

On isolated, respectively consecutive large values of arithmetic functions

by

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1. Throughout this paper we shall use the following notations: The set of positive integers is denoted by \mathbb{N} . If $f(x) = O(g(x))$, then we write $f(x) \ll g(x)$. p_i denotes the i th prime. $\omega(n)$ denotes the number of distinct prime divisors of n , while $\Omega(n)$ denotes the number of prime divisors of n counted with multiplicity. $d(n)$ and $\sigma(n)$ denote the number, resp. sum of positive divisors of n . $\mu(n)$ is the Möbius function and $\varphi(n)$ denotes Euler's function. The smallest and greatest prime factors of n are denoted by $p(n)$ and $P(n)$, respectively. If $f(n)$ is an arithmetic function and $x \geq 1$, then we write

$$M(f, x) = \max_{n \leq x} f(n), \quad T(f, x) = \max_{n \leq x} (f(n-1) + f(n))$$

and $G(f, x)$ will denote the greatest integer G such that there is a number $n \in \mathbb{N}$ with $n \leq x$ and

$$f(n) > \sum_{0 < |i| \leq G} f(n+i).$$

In the first half of this paper (Sections 2–7) we will study isolated large values of the arithmetic functions $\omega(n)$, $\Omega(n)$, $d(n)$ and $\sigma(n)$, i.e., the function $G(f, x)$ with these four functions in place of f ; see [1] and [5] for related results. (Since the first author studied a problem closely related to the estimate of $M(\varphi, x)$ in [4], we do not discuss the case $f(n) = \varphi(n)$ here.) In the second half of the paper we will study the converse of this problem by studying consecutive large values of these four functions, i.e., the function $T(f, x)$ with ω, Ω, d and σ in place of f . Note that $T(\Omega, x)$ and $T(\sigma, x)$ were studied by Erdős and Nicolas in [6]; here we will extend and sharpen their results.

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2. First we will prove

THEOREM 1. *There exist effectively computable positive constants c_1 and c_2 such that*

$$(2.1) \quad c_1 \frac{\log x}{(\log \log x)^2} < G(\omega, x) < c_2 \frac{\log x}{\log \log x \log \log \log x}.$$

Note that almost certainly the lower bound gives the right order of magnitude of $G(\omega, x)$ but, unfortunately, we have not been able to show this; this is an interesting but, possibly, deep problem.

PROOF. First we will prove the lower bound in (2.1). Define the positive integer $m = m(x)$ by

$$\prod_{i=1}^{m-1} p_i < x^{1/2} \leq \prod_{i=1}^m p_i$$

and write $P = \prod_{i=1}^m p_i$. Then by the prime number theorem we have

$$(2.2) \quad m = \left(\frac{1}{2} + o(1) \right) \frac{\log x}{\log \log x}$$

and

$$(2.3) \quad x^{1/2} \leq P = p_m \prod_{i=1}^{m-1} p_i \ll x^{1/2} \log x.$$

To prove the lower bound in (2.1), it suffices to show that writing

$$G = \left[c_1 \frac{\log x}{(\log \log x)^2} \right] + 1,$$

for c_1 small enough there is a $j \in \mathbb{N}$ such that

$$(2.4) \quad j \leq x/P \quad \text{and} \quad \omega(jP) > \sum_{0 < |i| \leq G} \omega(jP + i).$$

By (2.2) we have

$$(2.5) \quad \begin{aligned} \omega(jP) &\geq \omega(P) = m \\ &= \left(\frac{1}{2} + o(1) \right) \frac{\log x}{\log \log x} \quad \text{for all } j \in \mathbb{N}. \end{aligned}$$

On the other hand, if $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ denote the sets of primes p with $p \leq G$, $G < p \leq x^{1/3}$, resp. $x^{1/3} < p \leq 2x$, then by (2.2), (2.3) and

$$(2.6) \quad \sum_{p \leq y} \frac{1}{p} = \log \log y + C + O\left(\frac{1}{\log y} \right)$$

we have

$$\begin{aligned}
 \sum_{j \leq x/P} \sum_{0 < |i| \leq G} \omega(jP + i) &= \sum_{j \leq x/P} \sum_{0 < |i| \leq G} \sum_{p|jP+i} 1 \\
 &= \sum_{j \leq x/P} \sum_{p \in \mathcal{P}_1} \sum_{\substack{0 < |i| \leq G \\ p|jP+i}} 1 + \sum_{0 < |i| \leq G} \sum_{p \in \mathcal{P}_2} \sum_{\substack{j \leq x/P \\ p|jP+i}} 1 \\
 &\quad + \sum_{j \leq x/P} \sum_{0 < |i| \leq G} \sum_{\substack{p \in \mathcal{P}_3 \\ p|jP+i}} 1 \\
 &\ll \sum_{j \leq x/P} \sum_{p \leq G} \frac{G}{p} + \sum_{i=1}^G \sum_{G < p \leq x^{1/3}} \frac{x}{pP} + \sum_{j \leq x/P} \sum_{i=1}^G 1 \\
 &\ll G \frac{x}{P} \left(\sum_{p \leq x^{1/3}} \frac{1}{p} + 1 \right) \ll G \frac{x}{P} \log \log x,
 \end{aligned}$$

whence

$$(2.7) \quad \min_{j \leq x/P} \sum_{0 < |i| \leq G} \omega(jP + i) \ll G \log \log x < \frac{1}{3} \frac{\log x}{\log \log x}$$

if c_1 is small enough. If the minimum in (2.7) is assumed for, say, $j = j_0$, then it follows from (2.5) and (2.7) that (2.4) holds with j_0 in place of j and this completes the proof of the lower bound in (2.1).

To prove the upper bound in (2.1), it suffices to show that writing

$$H = \left[c_2 \frac{\log x}{\log \log x \log \log \log x} \right],$$

for c_2 large enough we have

$$(2.8) \quad \omega(n) \leq \sum_{0 < |i| \leq H} \omega(n + i)$$

for all $n \leq x$. Indeed, for $n \leq x$ we have

$$(2.9) \quad \omega(n) \leq M(\omega, x) = (1 + o(1)) \frac{\log x}{\log \log x}$$

and, on the other hand, by (2.6) we have

$$(2.10) \quad \sum_{0 < |i| \leq H} \omega(n + i) \geq \sum_{p \leq H} \sum_{\substack{0 < |i| \leq H \\ p|n+i}} 1 \gg H \log \log H.$$

If c_2 is large enough, then (2.8) follows from (2.9) and (2.10), and this completes the proof of Theorem 1.

3. $G(\Omega, x)$ can be estimated similarly:

THEOREM 2. *There exist effectively computable positive constants c_3, c_4 such that*

$$(3.1) \quad c_3 \frac{\log x}{\log \log x} < G(\Omega, x) < c_4 \frac{\log x}{\log \log \log x}.$$

Again, we think that the lower bound gives the right order of magnitude.

Proof. The proof is similar to the proof of Theorem 1, thus we will only sketch it. Write

$$G = \left\lceil c_3 \frac{\log x}{\log \log x} \right\rceil + 1$$

and define $u = u(x) \in \mathbb{N}$ by

$$2^u \leq x^{1/2} < 2^{u+1}$$

so that $u = \left\lfloor \frac{\log x}{2 \log 2} \right\rfloor$. Then for $j \in \mathbb{N}, 1 \leq j \leq x/2^u$ we have

$$(3.2) \quad \Omega(j \cdot 2^u) \geq \Omega(2^u) = u \gg \log x$$

and, on the other hand, it can be shown by an argument similar to the one in the proof of Theorem 1 that

$$\sum_{j \leq x/2^u} \sum_{0 < |i| \leq G} \Omega(j \cdot 2^u + i) \ll \frac{x}{2^u} G \log \log x$$

so that

$$(3.3) \quad \min_{j \leq x/2^u} \sum_{0 < |i| \leq G} \Omega(j \cdot 2^u + i) \ll G \log \log x.$$

If c_3 is small enough, then it follows from (3.2) and (3.3) that

$$\min_{j \leq x/2^u} \Omega(j \cdot 2^u) > \min_{j \leq x/2^u} \sum_{0 < |i| \leq G} \Omega(j \cdot 2^u + i),$$

which proves the lower bound in (3.1).

To prove the upper bound in (3.1), observe that for $n \leq x$ we have

$$(3.4) \quad \Omega(n) \leq M(\Omega, x) = \left\lceil \frac{\log x}{\log 2} \right\rceil$$

and, writing

$$H = \left\lceil c_4 \frac{\log x}{\log \log \log x} \right\rceil,$$

we have

$$(3.5) \quad \sum_{0 < |i| \leq H} \Omega(n+i) \gg H \log \log H,$$

which can be proved in the same way as (2.10). If c_4 is large enough, then it follows from (3.4) and (3.5) that

$$\Omega(n) \leq \sum_{0 < |i| \leq H} \Omega(n+i)$$

(for all $n \leq x$), which implies the upper bound in (3.1).

4. The function $d(n)$ gives the most interesting and most difficult problem. In this case, we will prove

THEOREM 3. *There are effectively computable absolute constants x_0 and c_5 such that for $x > x_0$ we have*

$$(4.1) \quad M(d, x) \exp\left(-11 \frac{\log x}{(\log \log x)^{3/2}}\right) < G(d, x) < c_5 M(d, x) \frac{\log \log x}{\log x}.$$

Note that it follows from the results of Ramanujan [9] that

$$(4.2) \quad M(d, x) = \exp\left(\log 2 \frac{\log x}{\log \log x} + O\left(\frac{\log x}{(\log \log x)^2}\right)\right).$$

It follows from (4.1) and (4.2) that

$$G(d, x) = \exp\left(\log 2 \frac{\log x}{\log \log x} + O\left(\frac{\log x}{(\log \log x)^{3/2}}\right)\right).$$

We expect the upper bound to be closer to the truth and perhaps we have

$$G(d, x) > M(d, x)(\log x)^{-c}.$$

5. The proof of the lower bound in (4.1) will be based on the following lemma:

LEMMA 1. *There is an effectively computable number x_0 such that if $x > x_0$, $a \in \mathbb{N}$, $Q \in \mathbb{N}$,*

$$(5.1) \quad aQ \leq x,$$

$$(5.2) \quad Q > \exp\left(8 \frac{\log x}{(\log \log x)^{1/2}}\right),$$

$b \in \mathbb{Z}$ and

$$(5.3) \quad |b| \leq a,$$

then there is a set $\mathcal{S}(a, b, Q)$ such that

$$(5.4) \quad \mathcal{S}(a, b, Q) \subset \{1, 2, \dots, Q\},$$

$$(5.5) \quad |\mathcal{S}(a, b, Q)| < Q \exp\left(-2 \frac{\log x}{\log \log x}\right)$$

and

$$(5.6) \quad \sum_{\substack{i \leq Q \\ i \notin \mathcal{S}(a, b, Q)}} d(ai + b) < d(b)Q \exp\left(4 \frac{\log x}{(\log \log x)^{3/2}}\right).$$

Proof of Lemma 1. Clearly, for all $m, n \in \mathbb{N}$ we have

$$(5.7) \quad d(mn) \leq d(m)d(n).$$

Thus writing $a = (a, b)a_1, b = (a, b)b_1$, for all $i \in \mathbb{N}$ we have

$$(5.8) \quad d(ai + b) \leq d((a, b))d(a_1i + b_1) \leq d(b)d(a_1i + b_1).$$

Set

$$K = \frac{\log x}{(\log \log x)^2}, \quad L = \exp((\log \log x)^{3/2}),$$

and for $n \in \mathbb{N}$ define $u = u(n), v = v(n), w = w(n)$ by

$$(5.9) \quad n = uvw, \quad P(u) \leq K < p(v) \leq P(v) \leq L < p(w).$$

(If there is no prime p with $p | n$ and $p \leq K, K < p \leq L$ or $L < p$, then we put $u = 1, v = 1$ and $w = 1$, respectively.) Then for large x and $n \leq 2x$, by the prime number theorem clearly we have

$$(5.10) \quad \begin{aligned} d(u) &= \prod_{p \leq K, p^\alpha \| n} d(p^\alpha) = \prod_{p \leq K, p^\alpha \| n} (\alpha + 1) \\ &\leq \prod_{p \leq K} \left(\frac{\log 2x}{\log 2} + 1\right) = \exp((1 + o(1))\pi(K) \log \log x) \\ &= \exp\left((1 + o(1)) \frac{\log x}{(\log \log x)^2}\right) < \exp\left(2 \frac{\log x}{(\log \log x)^2}\right). \end{aligned}$$

Moreover, it follows from the definition of w that for $n \leq 2x$ we have

$$2x \geq n \geq w = \prod_{p > L, p^\alpha \| n} p^\alpha \geq \prod_{p > L, p^\alpha \| n} L^\alpha = L^{\Omega(w)},$$

whence

$$(5.11) \quad \Omega(w) \leq \frac{\log 2x}{\log L} < 2 \frac{\log x}{(\log \log x)^{3/2}}.$$

For all $m \in \mathbb{N}$ we have $d(m) \leq 2^{\Omega(m)}$. Thus from (5.11) we obtain

$$(5.12) \quad d(w) \leq \exp\left(2 \log 2 \frac{\log x}{(\log \log x)^{3/2}}\right).$$

By (5.1), (5.3) and (5.7), it follows from (5.8), (5.9), (5.10) and (5.12) that for all $i \leq Q$ we have

$$(5.13) \quad \begin{aligned} d(ai + b) &\leq d(b)d(a_1i + b_1) \\ &\leq d(b)d(u(a_1i + b_1))d(v(a_1i + b_1))d(w(a_1i + b_1)) \\ &< d(b)d(v(a_1i + b_1)) \exp\left(3 \frac{\log x}{(\log \log x)^{3/2}}\right) \quad (\text{for } i \leq Q). \end{aligned}$$

Now write

$$z = \left\lceil 7 \frac{\log x}{(\log \log x)^2} \right\rceil,$$

and let $\mathcal{S}(a, b, Q)$ denote the set of the positive integers i such that $i \leq Q$ and

$$(5.14) \quad \Omega(v(a_1i + b_1)) > z.$$

Then (5.4) holds trivially. Write $v(a_1i + b_1)$ as the product of a square and a square-free number:

$$(5.15) \quad v(a_1i + b_1) = r^2q \quad (r, q \in \mathbb{N}, |\mu(q)| = 1).$$

It follows from (5.14) and (5.15) that either

$$(5.16) \quad \Omega(r) \geq z/3$$

or

$$(5.17) \quad \Omega(q) = \omega(q) \geq z/3.$$

Let \mathcal{T}_1 denote the set of the integers t with $t \in \mathbb{N}$, $K < p(t) \leq P(t) \leq L$, $\Omega(t) = [z/3]$, and let \mathcal{T}_2 denote the set of square-free elements of \mathcal{T}_1 so that for $t \in \mathcal{T}_2$ we have $K < p(t) \leq P(t) \leq L$, $\Omega(t) = \omega(t) = [z/3]$. Then for $i \in \mathcal{S}(n, b, Q)$ either there is an integer t such that $t \in \mathcal{T}_1$ and $t^2 \mid a_1i + b_1$, or there is an integer t such that $t \in \mathcal{T}_2$ and $t \mid a_1i + b_1$ (in fact, if (5.16) holds, then t can be chosen as any divisor of r with $\Omega(t) = [z/3]$, while if (5.17) holds, then t can be any divisor of q with $\Omega(t) = \omega(t) = [z/3]$). It follows that

$$(5.18) \quad \begin{aligned} |\mathcal{S}(a, b, Q)| &\leq \sum_{t \in \mathcal{T}_1} |\{i : i \leq Q, t^2 \mid a_1i + b_1\}| \\ &\quad + \sum_{t \in \mathcal{T}_2} |\{i : i \leq Q, t \mid a_1i + b_1\}|. \end{aligned}$$

Clearly, if $t \in \mathcal{T}_1 \cup \mathcal{T}_2$, then we have

$$(5.19) \quad \begin{aligned} t &= \prod_{p^\alpha \parallel t} p^\alpha > K^{\Omega(t)} = K^{[z/3]} = \exp\left((1 + o(1)) \frac{z}{3} \log K\right) \\ &= \exp\left((1 + o(1)) \frac{7}{3} \frac{\log x}{\log \log x}\right) \quad (\text{for } t \in \mathcal{T}_1 \cup \mathcal{T}_2) \end{aligned}$$

and

$$\begin{aligned}
 (5.20) \quad t &= \prod_{p^\alpha \parallel t} p^\alpha \leq L^{\Omega(t)} = L^{\lfloor z/3 \rfloor} = \exp\left((1 + o(1)) \frac{z}{3} \log L \right) \\
 &= \exp\left((1 + o(1)) \frac{7}{3} \frac{\log x}{(\log \log x)^{1/2}} \right) \\
 &< \left(3 \frac{\log x}{(\log \log x)^{1/2}} \right) < Q^{1/2} \quad (\text{for } t \in \mathcal{T}_1 \cup \mathcal{T}_2).
 \end{aligned}$$

By $(a_1, b_1) = 1$, for all $d \in \{1, 2, \dots, Q\}$ we have

$$(5.21) \quad |\{i : i \leq Q, d \mid a_1 i + b_1\}| \leq \frac{Q}{d} + 1 \leq 2 \frac{Q}{d} \quad (\text{for } d \leq Q).$$

It follows from (5.18), (5.20) and (5.21) that

$$(5.22) \quad |\mathcal{S}(a, b, Q)| \leq \sum_{t \in \mathcal{T}_1} 2 \frac{Q}{t^2} + \sum_{t \in \mathcal{T}_2} 2 \frac{Q}{t} = 2Q \left(\sum_{t \in \mathcal{T}_1} \frac{1}{t^2} + \sum_{t \in \mathcal{T}_2} \frac{1}{t} \right).$$

By (5.19) we have

$$(5.23) \quad \sum_{t \in \mathcal{T}_1} \frac{1}{t^2} \leq \sum_{t > \exp\left((1+o(1)) \frac{7}{3} \frac{\log x}{\log \log x} \right)} \frac{1}{t^2} = \exp\left(- (1 + o(1)) \frac{7}{3} \frac{\log x}{\log \log x} \right).$$

Moreover, by (2.6) and Stirling's formula we have

$$\begin{aligned}
 (5.24) \quad \sum_{t \in \mathcal{T}_2} \frac{1}{t} &= \sum_{K < p_{i_1} < p_{i_2} < \dots < p_{i_{\lfloor z/3 \rfloor}} \leq L} \frac{1}{p_{i_1} p_{i_2} \dots p_{i_{\lfloor z/3 \rfloor}}} \\
 &< \frac{1}{(\lfloor z/3 \rfloor)!} \left(\sum_{p \leq L} \frac{1}{p} \right)^{\lfloor z/3 \rfloor} = \exp\left(- (1 + o(1)) \frac{7}{3} \frac{\log x}{\log \log x} \right).
 \end{aligned}$$

(5.5) follows from (5.22), (5.23) and (5.24).

It remains to show that (5.6) also holds. By (5.13) we have

$$\begin{aligned}
 (5.25) \quad \sum_{i \leq Q, i \notin \mathcal{S}(a, b, Q)} d(ai + b) \\
 < d(b) \exp\left(3 \frac{\log x}{(\log \log x)^{3/2}} \right) \sum_{i \leq Q, i \notin \mathcal{S}(a, b, Q)} d(v(a_1 i + b_1)).
 \end{aligned}$$

Here the last sum is

$$\sum_{i \leq Q, i \notin \mathcal{S}(a, b, Q)} |\{d : d \mid v(a_1 i + b_1)\}|.$$

If $d \mid v(a_1 i + b_1)$ for some $i \leq Q$, $i \notin \mathcal{S}(a, b, Q)$, then, by the definitions of $v(n)$ and $\mathcal{S}(a, b, Q)$, for all $p \mid d$ we have $p \leq L$; moreover, $\Omega(d) \leq z$. Thus

by (5.2), for such a d we have

$$d = \prod_{p^\alpha \parallel d} p^\alpha \leq L^{\Omega(d)} \leq L^z = \exp\left((1 + o(1))7 \frac{\log x}{(\log \log x)^{1/2}}\right) < Q$$

so that for this fixed d , by (5.21) we have

$$|\{i : i \leq Q, i \notin \mathcal{S}(a, b, Q), d \mid a_1 i + b_1\}| \leq 2 \frac{Q}{d},$$

whence, by Mertens' formula,

$$\begin{aligned} (5.26) \quad \sum_{i \leq Q, i \notin \mathcal{S}(a, b, Q)} d(v(a_1 i + b_1)) &\leq \sum_{P(d) \leq L, \Omega(d) \leq z} 2 \frac{Q}{d} < 2Q \sum_{P(d) \leq L} \frac{1}{d} \\ &= 2Q \prod_{p \leq L} \sum_{\alpha=0}^{\infty} \frac{1}{p^\alpha} = 2Q \prod_{p \leq L} \left(1 - \frac{1}{p}\right)^{-1} \\ &\ll Q \log L = Q(\log \log x)^{3/2}. \end{aligned}$$

(5.6) follows from (5.25) and (5.26), and this completes the proof of the lemma.

6. Completion of the proof of Theorem 3. First we will prove the lower bound in (4.1). Set

$$(6.1) \quad Q = \left\lceil \exp\left(8 \frac{\log x}{(\log \log x)^{1/2}}\right) \right\rceil + 1$$

and

$$(6.2) \quad G = \left\lceil M(d, x) \exp\left(-11 \frac{\log x}{(\log \log x)^{3/2}}\right) \right\rceil + 1.$$

Define the positive integer m by

$$(6.3) \quad \prod_{i=1}^m p_i \leq \frac{x}{Q} < \prod_{i=1}^{m+1} p_i$$

so that, by the prime number theorem, we have

$$\begin{aligned} (6.4) \quad m &= \frac{\log(x/Q)}{\log \log(x/Q)} (1 + O((\log \log(x/Q))^{-1})) \\ &= \frac{\log x}{\log \log x} (1 - 8(\log \log x)^{-1/2} + O((\log \log x)^{-1})). \end{aligned}$$

Write

$$(6.5) \quad P = \prod_{i=1}^m p_i.$$

In order to prove the lower bound in (4.1), it suffices to show that for large x there is a $j \in \mathbb{N}$ such that

$$(6.6) \quad j \leq x/P \quad \text{and} \quad d(jP) > \sum_{0 < |i| \leq G} d(jP + i).$$

By (4.2) and (6.4), for large x we have

$$(6.7) \quad \begin{aligned} d(jP) &\geq d(P) = 2^m \\ &= \exp\left(\log 2 \frac{\log x}{\log \log x} (1 - 8(\log \log x)^{-1/2} + O((\log \log x)^{-1}))\right) \\ &= M(d, x) \exp\left(- (1 + o(1)) 8 \log 2 \frac{\log x}{(\log \log x)^{3/2}}\right) \end{aligned}$$

(for all $j \leq x/P$).

On the other hand, let $\mathcal{S} = \bigcup_{0 < |b| \leq G} \mathcal{S}(P, b, Q)$ where G, P, Q are defined by (6.2), (6.5) and (6.1), respectively, and the set $\mathcal{S}(P, b, Q)$ is defined in Lemma 1. Then by Lemma 1 and (4.2), for large x we have

$$(6.8) \quad |\mathcal{S}| \leq \sum_{0 < |b| \leq Q} |\mathcal{S}(P, b, Q)| \leq 2GQ \exp\left(- 2 \frac{\log x}{\log \log x}\right) < \frac{Q}{2}.$$

Let $\mathcal{J} = \{1, 2, \dots, Q\} \setminus \mathcal{S}$ so that by (6.8) we have

$$(6.9) \quad |\mathcal{J}| = Q - |\mathcal{S}| > Q/2.$$

Then by Lemma 1 and the definition of \mathcal{J} we have

$$(6.10) \quad \begin{aligned} \sum_{i \in \mathcal{J}} \sum_{0 < |b| \leq G} d(Pi + b) &\leq \sum_{0 < |b| \leq G} \sum_{i \leq Q, i \notin \mathcal{S}(P, b, Q)} d(Pi + b) \\ &< \sum_{0 < |b| \leq G} d(b)Q \exp\left(4 \frac{\log x}{(\log \log x)^{3/2}}\right) \\ &= Q \exp\left(4 \frac{\log x}{(\log \log x)^{3/2}}\right) \sum_{0 < |b| \leq G} d(b) \\ &\ll Q \exp\left(4 \frac{\log x}{(\log \log x)^{3/2}}\right) G \log G. \end{aligned}$$

It follows from (6.9) and (6.10) that there is a $j \in \mathbb{N}$ with

$$(6.11) \quad j \leq Q \leq x/P$$

such that

$$(6.12) \quad \sum_{0 < |b| \leq G} d(Pj + b) \ll \exp\left(4 \frac{\log x}{(\log \log x)^{3/2}}\right) G \log G$$

$$< M(d, x) \exp\left(-6 \frac{\log x}{(\log \log x)^{3/2}}\right).$$

(6.6) follows from (6.7), (6.11) and (6.12), and this completes the proof of the lower bound in (4.1).

To prove the upper bound in (4.1), write

$$H = \left[c_6 \frac{\log \log x}{\log x} M(d, x) \right].$$

Then for sufficiently large c_6 and all $n \leq x$, by (4.2) we have

$$\begin{aligned} \sum_{0 < |i| \leq H} d(n + i) &= \sum_{0 < |i| \leq H} \sum_{d|n+i} 1 \geq \sum_{0 < |i| \leq H} \sum_{\substack{d|n+i \\ d \leq H}} 1 \\ &= \sum_{d=1}^H \sum_{\substack{0 < |i| \leq H \\ d|n+i}} 1 \gg \sum_{d=1}^H \frac{H}{d} \\ &\gg H \log H > M(d, x) \geq d(n) \end{aligned}$$

which proves the upper bound in (4.1).

7. The sharpest estimate can be given for $G(\sigma, x)$:

THEOREM 4. *We have*

$$(7.1) \quad G(\sigma, x) = (1 + o(1)) 3\pi^{-2} e^\gamma \log \log x$$

where $\gamma = 0.57722\dots$ is Euler's constant.

Proof. First we will show that

$$(7.2) \quad G(\sigma, x) \geq (1 + o(1)) 3\pi^{-2} e^\gamma \log \log x.$$

Let $0 < \varepsilon < 1$, and write

$$G = [(1 - \varepsilon) 3\pi^{-2} e^\gamma \log \log x].$$

Moreover, for $p \leq \frac{1}{2} \log x$ define α_p by

$$(7.3) \quad p^{\alpha_p - 1} \leq (\log x)^{1/2} < p^{\alpha_p}$$

and let

$$R = \prod_{p \leq \frac{1}{2} \log x} p^{\alpha_p}.$$

Then by the prime number theorem we have

$$\begin{aligned}
 R &= \prod_{p \leq (\log x)^{1/2}} p^{\alpha_p - 1} \prod_{p \leq \frac{1}{2} \log x} p \leq \prod_{p \leq (\log x)^{1/2}} (\log x)^{1/2} \prod_{p \leq \frac{1}{2} \log x} p \\
 &= \exp\left(\frac{1}{2} \log \log x \pi((\log x)^{1/2}) + \left(\frac{1}{2} + o(1)\right) \log x\right) = \exp\left(\left(\frac{1}{2} + o(1)\right) \log x\right),
 \end{aligned}$$

whence

$$(7.4) \quad R \leq x.$$

To prove (7.2), it suffices to show that for all ε and $x > x_0(\varepsilon)$ there is a $j \in \mathbb{N}$ such that

$$(7.5) \quad j \leq x/R \quad \text{and} \quad \sigma(jR) > \sum_{0 < |i| \leq G} \sigma(jR + i).$$

For all $m, n \in \mathbb{N}$ we have

$$(7.6) \quad \frac{\sigma(mn)}{mn} \geq \frac{\sigma(m)}{m}.$$

Thus by Mertens' formula, for all $j \in \mathbb{N}$ we have

$$\begin{aligned}
 (7.7) \quad & \frac{\sigma(jR)}{jR} \\
 & \geq \frac{\sigma(R)}{R} = \prod_{p \leq \frac{1}{2} \log x} \sigma(p^{\alpha_p}) p^{-\alpha_p} \\
 & = \prod_{p \leq \frac{1}{2} \log x} \left(\sum_{\alpha=0}^{\alpha_p} p^{-\alpha} \right) = \prod_{p \leq \frac{1}{2} \log x} \left(1 - \frac{1}{p} \right)^{-1} (1 - p^{-(\alpha_p+1)}) \\
 & > \prod_{p \leq \frac{1}{2} \log x} \left(1 - \frac{1}{p} \right)^{-1} \prod_{p \leq (\log x)^{1/4}} (1 - (\log x)^{-1/2}) \prod_{(\log x)^{1/4} < p} (1 - p^{-2}) \\
 & = (1 + o(1)) e^\gamma \log \log x (1 - (\log x)^{-1/2})^{\pi((\log x)^{1/4})} (1 + o(1)) \\
 & = (1 + o(1)) e^\gamma \log \log x.
 \end{aligned}$$

On the other hand, uniformly for $j \in \mathbb{N}$, $j \leq x/R$ we have

$$\begin{aligned}
 (7.8) \quad & \frac{1}{jR} \sum_{0 < |i| \leq G} \sigma(jR + i) = \sum_{0 < |i| \leq G} \frac{\sigma(jR + i)}{jR + i} \cdot \frac{jR + i}{jR} \\
 & = \left(1 + O\left(\frac{G}{R}\right) \right) \sum_{0 < |i| \leq G} \frac{\sigma(jR + i)}{jR + i} \\
 & = (1 + o(1)) \sum_{0 < |i| \leq G} \frac{\sigma(jR + i)}{jR + i}.
 \end{aligned}$$

Now consider

$$(7.9) \quad \sum_{j \leq x/R} \sum_{0 < |i| \leq G} \frac{\sigma(jR+i)}{jR+i} = \sum_{j \leq x/R} \sum_{0 < |i| \leq G} \sum_{d|jR+i} \frac{1}{d}.$$

We split this sum into two parts according to the magnitude of d . First we have

$$(7.10) \quad \begin{aligned} \sum_{d \leq G} \sum_{j \leq x/R} \sum_{\substack{0 < |i| \leq G \\ d|jR+i}} \frac{1}{d} &= \sum_{d \leq G} \frac{1}{d} \sum_{j \leq x/R} \sum_{\substack{0 < |i| \leq G \\ d|jR+i}} 1 \\ &\leq \sum_{d \leq G} \frac{1}{d} \sum_{j \leq x/R} 2 \left(\frac{G}{d} + 1 \right) \\ &= 2 \left[\frac{x}{R} \right] \sum_{d \leq G} \left(\frac{G}{d^2} + \frac{1}{d} \right) \\ &< \left[\frac{x}{R} \right] \left(\frac{\pi^2}{3} G + O(\log G) \right). \end{aligned}$$

Moreover, for $G < d \leq 2x$, $0 < |i| \leq G$ we have

$$\begin{aligned} \sum_{\substack{j \leq x/R \\ d|jR+i}} \frac{1}{d} &= \frac{1}{d} \left| \left\{ j : j \leq x/R, \frac{d}{(d, (R, i))} \mid \left(j \frac{R}{(R, i)} + \frac{i}{(R, i)} \right) \right\} \right| \\ &\leq \frac{1}{d} \left(\frac{x(d, (R, i))}{Rd} + 1 \right) \leq \frac{1}{d} \left(\frac{x(d, i)}{Rd} + 1 \right). \end{aligned}$$

Thus writing $(d, i) = r$, $d = rs$, $i = rt$, we have

$$(7.11) \quad \begin{aligned} \sum_{G < d \leq 2x} \sum_{0 < |i| \leq G} \sum_{\substack{j \leq x/R \\ d|jR+i}} \frac{1}{d} &\leq \sum_{G < d \leq 2x} \left(\frac{2G}{d} + 2 \frac{x}{R} \sum_{i=1}^G \frac{(d, i)}{d^2} \right) \\ &\ll G \log x + \frac{x}{R} \sum_{r=1}^G \sum_{t \leq G/r} \sum_{G/r < s \leq 2x/r} \frac{r}{r^2 s^2} \\ &\ll G \log x + \frac{x}{R} \sum_{r=1}^G \sum_{t \leq G/r} \frac{1}{G} \\ &\ll G \log x + \frac{x}{R} \log G \ll \frac{x}{R} \log G. \end{aligned}$$

By (7.9), (7.10) and (7.11), we have

$$\sum_{j \leq x/R} \sum_{0 < |i| \leq G} \frac{\sigma(jR+i)}{jR+i} \leq \left[\frac{x}{R} \right] (1 + o(1)) \frac{\pi^2}{3} G.$$

It follows that there is a $j \leq x/R$ with

$$\sum_{0 < |i| \leq G} \frac{\sigma(jR+i)}{jR+i} \leq (1+o(1)) \frac{\pi^2}{3} G.$$

Combining this with (7.7), we obtain

$$(7.12) \quad \frac{1}{jR} \sum_{0 < |i| \leq G} \sigma(jR+i) \leq (1+o(1)) \frac{\pi^2}{3} G \\ = (1+o(1))(1-\varepsilon)e^\gamma \log \log x.$$

(7.5) follows from (7.7) and (7.12), which completes the proof of (7.2).

To prove that

$$G(\sigma, x) \leq (1+o(1))3\pi^{-2}e^\gamma \log \log x,$$

we have to show that if $\varepsilon > 0$ and we write

$$H = [(1+\varepsilon)3\pi^{-2}e^\gamma \log \log x],$$

then for $x > x_0(\varepsilon)$, $H < n \leq x$ we have

$$\sigma(n) < \sum_{0 < |i| \leq H} \sigma(n+i)$$

or, in equivalent form,

$$(7.13) \quad \frac{\sigma(n)}{n} < \frac{1}{n} \sum_{0 < |i| \leq H} \sigma(n+i).$$

Since (7.13) is nearly trivial for $n/H = O(1)$, we may assume that $n/H \rightarrow \infty$. It is well-known (and, by using Mertens' formula, it can be shown easily) that for $n \leq x$ we have

$$(7.14) \quad \frac{\sigma(n)}{n} \leq (1+o(1))e^\gamma \log \log x.$$

On the other hand, for $n/H \rightarrow \infty$ we have

$$(7.15) \quad \frac{1}{n} \sum_{0 < |i| \leq H} \sigma(n+i) = (1+o(1)) \sum_{0 < |i| \leq H} \frac{\sigma(n+i)}{n+i} \\ = (1+o(1)) \sum_{0 < |i| \leq H} \sum_{d|n+i} \frac{1}{d} \\ \geq (1+o(1)) \sum_{d=1}^H \frac{1}{d} \sum_{\substack{0 < |i| \leq H \\ d|n+i}} 1$$

$$\begin{aligned}
 &= (1 + o(1)) \sum_{d=1}^H \frac{1}{d} \left(2 \frac{H}{d} + O(1) \right) \\
 &= (1 + o(1)) \left(2H \sum_{d=1}^H \frac{1}{d^2} + O \left(\sum_{d=1}^H \frac{1}{d} \right) \right) \\
 &= (1 + o(1)) \left(\frac{\pi^2}{3} H + O(\log H) \right) \\
 &= (1 + o(1))(1 + \varepsilon)e^\gamma \log \log x.
 \end{aligned}$$

If $x > x_0(\varepsilon)$, then (7.13) follows from (7.14) and (7.15), which completes the proof of the theorem.

8. Our results and methods presented above are of different nature for each of the four functions $\omega(n)$, $\Omega(n)$, $d(n)$ and $\sigma(n)$. The following facts explain some of these differences:

If $f(n)$ is an arithmetic function and $n \in \mathbb{N}$, then let $g(f, n)$ denote the smallest integer g such that

$$f(n) < \sum_{i=1}^g f(n+i).$$

If $f(n) = \omega(n), \Omega(n), d(n)$ or $\sigma(n)$, then the density of the integers n with $f(n) < f(n+1)$ is $1/2$. Moreover, if $f(n) = \omega(n)$ or $\Omega(n)$, then for almost all n we have $f(n) = (1 + o(1)) \log \log n$. It follows that for almost all n we have $g(\omega, n) \leq 2$ and $g(\Omega, n) \leq 2$ and, indeed, the density of the integers n with $g(\omega, n) = 1, g(\omega, n) = 2, g(\Omega, n) = 1, \text{ resp. } g(\Omega, n) = 2$ is $1/2$.

This is not so for the functions $\sigma(n)$ and $d(n)$. Indeed, considering the function $d(n)$, for every $t \in \mathbb{N}$ the density of the integers n with $g(d, n) = t$ is $1/2^t$, and for almost all n we have $d(n + g(d, n)) > d(n)$. If we consider the function $\sigma(n)$, then again, the density of the integers n with $g(\sigma, n) = t$ is positive for all $n \in \mathbb{N}$ but, on the other hand, the density of the integers n with $\sigma(n + g(\sigma, n)) > \sigma(n)$ is < 1 (and > 0).

One might like to study the analogous questions for $f(n) = P(n)$. We can show that

$$\lim_{x \rightarrow \infty} G(P, x) = \infty;$$

it would be interesting to estimate $g(P, x)$.

9. In the rest of this paper we will study consecutive large values of arithmetic functions. Erdős and Nicolas [6] proved that

$$(9.1) \quad T(\Omega, x) = (1 + o(1)) \frac{\log x}{\log 2}$$

where the error term is ineffective and

$$(9.2) \quad T(\sigma, x) \leq (1 + o(1))e^\gamma x \log \log x$$

where γ is Euler’s constant. In this paper our goal is to extend and sharpen these results. Note that the estimate of $T(\omega, x)$ seems to be very difficult, in particular, we have not been able to show that

$$\limsup_{x \rightarrow \infty} (T(\omega, x) - M(\omega, x)) = \infty$$

(which is certainly true), and the study of $T(d, x)$ seems to be even more difficult.

In Sections 9 and 10, we will study $T(\Omega, x)$, while in Sections 11 and 12 we will estimate $T(\sigma, x)$. First we give a lower bound for $T(\Omega, x)$:

THEOREM 5. *For all $\varepsilon > 0$ there are infinitely many numbers $x \in \mathbb{N}$ such that*

$$(9.3) \quad T(\Omega, x) > M(\Omega, x) + \exp\left(\left(\log 2 - \varepsilon\right) \frac{\log \log x}{\log \log \log x}\right) \\ = \left[\frac{\log x}{\log 2}\right] + \exp\left(\left(\log 2 - \varepsilon\right) \frac{\log \log x}{\log \log \log x}\right).$$

PROOF. The proof will be based on the following result of A. S. Bang [2] (see also [3, p. 385]):

LEMMA 2. *If $n \in \mathbb{N}$ and $n \neq 1, 6$, then there is a prime p such that $p \mid 2^n - 1$ but $p \nmid 2^m - 1$ for $m = 1, 2, \dots, n - 1$.*

To prove (9.3), consider a highly composite number n , i.e., assume that

$$d(n) = M(d, n) \quad \left(= \exp\left(\left(1 + o(1)\right) \log 2 \frac{\log n}{\log \log n}\right) \right).$$

Write $x = 2^n$. Then by Lemma 2 we have

$$T(\Omega, x) \geq \Omega(x - 1) + \Omega(x) = \Omega(2^n - 1) + n \\ \geq |\{i : (2^i - 1) \mid (2^n - 1), i \neq 1, 6\}| + M(\Omega, x) \\ \geq (d(n) - 2) + M(\Omega, x) \\ = \exp\left(\left(1 + o(1)\right) \log 2 \frac{\log n}{\log \log n}\right) + M(\Omega, x) \\ > \exp\left(\left(\log 2 - \varepsilon\right) \frac{\log \log x}{\log \log \log x}\right) + M(\Omega, x)$$

for x large enough, which proves (9.3).

10. In this section we will give upper bounds for consecutive values of the Ω function.

Erdős and Nicolas proved (9.1) by using a result of Ridout. Next we will show by using a result of Mahler that (9.1) can be extended to k consecutive values of the Ω function.

THEOREM 6. For every $\varepsilon > 0$ and $k \in \mathbb{N}$ there is an (ineffective) number $x_0 = x_0(\varepsilon, k)$ such that for $x > x_0$ we have

$$(10.1) \quad \left(\left[\frac{\log x}{\log 2} \right] \leq \right) \max_{n \leq x} (\Omega(n - k + 1) + \Omega(n - k + 2) + \dots + \Omega(n)) < \left(\frac{1}{\log 2} + \varepsilon \right) \log x.$$

PROOF. If $\mathcal{S} = \{q_1, q_2, \dots, q_t\}$ is a finite set of prime numbers and $a \in \mathbb{N}$, then denote the largest divisor of a composed solely of powers of primes from \mathcal{S} by $[a]_{\mathcal{S}}$ so that a can be represented in the form

$$(10.2) \quad a = [a]_{\mathcal{S}} v, \quad v \in \mathbb{N}, \quad \left(v, \prod_{p \in \mathcal{S}} p \right) = 1.$$

The proof of (10.1) will be based on the following result of Mahler [8, p. 159, Theorem 5, II]:

LEMMA 3. If \mathcal{S} is a finite set of prime numbers, $k \in \mathbb{N}$ and $\varepsilon > 0$, then there is an (ineffective) number $n_0 = n_0(\mathcal{S}, k, \varepsilon)$ such that for $n > n_0$ we have

$$[(n - k + 1)(n - k + 2) \dots n]_{\mathcal{S}} < n^{1+\varepsilon}.$$

In order to prove (10.1), clearly it suffices to show that for $n > n_1(k, \varepsilon)$ we have

$$(10.3) \quad \Omega(n - k + 1) + \Omega(n - k + 2) + \dots + \Omega(n) < \left(\frac{1}{\log 2} + \varepsilon \right) \log n.$$

Define t by

$$(10.4) \quad \frac{k}{\log p_{t+1}} \leq \frac{\varepsilon}{2} < \frac{k}{\log p_t}$$

and let $\mathcal{S} = \{p_1, p_2, \dots, p_t\}$. Write $u = [(n - k + 1)(n - k + 2) \dots n]_{\mathcal{S}}$ and define v by

$$(10.5) \quad (n - k + 1)(n - k + 2) \dots n = uv$$

so that $v \in \mathbb{N}$,

$$(10.6) \quad \left(v, \prod_{i=1}^t p_i \right) = 1$$

and

$$(10.7) \quad \Omega(n - k + 1) + \Omega(n - k + 2) + \dots + \Omega(n) = \Omega(u) + \Omega(v).$$

By Lemma 3 (with $\varepsilon/4$ in place of ε), for $n > n_2(k, \varepsilon)$ we have

$$u < n^{1+\varepsilon/4},$$

whence

$$(10.8) \quad \Omega(u) \leq \left\lceil \frac{\log n^{1+\varepsilon/4}}{\log 2} \right\rceil < \left(\frac{1}{\log 2} + \frac{\varepsilon}{2} \right) \log n.$$

Moreover, by (10.5) and (10.6) we have

$$n^k \geq uv \geq v = \prod_{p^\alpha \parallel v} p^\alpha \geq p_{t+1}^{\Omega(v)},$$

whence

$$(10.9) \quad \Omega(v) \leq \frac{k \log n}{\log p_{t+1}}.$$

(10.3) follows from (10.4), (10.7), (10.8) and (10.9), and this completes the proof of Theorem 6.

Since both (9.1) and Theorem 6 are ineffective, one might like to give an effective upper bound for consecutive values of the Ω function. Here we restrict ourselves to the case $k = 2$ (the case $k > 2$ could be handled similarly).

THEOREM 7. *There are effectively computable positive numbers n_3 and c_7 such that for $n > n_3$, $n \in \mathbb{N}$ we have*

$$(10.10) \quad \Omega(n-1) + \Omega(n) < \left(\frac{1}{\log 2} + \frac{1}{\log 3} - c_7 \right) \log n.$$

Proof. The proof will be based on the following result of Stewart [10, Theorem 2]:

LEMMA 4. *If $\mathcal{S} = \{q_1, q_2, \dots, q_r\}$ is a finite set of distinct prime numbers and $k \in \mathbb{N}$, then there are positive numbers c_8 and c_9 which are effectively computable in terms of q_1, q_2, \dots, q_r and k such that for all $n \in \mathbb{N}$, $n > k$ we have*

$$[(n-k+1)(n-k+2) \dots n]_{\mathcal{S}} < c_8 n^{t-c_9}$$

where $t = \min(k, r)$.

Note that

$$(10.11) \quad m \in \mathbb{N}, z > 1, \left(m, \prod_{p < z} p \right) = 1 \quad \text{imply} \quad \Omega(m) \leq \frac{\log m}{\log z}$$

since by $(m, \prod_{p < z} p) = 1$ we have

$$m = \prod_{p^\alpha \parallel m} p^\alpha \geq \prod_{p^\alpha \parallel m} z^\alpha = z^{\Omega(m)}.$$

For $n \in \mathbb{N}$, define the non-negative integers a, b and the positive integers u, v, z by

$$(10.12) \quad (n-1)n = 2^a 3^b v = uv, \quad 2^\alpha \parallel (n-1)n, \quad 3^b \parallel (n-1)n, \quad z = 2^a,$$

so that clearly we have

$$(10.13) \quad \Omega(n-1) + \Omega(n) = \Omega(u) + \Omega(v)$$

and

$$(10.14) \quad z \leq n.$$

By (10.11) and (10.12) we have

$$(10.15) \quad \Omega(v) \leq \frac{\log v}{\log 5} < \frac{\log(n^2/u)}{\log 5}.$$

Finally, by using Lemma 4 with $\mathcal{S} = \{2, 3\}$, we see that there are effectively computable positive numbers c_8 and c_9 such that

$$u = [(n-1)n]_{\mathcal{S}} < c_8 n^{2-c_9}$$

so that there are effectively computable positive numbers c_{10} and n_4 such that

$$(10.16) \quad u < n^{2-c_{10}} \quad \text{for } n > n_4.$$

By (10.12) and (10.14), we have

$$(10.17) \quad \begin{aligned} \Omega(u) &= \Omega(z) + \Omega(u/z) = \frac{\log z}{\log 2} + \frac{\log(u/z)}{\log 3} \\ &= \left(\frac{1}{\log 2} - \frac{1}{\log 3} \right) \log z + \frac{\log u}{\log 3} \\ &\leq \left(\frac{1}{\log 2} - \frac{1}{\log 3} \right) \log n + \frac{\log u}{\log 3}. \end{aligned}$$

It follows from (10.13), (10.15), (10.16) and (10.17) that for $n > n_4$ we have

$$\begin{aligned} \Omega(n-1) + \Omega(n) &= \left(\left(\frac{1}{\log 2} - \frac{1}{\log 3} \right) \log n + \frac{\log u}{\log 3} \right) + \frac{\log(n^2/u)}{\log 5} \\ &= \left(\frac{1}{\log 2} + \frac{1}{\log 3} \right) \log n - \left(\frac{1}{\log 3} - \frac{1}{\log 5} \right) \log(n^2/u) \\ &< \left(\frac{1}{\log 2} + \frac{1}{\log 3} - c_{10} \left(\frac{1}{\log 3} - \frac{1}{\log 5} \right) \right) \log n, \end{aligned}$$

which proves (10.10).

11. Finally, we will sharpen (9.2) by proving

THEOREM 8. *For $x \rightarrow \infty$ we have*

$$(11.1) \quad T(\sigma, x) \leq x \left(M \left(\frac{\sigma(n)}{n}, x \right) + 1 + O((\log \log x)^{-1}) \right).$$

Note that it is well-known [7, §22.9] that

$$(11.2) \quad M\left(\frac{\sigma(n)}{n}, x\right) = (1 + o(1))e^\gamma \log \log x.$$

Moreover, note that (11.1) is the best possible apart from the error term, as the following example shows: since

$$\lim_{x \rightarrow \infty} M\left(\frac{\sigma(n)}{n}, x\right) = \infty,$$

there are infinitely many integers x such that

$$(11.3) \quad \frac{\sigma(n)}{n} \leq \frac{\sigma(x)}{x} = M\left(\frac{\sigma(n)}{n}, x\right) \quad \text{for all } n \leq x.$$

If x satisfies (11.3), then

$$\begin{aligned} T(\sigma, x) &\geq \sigma(x-1) + \sigma(x) = \sum_{d|x-1} d + xM\left(\frac{\sigma(n)}{n}, x\right) \\ &\geq 1 + (x-1) + xM\left(\frac{\sigma(n)}{n}, x\right) = x\left(M\left(\frac{\sigma(n)}{n}, x\right) + 1\right). \end{aligned}$$

To prove Theorem 8, we need the following lemma:

LEMMA 5. *For all $\omega > 0$ there exist numbers $\varepsilon = \varepsilon(\omega) > 0$ and $x_0 = x_0(\omega)$ such that for $x > x_0$ we have*

$$\prod_{p \leq \varepsilon \log x} \left(1 - \frac{1}{p}\right)^{-1} < M\left(\frac{\sigma(n)}{n}, x\right) - \omega.$$

PROOF OF LEMMA 5. Clearly, it suffices to show that there is a number $R \in \mathbb{N}$ with

$$(11.4) \quad R \leq x$$

and

$$(11.5) \quad \frac{\sigma(R)}{R} > \prod_{p \leq \varepsilon \log x} \left(1 - \frac{1}{p}\right)^{-1} + \omega.$$

Indeed, define R in the same way as in the proof of Theorem 4. Then (11.4) holds by (7.4). Moreover, by (2.6) and Mertens' formula we have

$$\frac{\sigma(R)}{R} = \prod_{p \leq \frac{1}{2} \log x} \frac{\sigma(p^{\alpha_p})}{p^{\alpha_p}} = \prod_{p \leq \frac{1}{2} \log x} \left(1 - \frac{1}{p}\right)^{-1} (1 - p^{-(\alpha_p+1)})$$

$$\begin{aligned}
 &> \prod_{p \leq \varepsilon \log x} \left(1 - \frac{1}{p}\right)^{-1} \prod_{\varepsilon \log x < p \leq \frac{1}{2} \log x} \left(1 - \frac{1}{p}\right)^{-1} (1 - (\log x)^{-1/2})^{\pi((\log x)^{1/2})} \\
 &\quad \times \prod_{(\log x)^{1/2} < p} \left(1 - \frac{1}{p^2}\right) \\
 &= \prod_{p \leq \varepsilon \log x} \left(1 - \frac{1}{p}\right)^{-1} \exp\left((1 + o(1))\left(\sum_{\varepsilon \log x < p \leq \frac{1}{2} \log x} \frac{1}{p} - \frac{2}{\log \log x}\right)\right) \\
 &= \prod_{p \leq \varepsilon \log x} \left(1 - \frac{1}{p}\right)^{-1} \exp\left(\frac{(1 + O(1)) \log(1/(2\varepsilon)) + O(1)}{\log \log x}\right) \\
 &= \prod_{p \leq \varepsilon \log x} \left(1 - \frac{1}{p}\right)^{-1} + (1 + o(1))e^\gamma \log \log x \cdot \frac{\log(1/(2\varepsilon)) + O(1)}{\log \log x} \\
 &= \prod_{p \leq \varepsilon \log x} \left(1 - \frac{1}{p}\right)^{-1} + (1 + o(1))e^\gamma (\log(1/(2\varepsilon)) + O(1)),
 \end{aligned}$$

which implies (11.5) if ε is small enough in terms of ω , and this completes the proof of the lemma.

12. Completion of the proof of Theorem 8. We have to show that for all $m \leq x$ we have

$$\begin{aligned}
 (12.1) \quad &\sigma(m-1) + \sigma(m) \\
 &\leq x \left(M\left(\frac{\sigma(n)}{n}, x\right) + 1 + O((\log \log x)^{-1}) \right) \quad \text{for all } m \leq x.
 \end{aligned}$$

For all $m \leq x$ we have

$$\sigma(m-1) + \sigma(m) < m \left(\frac{\sigma(m-1)}{m-1} + \frac{\sigma(m)}{m} \right) \leq x \left(\frac{\sigma(m-1)}{m-1} + \frac{\sigma(m)}{m} \right).$$

Thus in order to prove (12.1), it suffices to show that

$$(12.2) \quad \frac{\sigma(m-1)}{m-1} + \frac{\sigma(m)}{m} \leq M\left(\frac{\sigma(n)}{n}, x\right) + 1 + O((\log \log x)^{-1}).$$

If

$$(12.3) \quad \max\left(\frac{\sigma(m-1)}{m-1}, \frac{\sigma(m)}{m}\right) \leq \frac{1}{2} \log \log x,$$

then for x large enough, (12.2) follows from (11.2) and (12.3). Thus we may assume that

$$(12.4) \quad \max\left(\frac{\sigma(m-1)}{m-1}, \frac{\sigma(m)}{m}\right) > \frac{1}{2} \log \log x.$$

Now we will define a finite sequence $(u_0, v_0), (u_1, v_1), \dots, (u_z, v_z)$ of pairs of positive integers by the following recursion: Define u_0, v_0 so that one of them is equal to $n - 1$ and the other is n , and

$$\frac{\sigma(v_0)}{v_0} \geq \frac{\sigma(u_0)}{u_0}.$$

Then by (12.4) we have

$$(12.5) \quad \frac{\sigma(v_0)}{v_0} > \frac{1}{2} \log \log x.$$

On the other hand, by $(m - 1, m) = 1$ and (11.2) we have

$$(12.6) \quad \frac{\sigma(u_0)}{u_0} \cdot \frac{\sigma(v_0)}{v_0} = \frac{\sigma(u_0 v_0)}{u_0 v_0} = \frac{\sigma((m - 1)m)}{(m - 1)m} \\ \leq M\left(\frac{\sigma(n)}{n}, x^2\right) = (1 + o(1))e^\gamma \log \log x.$$

It follows from (12.5) and (12.6) that for x large enough we have

$$(12.7) \quad \frac{\sigma(u_0)}{u_0} < 6.$$

Assume now that i is a non-negative integer and $(u_0, v_0), \dots, (u_i, v_i)$ have been defined so that

$$(12.8) \quad (u_j, v_j) = 1 \quad \text{for } j = 0, 1, \dots, i,$$

$$(12.9) \quad u_j v_j \mid u_{j-1} v_{j-1} \quad \text{for } j = 1, 2, \dots, i,$$

$$(12.10) \quad v_j < v_{j-1} \quad \text{for } j = 1, 2, \dots, i,$$

$$(12.11) \quad P(v_j) < P(v_{j-1}) \quad \text{for } j = 1, 2, \dots, i,$$

$$(12.12) \quad p(u_j) > p(u_{j-1}) \quad \text{for } j = 1, 2, \dots, i,$$

$$(12.13) \quad \frac{\sigma(v_j)}{v_j} > \frac{\sigma(v_{j-1})}{v_{j-1}} \quad \text{for } j = 1, 2, \dots, i,$$

and

$$(12.14) \quad \frac{\sigma(u_j)}{u_j} + \frac{\sigma(v_j)}{v_j} \geq \frac{\sigma(u_{j-1})}{u_{j-1}} + \frac{\sigma(v_{j-1})}{v_{j-1}} \quad \text{for } j = 1, 2, \dots, i$$

(note that (12.8) holds trivially for $j = 0$). If $P(v_i) < p(u_i)$, then the construction terminates, i.e., we put $z = i$ so that we have

$$(12.15) \quad P(v_z) < p(u_z).$$

(Note that (12.11) ensures that the construction terminates in finitely many steps.) If $P(v_i) > p(u_i)$, then write v_i and u_i as the product of prime powers:

$$v_i = q_1^{\alpha_1} \dots q_s^{\alpha_s}, \quad q_1 < \dots < q_s, \\ u_i = r_1^{\beta_1} \dots r_t^{\beta_t}, \quad r_1 < \dots < r_t$$

where

$$(12.16) \quad P(v_i) = q_s > p(u_i) = r_1 .$$

(Note that $P(v_i) \neq p(u_i)$ by (12.8).) Then define u_{i+1} and v_{i+1} by

$$\begin{aligned} v_{i+1} &= q_1^{\alpha_1} \dots q_{s-1}^{\alpha_{s-1}} r_1 , \\ u_{i+1} &= q_s^{\alpha_s} r_2^{\beta_2} \dots r_t^{\beta_t} . \end{aligned}$$

Then by (12.16), each of (12.8)–(12.12) holds trivially with $i + 1$ in place of j .

Moreover, if p is a prime number and $\alpha \in \mathbb{N}$, then

$$\frac{\sigma(p^\alpha)}{p^\alpha} = 1 + \frac{1}{p} + \dots + \frac{1}{p^\alpha} < \left(1 - \frac{1}{p}\right)^{-1} \leq \left(1 - \frac{1}{2}\right)^{-1} = 2$$

and thus it follows from (12.5) and (12.13) that

$$(12.17) \quad \omega(v_i) \rightarrow \infty .$$

By (12.16) and (12.17) we have

$$(12.18) \quad q_s \geq r_1 + 2 .$$

It follows that

$$\begin{aligned} (12.19) \quad \frac{\sigma(v_{i+1})}{v_{i+1}} &= \frac{\sigma(v_i)}{v_i} \cdot \frac{q_s^{\alpha_s}}{\sigma(q_s^{\alpha_s})} \cdot \frac{\sigma(r_1)}{r_1} \\ &= \frac{\sigma(v_i)}{v_i} \left(\sum_{k=0}^{\alpha_s} q_s^{-k}\right)^{-1} \left(1 + \frac{1}{r_1}\right) \\ &> \frac{\sigma(v_i)}{v_i} \left(1 - \frac{1}{q_s}\right) \left(1 + \frac{1}{r_1}\right) \\ &= \frac{\sigma(v_i)}{v_i} \left(1 + \frac{q_s - r_1 - 1}{q_s r_1}\right) > \frac{\sigma(v_i)}{v_i} , \end{aligned}$$

which proves (12.13) with $i + 1$ in place of j .

Finally, by (12.18) we have

$$\begin{aligned} (12.20) \quad \frac{\sigma(u_{i+1})}{u_{i+1}} &= \frac{\sigma(u_i)}{u_i} \cdot \frac{r_1^{\beta_1}}{\sigma(r_1^{\beta_1})} \cdot \frac{\sigma(q_s^{\alpha_s})}{q_s^{\alpha_s}} \\ &= \frac{\sigma(u_i)}{u_i} \left(\sum_{k=0}^{\beta_1} r_1^{-k}\right)^{-1} \left(\sum_{k=0}^{\alpha_s} q_s^{-k}\right) \\ &> \frac{\sigma(u_i)}{u_i} \left(1 - \frac{1}{r_1}\right) \left(1 + \frac{1}{q_s}\right) \\ &= \frac{\sigma(u_i)}{u_i} \left(1 - \frac{q_s - r_1 + 1}{q_s r_1}\right) \end{aligned}$$

$$\begin{aligned}
 &= \frac{\sigma(u_i)}{u_i} \left(1 - \frac{(q_s - r_1 - 1) + 2}{r_1 q_s} \right) \\
 &\geq \frac{\sigma(u_i)}{u_i} \left(1 - \frac{3(q_s - r_1 - 1)}{r_1 q_s} \right).
 \end{aligned}$$

Combining (12.19) and (12.20) we obtain

$$\begin{aligned}
 (12.21) \quad &\frac{\sigma(u_{i+1})}{u_{i+1}} + \frac{\sigma(v_{i+1})}{v_{i+1}} \\
 &> \frac{\sigma(u_i)}{u_i} \left(1 - \frac{3(q_s - r_1)}{q_s r_1} \right) + \frac{\sigma(v_i)}{v_i} \left(1 + \frac{q_s - r_1 - 1}{q_s r_1} \right) \\
 &= \left(\frac{\sigma(u_i)}{u_i} + \frac{\sigma(v_i)}{v_i} \right) + \frac{q_s - r_1 - 1}{q_s r_1} \left(\frac{\sigma(v_i)}{v_i} - 3 \frac{\sigma(u_i)}{u_i} \right).
 \end{aligned}$$

By (7.6), (12.5), (12.7), (12.9) and (12.13) we have

$$\frac{\sigma(v_i)}{v_i} \rightarrow \infty, \quad \frac{\sigma(u_i)}{u_i} = O(1),$$

whence

$$(12.22) \quad \frac{\sigma(v_i)}{v_i} - 3 \frac{\sigma(u_i)}{u_i} > 0$$

if x is large enough. (12.14) (with $i + 1$ in place of j) follows from (12.21) and (12.22), and this completes the proof of the existence of a sequence $(u_0, v_0), (u_1, v_1), \dots, (u_z, v_z)$ with the desired properties.

It follows from (12.5) and (12.13) that

$$(12.23) \quad \frac{\sigma(v_z)}{v_z} > \frac{\sigma(v_0)}{v_0} > \frac{1}{2} \log \log x$$

where, by (12.10), we have

$$(12.24) \quad v_z < v_0 \leq n.$$

Moreover, by (11.2), (12.8), (12.9) and $u_0 v_0 = (m - 1)m \leq x^2$ we have

$$\begin{aligned}
 (12.25) \quad &\frac{\sigma(u_z)}{u_z} \cdot \frac{\sigma(v_z)}{v_z} = \frac{\sigma(u_z v_z)}{u_z v_z} \leq M \left(\frac{\sigma(n)}{n}, x^2 \right) \\
 &= (1 + o(1)) e^\gamma \log \log x.
 \end{aligned}$$

It follows from (12.23) and (12.25) that

$$(12.26) \quad \frac{\sigma(u_z)}{u_z} < 6.$$

If

$$(12.27) \quad \frac{\sigma(v_z)}{v_z} \leq M \left(\frac{\sigma(n)}{n}, x \right) - 5,$$

then (12.2) follows from (12.14), (12.26) and (12.27). Thus we may assume that

$$(12.28) \quad \frac{\sigma(v_z)}{v_z} > M\left(\frac{\sigma(n)}{n}, x\right) - 5,$$

whence, by (11.2),

$$(12.29) \quad \frac{\sigma(v_z)}{v_z} \geq (1 + o(1))e^\gamma \log \log x.$$

Write v_z and u_z as a product of prime powers:

$$\begin{aligned} v_z &= q_1^{\alpha_1} \dots q_s^{\alpha_s}, & q_1 &< \dots < q_s, \\ u_z &= r_1^{\beta_1} \dots r_t^{\beta_t}, & r_1 &< \dots < r_t. \end{aligned}$$

By (12.15) we have

$$(12.30) \quad q_s = P(v_z) < p(u_z) = r_1.$$

By Mertens' formula, clearly we have

$$(12.31) \quad \begin{aligned} \frac{\sigma(v_z)}{v_z} &= \prod_{i=1}^s \frac{\sigma(q_i^{\alpha_i})}{q_i^{\alpha_i}} = \prod_{i=1}^s \left(\sum_{k=0}^{\alpha_i} q_i^{-k} \right) \\ &< \prod_{i=1}^s \left(1 - \frac{1}{q_i} \right)^{-1} \leq \prod_{i=1}^s \left(1 - \frac{1}{p_i} \right)^{-1} = \prod_{p \leq p_s} \left(1 - \frac{1}{p} \right)^{-1}. \end{aligned}$$

By using Lemma 5 with $\omega = 5$, we deduce from (12.28) and (12.31) that there is a positive number $\varepsilon > 0$ such that

$$(12.32) \quad q_s \geq p_s > \varepsilon \log x.$$

It follows from

$$u_0 v_0 = (m - 1)m \leq x^2$$

and (12.9) that

$$(12.33) \quad u_z \leq x^2.$$

This implies that

$$(12.34) \quad t = \omega(u_z) < 3 \log x$$

since by the prime number theorem, otherwise we had

$$\begin{aligned} u_z &= r_1^{\beta_1} \dots r_t^{\beta_t} \geq r_1 \dots r_t \geq \prod_{i \leq 3 \log x} p_i \\ &\geq \prod_{p \leq 3 \log x} p = \exp((1 + o(1))3 \log x) > x^2, \end{aligned}$$

which contradicts (12.33).

It follows from (12.30), (12.32) and (12.34) that

$$\begin{aligned}
 (12.35) \quad \frac{\sigma(u_z)}{u_z} &= \prod_{i=1}^t \frac{\sigma(r_i^{\beta_i})}{r_i^{\beta_i}} = \prod_{i=1}^t \left(\sum_{k=0}^{\beta_i} r_i^{-k} \right) \\
 &< \prod_{i=1}^t \left(1 - \frac{1}{r_i} \right)^{-1} \leq \prod_{i=1}^t \left(1 - \frac{1}{p_{s+i}} \right)^{-1} \\
 &\leq \prod_{i < 3 \log x} \left(1 - \frac{1}{p_{s+i}} \right)^{-1} \\
 &< \prod_{\varepsilon \log x < p < 4 \log x} \left(1 - \frac{1}{p} \right)^{-1} \\
 &= \exp \left((1 + o(1)) \sum_{\varepsilon \log x < p < 4 \log x} \frac{1}{p} \right) \\
 &= \exp(O((\log \log x)^{-1})) = 1 + O((\log \log x)^{-1}).
 \end{aligned}$$

It follows from (12.10), (12.14) and (12.35) that

$$\begin{aligned}
 \frac{\sigma(m-1)}{m-1} + \frac{\sigma(m)}{m} &= \frac{\sigma(u_0)}{u_0} + \frac{\sigma(v_0)}{v_0} \leq \frac{\sigma(u_z)}{u_z} + \frac{\sigma(v_z)}{v_z} \\
 &\leq (1 + O((\log \log x)^{-1})) + \max_{n \leq x} \frac{\sigma(n)}{n} \\
 &= M \left(\frac{\sigma(n)}{n}, x \right) + 1 + O((\log \log x)^{-1}),
 \end{aligned}$$

which proves (12.2), and this completes the proof of the theorem.

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