]{xN^{-4}})^{39/79} = x^{55/474} < x^{0.117}, \end{aligned}$$

$$\begin{aligned} S_3 &= \min(\sqrt[120]{x^{11}N^{42}},\ \sqrt[156]{x^{23}N^{-66}}) \\ &\le (\sqrt[120]{x^{11}N^{42}})^{110/201}(\sqrt[156]{x^{23}N^{-66}})^{91/201} = x^{47/402} < x^{0.117}, \end{aligned}$$

$$\begin{aligned} S_4 &= \min(\sqrt[120]{x^{11}N^{42}}, \sqrt[6]{xN^{-4}}) \\ &\le (\sqrt[120]{x^{11}N^{42}})^{40/61}(\sqrt[6]{xN^{-4}})^{21/61} = x^{43/366} < x^{0.118}, \end{aligned}$$

$$\begin{aligned} S_5 &= \min(\sqrt[12]{xN^{6}},\ \sqrt[156]{x^{23}N^{-66}}) \\ &\le (\sqrt[12]{xN^{6}})^{11/24}(\sqrt[156]{x^{23}N^{-66}})^{13/24} = x^{17/144} < x^{0.119}, \end{aligned}$$

$$S_6 = \min(\sqrt[12]{xN^{6}}, \sqrt[6]{xN^{-4}}) \le (\sqrt[12]{xN^{6}})^{4/7}(\sqrt[6]{xN^{-4}})^{3/7} = x^{5/42} < x^{\theta}.$$

Thus we have

$$x^{-\varepsilon}S(M,N) \ll \sqrt[90]{x^{11}M^{-11}N^{8}} + \sqrt[15]{x^{2}M^{-2}N^{-1}} + x^{\theta}. \tag{9}$$

From (6), (8) and (9) we finally achieve

$$x^{-\varepsilon}S(M,N) \ll \sum_{1\le i\le 4} K_i + x^{\theta},$$

where

$$\begin{aligned} K_1 &= \min(\sqrt[24]{x^2M^7N^5},\ \sqrt[156]{x^{23}M^{-23}N^{-43}},\ \sqrt[90]{x^{11}M^{-11}N^{8}}) \\ &= \min(A_1, B_1, C_1) \le A_1^{219/692}B_1^{232.5/692}C_1^{240.5/692} = x^{657/5536}, \end{aligned}$$

$$\begin{aligned} K_2 &= \min(\sqrt[24]{x^2M^7N^5},\ \sqrt[156]{x^{23}M^{-23}N^{-43}},\ \sqrt[15]{x^{2}M^{-2}N^{-1}}) \\ &= \min(A_2, B_2, C_2) \le A_2^{168/530}B_2^{52/530}C_2^{310/530} = x^{189/1590}, \end{aligned}$$

$$\begin{aligned} K_3 &= \min(\sqrt[24]{x^2M^7N^5}, \sqrt[6]{xM^{-1}N^{-3}}, \sqrt[15]{x^2M^{-2}N^{-1}}) \\ &= \min(A_3, B_3, C_3) \le A_3^{20/63}B_3^{3/63}C_3^{40/63} = x^{5/42}, \\ K_4 &= \min(\sqrt[24]{x^2M^7N^5}, \sqrt[6]{xM^{-1}N^{-3}}, \sqrt[90]{x^{11}M^{-11}N^8}) \\ &= \min(A_4, B_4, C_4) \le A_4^{164/515}B_4^{111/515}C_4^{240/515} = x^{123/1030}. \end{aligned}$$

This completes the proof.

## References


[1] E. Krätzel, *Zweifache Exponentialsummen und dreidimensionale Gitterpunktprobleme*, in: Elementary and Analytic Theory of Numbers, Banach Center Publ. 17, PWN–Polish Scientific Publishers, Warszawa, 1985, 337–369.

[2] H.-Q. Liu, *On the number of abelian groups of a given order*, Acta Arith. 59 (1991), 261–277.

[3] —, *The number of squarefull numbers in an interval*, ibid. 64 (1993), 129–149.

[4] —, *On the number of abelian groups of a given order* (*supplement*), ibid. 64 (1993), 285–296.

[5] S. H. Min, *Methods in Number Theory*, Vol. 2, Science Press, 1981 (in Chinese).

[6] P. G. Schmidt, *Zur Anzahl Abelscher Gruppen gegebener Ordnung*, J. Reine Angew. Math. 229 (1968), 34–42.

[7] P. Shiu, *The distribution of cube-full numbers*, Glasgow Math. J. 33 (1991), 287–295.



206-10, BAO GUO ST.
HARBIN, 150066
CHINA


*Received on 26.3.1993* (2406)