

On the positivity of the number of t -core partitions

by

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1. Introduction. A partition of a positive integer n is a nonincreasing sequence of positive integers with sum n . Here we define a special class of partitions.

DEFINITION 1. Let $t \geq 1$ be a positive integer. Any partition of n whose Ferrers graph have no hook numbers divisible by t is known as a t -core partition of n .

The hooks are important in the representation theory of finite symmetric groups and the theory of cranks associated with Ramanujan's congruences for the ordinary partition function [3, 4, 6].

If $t \geq 1$ and $n \geq 0$, then we define $c_t(n)$ to be the number of partitions of n that are t -core partitions. The arithmetic of $c_t(n)$ is studied in [3, 4]. The power series generating function for $c_t(n)$ is given by the infinite product:

$$\sum_{n=0}^{\infty} c_t(n)q^n = \prod_{n=1}^{\infty} \frac{(1 - q^{tn})^t}{1 - q^n}.$$

Note that these generating functions are quotients of Dedekind η -functions. The Dedekind η -function,

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n),$$

is a modular form with weight $1/2$. Here $q = e^{2\pi i\tau}$ is the uniformizing variable at $i\infty$.

If $p \geq 5$ is a prime, then let $\delta_p = (p^2 - 1)/24$. Let $a_p(n)$ be defined by

$$\frac{\eta^p(p\tau)}{\eta(\tau)} = \sum_{n \geq \delta_p} a_p(n)q^n.$$

The arithmetic behavior of $a_p(n)$ determines the number of p -core partitions of n because $c_p(n - \delta_p) = a_p(n)$. Fortunately, this η -quotient is a modular

form of weight $(p-1)/2$ on $\Gamma_0(p)$ with character $\varepsilon(n) = \left(\frac{n}{p}\right)$. Exact formulae for $c_5(n)$ and $c_7(n)$ appear in [4].

Given a positive integer t , is $c_t(n) > 0$ for all $n \geq 0$? In other words, does every positive integer n admit at least one t -core partition? The results in [4] show that $c_5(n)$ and $c_7(n)$ are positive for all $n \geq 0$. For $t \geq 5$ prime, Olsson has asked if $c_t(n) > 0$ for all n . One easily verifies that $c_2(n)$ and $c_3(n)$ are zero infinitely often. Here are the first few terms of the relevant generating functions.

$$\prod_{n=1}^{\infty} \frac{(1 - q^{2n})^2}{1 - q^n} = \sum_{n \geq 0} c_2(n)q^n = 1 + q + q^3 + q^6 + q^{10} + q^{15} + q^{21} + \dots,$$

$$\prod_{n=1}^{\infty} \frac{(1 - q^{3n})^3}{1 - q^n} = \sum_{n \geq 0} c_3(n)q^n$$

$$= 1 + q + 2q^2 + 2q^4 + q^5 + 2q^6 + q^8 + 2q^9 + 2q^{10} + 2q^{12} + \dots$$

In fact, it is a classical fact that

$$\prod_{n=1}^{\infty} \frac{(1 - q^{2n})^2}{1 - q^n} = \sum_{n \geq 0} q^{t_n}.$$

Here $t_n = n(n+1)/2$ are the usual triangular numbers.

It has been conjectured that if $t \geq 4$, then $c_t(n) > 0$ for all $n \geq 0$ [3, 6]. We will prove that if $t \geq 4$ and even, then $c_t(n)$ is always positive. Moreover, we will see that for any fixed $t \geq 4$, there are only finitely many n such that $c_t(n) = 0$.

The reader should note that $c_t(n) \leq c_{tk}(n)$ for all n . If a partition has no hook numbers divisible by t , then it certainly has no hook numbers divisible by any multiple of t . Hence the conjecture reduces to a study of $c_p(n)$ when p is a prime. For t that are divisible by 2 and 3 we will use the theory of quadratic forms, modular forms and Gauss' *Eureka* theorem to prove the positivity of $c_t(n)$. When $p \geq 5$ is prime, we will apply Deligne's theorem on the estimates of Fourier coefficients of cusp forms.

2. An application of Deligne's theorem. In this section we are interested in the modular forms

$$\frac{\eta^p(p\tau)}{\eta(\tau)} = \sum_{n \geq \delta_p} a(n)q^n$$

where $p \geq 5$ is prime and $\delta_p = (p^2 - 1)/24$. This form is a modular form on $\Gamma_0(p)$ of type $((p-1)/2, \varepsilon)$ where $\varepsilon(n) = \left(\frac{n}{p}\right)$. Clearly, δ_p is the order of this modular form at $i\infty$. These forms do not vanish at $\tau = 0$.

Consequently, since $\Gamma_0(p)$ has only 2 cusps, at $\tau = i\infty$ and $\tau = 0$, we get the decomposition

$$\frac{\eta^p(p\tau)}{\eta(\tau)} = \alpha_E E(\tau) + f_{\text{cusp}}$$

with $\alpha_E \neq 0$. $E(\tau)$ is the classical Eisenstein series centered at $\tau = 0$ defined by

$$E(\tau) = \sum_{n \geq 1} \sum_{d|n} \varepsilon\left(\frac{n}{d}\right) d^{(p-3)/2} q^n.$$

There are no cusp forms on $\Gamma_0(5)$ with character $\varepsilon(n) = \left(\frac{n}{5}\right)$. Consequently, it is easy to see that the generating function $\eta^5(5\tau)/\eta(\tau)$ is exactly the Eisenstein series.

In fact, we use these facts to show that if $p \geq 7$, then there must be a cusp form in the decomposition of $\eta^p(p\tau)/\eta(\tau)$ into Eisenstein series and cusp forms. This is true since

$$a(n) = 0 \quad \text{when } 1 \leq n \leq \delta_p - 1.$$

If $p \geq 7$, then $\delta_p \geq 1$. Since the Fourier coefficient of q in $E(\tau)$ is 1, we need a cusp form to cancel it.

Now we return to the conjecture.

THEOREM 1 (Deligne). *Let $f(\tau) = \sum_{n \geq 1} a(n)q^n$ be a cusp form of weight k for Γ , some congruence subgroup of $\text{SL}_2(\mathbb{Z})$. Then*

$$a(n) = O_\varepsilon(n^{(k-1)/2+\varepsilon}) \quad \text{for all } \varepsilon > 0.$$

Using these estimates, it is clear that the Fourier coefficients of the Eisenstein series $E(\tau)$ dominate the Fourier coefficients of a cusp form of the same weight beyond a certain point. Hence, since the Fourier coefficients $a(n)$ of $\eta^p(p\tau)/\eta(\tau)$ are nonnegative, it is clear that $\alpha_E > 0$. Consequently, at most finitely many Fourier coefficients $a(n)$ are zero.

This proves the following theorem which is also contained in [6].

THEOREM 2. *If t is a positive integer with at least one prime factor $p \geq 5$, then*

$$c_t(n) = 0 \quad \text{for at most finitely many } n \geq 0.$$

One can verify the positivity of $c_p(n)$ by explicitly calculating the decomposition of $\eta^p(p\tau)/\eta(\tau)$ into Eisenstein series and normalized eigenforms. One can explicitly calculate the point in the Fourier expansion beyond which the Eisenstein series dominates the cusp forms. Hence, for all primes p , the above theorem is easily strengthened by a finite calculation. The author has verified that $c_{11}(n)$ is always positive.

3. Remaining cases. Deligne’s theorem provides a partial solution to the positivity conjecture for all $t \geq 5$ that have prime divisors other than 2 and 3. Here we settle the remaining cases.

In the next theorem we show that $c_t(n)$ is positive when t is a multiple of 4. To do this, it suffices to show that $c_4(n)$ is always positive.

THEOREM 3. *If t is a multiple of 4, then $c_t(n) > 0$ for all $n \geq 0$.*

PROOF. Consider the η -quotient

$$\frac{\eta^4(32\tau)}{\eta(8\tau)} = q^5 \sum_{n=0}^{\infty} c_4(n)q^{8n} = \sum_{n=0}^{\infty} c_4(n)q^{8n+5}.$$

The first few terms in its Fourier expansion are

$$\frac{\eta^4(32\tau)}{\eta(8\tau)} = q^5 + q^{13} + 2q^{21} + 3q^{29} + q^{37} + \dots$$

This η -quotient is a modular form of weight $3/2$ on $\Gamma_0(32)$. The Serre–Stark basis theorem for modular forms of weight $1/2$ states that all modular forms with weight $1/2$ on $\Gamma_0(4N)$ are linear combinations of theta series. In this case, we find that this η -quotient is a product of theta series on $\Gamma_0(32)$. The relevant theta series are

$$\Theta_1(\tau) = \frac{1}{2} \sum_{n \equiv 1 \pmod{2}} q^{n^2} = q + q^9 + q^{25} + q^{49} + \dots$$

and

$$\Theta_2(\tau) = \frac{1}{2} \sum_{n \equiv 1 \pmod{2}} q^{2n^2} = q^2 + q^{18} + q^{50} + q^{98} + \dots$$

We obtain the following identity:

$$\frac{\eta^4(32\tau)}{\eta(8\tau)} = \Theta_1(\tau)\Theta_2^2(\tau).$$

Let Q be a ternary quadratic form and let $r(Q, n)$ denote the number of representations of n by the quadratic form Q by nonnegative integers. The above identity implies that

$$c_4(n) = r(x^2 + 2y^2 + 2z^2, 8n + 5).$$

This follows from the fact that if $x^2 + 2y^2 + 2z^2$ represents an integer of the form $8n + 5$, then x, y and z are all odd. Consequently, we must show that $r(x^2 + 2y^2 + 2z^2, 8n + 5) > 0$ for all $n \geq 0$.

The methods in [5] complete the proof. The genus of the ternary form $x^2 + 2y^2 + 2z^2$ consists of one equivalence class. By Corollary 44a of [5], every integer n is represented by some form in the genus of a ternary quadratic form Q if

$$Q \equiv n \pmod{p^{r+1}}$$

is solvable for every prime $p \mid 2d$ where d is the discriminant of Q . If p is odd, then p^r is the highest power of p dividing n . If $p = 2$, then p^r is the highest power of 2 dividing $4n$.

The discriminant of $Q = x^2 + 2y^2 + 2z^2$ is 4. Consequently, every integer of the form $8n + 5$ is represented by Q since the relevant congruence

$$Q = x^2 + 2y^2 + 2z^2 \equiv 8n + 5 \equiv 5 \pmod{8}$$

is solvable and since the genus of Q is 1 (i.e. equivalent ternary quadratic forms represent the same integers). This completes the proof. ■

Now we show that if $t \geq 4$ is even, then $c_t(n)$ is always positive. As a consequence of the last theorem, it suffices to assume that $t = 4s + 2$ where $s \geq 1$. This theorem is a corollary to Gauss' famous *Eureka* theorem. This theorem asserts that every nonnegative integer can be represented as a sum of 3 triangular numbers. Recall that the n th triangular number is defined by

$$t_n = \frac{n(n + 1)}{2}.$$

Here are the first few triangular numbers:

$$t_0 = 0, \quad t_1 = 1, \quad t_2 = 3, \quad t_3 = 6 \quad \text{and} \quad t_4 = 10.$$

Let $\Delta_k(n)$ be the number of representations of n as a sum of k triangular numbers. Define the formal power series generating function for the triangular number t_n by

$$\Psi(q) = \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^2}{1 - q^n} = \sum_{n=0}^{\infty} q^{t_n}.$$

Consequently, we obtain the following identity of formal power series:

$$\Psi^k(q) = \sum_{n=0}^{\infty} \Delta_k(n)q^n.$$

Now we state the famous *Eureka* theorem.

THEOREM 4 (Gauss). *If $\Delta_k(n)$ denotes the number of representations of n as a sum of k triangular numbers and $k \geq 3$, then $\Delta_k(n) > 0$ for all $n \geq 0$.*

THEOREM 5. *If $s \geq 1$ and $t = 4s + 2$, then $c_t(n) > 0$ for all $n \geq 0$.*

Proof. We simply manipulate the generating function for $c_t(n)$ by splitting it into a product of two formal power series:

$$\begin{aligned}
 \sum_{n=0}^{\infty} c_{4s+2}(n)q^n &= \prod_{n=1}^{\infty} \frac{(1 - q^{n(4s+2)})^{4s+2}}{1 - q^n} \\
 &= \prod_{n=1}^{\infty} \left(\frac{(1 - q^{n(4s+2)})^2}{1 - q^{n(2s+1)}} \right)^{2s+1} \prod_{n=1}^{\infty} \frac{(1 - q^{n(2s+1)})^{2s+1}}{1 - q^n} \\
 &= \psi^{2s+1}((2s + 1)q) \sum_{n=0}^{\infty} c_{2s+1}(n)q^n \\
 &= \sum_{n=0}^{\infty} \Delta_{2s+1}(n)q^{n(2s+1)} \sum_{n=0}^{\infty} c_{2s+1}(n)q^n.
 \end{aligned}$$

Since $s \geq 1$, we know that $\Delta_{2s+1}(n) > 0$ for all $n \geq 0$. Furthermore, if $0 \leq n \leq 2s$, then $0 < p(n) = c_{2s+1}(n)$ where $p(n)$ is the usual partition function (i.e. there are not enough nodes to make a $2s + 1$ hook when $0 \leq n \leq 2s$). It is now easy to see that the product power series has positive coefficients. This completes the proof that $c_{4s+2}(n) > 0$ for all $n \geq 0$. ■

The only values of $t \geq 4$ that we have not considered are powers of 3. We complete our discussion of the positivity of $c_t(n)$ by demonstrating that $c_9(n)$ is always positive.

THEOREM 6. *If $c_9(n)$ is the number of 9-core partitions of n , then $c_9(n) > 0$ for all $n \geq 0$.*

PROOF. The generating function for $c_9(n)$ is

$$\begin{aligned}
 \prod_{n=1}^{\infty} \frac{(1 - q^{9n})^9}{1 - q^n} &= \prod_{n=1}^{\infty} \frac{(1 - q^{3n})^3}{1 - q^n} \prod_{n=1}^{\infty} \frac{(1 - q^{9n})^9}{(1 - q^{3n})^3} \\
 &= \sum_{n=0}^{\infty} c_3(n)q^n \prod_{n=1}^{\infty} \frac{(1 - q^{9n})^9}{(1 - q^{3n})^3}.
 \end{aligned}$$

It is well known that $\eta^9(3\tau)/\eta^3(\tau)$ is the normalized weight 3 Eisenstein series on $\Gamma_0(3)$ with character $\varepsilon(n) = \left(\frac{n}{3}\right)$ that vanishes at $\tau = i\infty$. Its Fourier expansion is

$$\frac{\eta^9(3\tau)}{\eta^3(\tau)} = q + 3q^2 + 9q^3 + \dots = \sum_{n=1}^{\infty} \sigma_{2,\varepsilon}(n)q^n.$$

Here $\sigma_{2,\varepsilon}(n)$ is the generalized divisor function defined by

$$\sigma_{2,\varepsilon}(n) = \sum_{d|n} \varepsilon\left(\frac{n}{d}\right)d^2.$$

Consequently, we obtain the identity

$$\frac{\eta^9(9\tau)}{\eta^3(3\tau)} = q^3 \prod_{n=1}^{\infty} \frac{(1 - q^{9n})^9}{(1 - q^{3n})^3} = \sum_{n=1}^{\infty} \sigma_{2,\varepsilon}(n) q^{3n}.$$

This identity implies that

$$\prod_{n=1}^{\infty} \frac{(1 - q^{9n})^9}{(1 - q^{3n})^3} = \sum_{n=1}^{\infty} \sigma_{2,\varepsilon}(n) q^{3(n-1)} = \sum_{n=0}^{\infty} \sigma_{2,\varepsilon}(n+1) q^{3n}.$$

The coefficients of the last power series are positive because the generalized divisor function $\sigma_{2,\varepsilon}(n)$ is always positive.

Combining these facts we obtain

$$\sum_{n=0}^{\infty} c_9(n) q^n = \{1 + q + 2q^2 + \dots\} \sum_{n=0}^{\infty} \sigma_{2,\varepsilon}(n+1) q^{3n}.$$

Since the first 3 coefficients of the power series in braces are positive and $\sigma_{2,\varepsilon}(n)$ is always positive, we find that $c_9(n)$ is always positive. This completes the proof. ■

In summary we have proven the following theorem.

THEOREM 7. *Fix an integer $t \geq 4$. If $c_t(n)$ is the number of t -core partitions of n , then there are at most finitely many n such that $c_t(n) = 0$. Furthermore, if $t \geq 4$ is even or divisible by 5, 7, 9 or 11, then $c_t(n)$ is always positive.*

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